Journal of Mathematical Research & Exposition
May, 2008, Vol. 28, No. 2, pp. 331–339
DOI:10.3770/j.issn:1000-341X.2008.02.012
Http://jmre.dlut.edu.cn

The Problems of Best Approximation in β-Normed Spaces $(0 < \beta < 1)$

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Abstract This paper deals with the problems of best approximation in β-normed spaces. With the tool of conjugate cone introduced in [1] and via the Hahn-Banach extension theorem of β-subseminorm in [2], the characteristics that an element in a closed subspace is the best approximation are given in Section 2. It is obtained in Section 3 that all convex sets or subspaces of a β-normed space are semi-Chebyshev if and only if the space is itself strictly convex. The fact that every finite dimensional subspace of a strictly convex β-normed space must be Chebyshev is proved at last.

Keywords locally β-convex space; β-normed space; normed conjugate cone; the best approximation.

Document code A
MR(2000) Subject Classification 41A50; 41A65; 46A16
Chinese Library Classification O174.41

1. Introduction

In this paper, $0 < \beta \leq 1$ is a constant, $\mathbf{R}^+$ is the set of positive numbers, X is a linear space, and θ is used to denote the zero element or zero functional. If $A \subset X$ satisfies

$$[x, y]_\beta = \{\lambda x + \mu y : \lambda, \mu \geq 0, \lambda^\beta + \mu^\beta = 1\} \subset A, \quad \forall x, y \in A, \tag{1}$$

then A is said to be β-convex, where $[x, y]_\beta$ is the β-curve segment with the end points x and y while $[x, y]$ is used to denote the relative line segment.

Definition 1.1[1−4] *Suppose X is a topological linear space and $0 < \beta \leq 1$. X is called locally β-convex if there exists a θ-neighborhood basis consisting of β-convex sets.*

A real-valued functional f on X is called a *β-subseminorm if*

1) $f(x) \geq 0$, $x \in X$;

2) $f(tx) = t^\beta f(x)$, $t \in \mathbf{R}^+, x \in X$;

3) $f(x + y) \leq f(x) + f(y)$, $x, y \in X$.

The algebraic β-conjugate cone X'_β consisting of all β-subseminorms on X was first introduced in [1]. If X is a topological linear space, then X^*_β is used to denote the topological β-conjugate cone consisting of all continuous β-subseminorms on X. If X is locally β-convex, then X^*_β is

Received date: 2006-04-21; **Accepted date**: 2006-08-28
Foundation item: the Foundation of the Education Department of Jiangsu Province (No. 05KJB110001).

large enough to separate the points of X by [1], i.e., for each pair $x, y \in X, x \neq y$, there is $\theta \neq f \in X^*$ such that $f(x) \neq f(y)$. Then X_β^* is called the conjugate cone of X shortly. If the positive β-homogeneity 2) is replaced by the absolute β-homogeneity

2′). $f(tx) = |t|^\beta f(x)$, $t \in \mathbf{R}, x \in X$,

then f is called a β-seminorm. If a β-seminorm $\|\cdot\|_\beta$ also satisfies the zero hypothesis

1′). $\|x\|_\beta = 0 \Leftrightarrow x = \theta$,

then the F-norm $\|\cdot\|_\beta$ is called a β-norm and $(X, \|\cdot\|_\beta)$ a β-normed space. A complete β-normed space is called a β-Banach space[5]. Now with the θ-neighborhood basis consisting of β-convex sets

$$U_\varepsilon = \{x \in X : \ \|x\|_\beta < \varepsilon\} (\varepsilon > 0),$$

$(X, \|\cdot\|_\beta)$ forms a locally β-convex space. Sequence space l^β, function space $L^\beta(\mu)$ and Hardy space H^β are three typical β-Banach spaces. Suppose $(X, \|\cdot\|_\beta)$ is a β-normed space. Then under the norm

$$\|f\| = \sup\{f(x) : \|x\|_\beta \leq 1\}, f \in X_\beta^* \tag{2}$$

$(X_\beta^*, \|\cdot\|)$ forms a normed topological cone in the sense of [6], called the normed conjugate cone of $(X, \|\cdot\|_\beta)$. Then with the induced metric $\rho : X_\beta^* \times X_\beta^* \to R^+$

$$\rho(f, g) = \inf\{t > 0 : \exists h, l \in X_\beta^*, \|h\|, \|l\| \leq t \ \text{ s.t. } \ f + h = g + l\} \tag{3}$$

X_β^* forms a Quasi-translation invariant topological cone in the sense of [6].

Generally the problem of best approximation is discussed in normed spaces[7−8] and it is very difficult to study in ordinary F-normed spaces. But β-norm is one of the best F-norms, so it is possible to get some more pretty results if we study this problem in β-normed spaces. In this paper, with the tools of conjugate cone X_β^* introduced in [1] and via the Hahn-Banach Theorems about β-subseminorms obtained in [2], we study the problem of best approximation in β-normed spaces, the characteristics of an element to be the best approximation in a closed subspace are given in Section 2. It is obtained in Section 3 that all convex sets or subspaces of a β-normed space are semi-Chebyshev if and only if the space is itself strictly convex. The fact that every finite dimensional subspace of a strictly convex β-normed space must be Chebyshev is proved at last.

2. The characteristics of best approximation elements in closed subspaces

Let $(X, \|\cdot\|_\beta)$ be a β-normed space, E be a subset of X and $x \in X$. Then

$$d(E, x) = \inf\{\|y - x\|_\beta : y \in E\} \tag{4}$$

is called the distance from x to E. If there is a $y_0 \in E$ such that $d(E, x) = \|y_0 - x\|_\beta$, then y_0 is called an element of best approximation to x in E, and the set of all such elements is denoted by $P_E(x)$. If $P_E(x) \neq \phi$ for every $x \in X$, then E is called a proximal set; if $P_E(x)$ is at most a singleton for every $x \in X$, then E is called a semi-Chebyshev set; if $P_E(x)$ contains exactly one element for every $x \in X$, then E is called a Chebyshev set. The Chebyshev set is naturally the best set. If $x \in A \subset X$ satisfies that $\forall y \in X, y \neq x$, there is $z \in (x,y)$ such that $[x,z] \subset A$, then x is called an algebraic interior point of A. Suppose $\theta \in A$. A is said to be an absorbing set if $\forall x \in X, x \neq \theta$, there is $t>0$ such that $[0,t]x \subset A$; A is said to be a star shaped set if $\forall x \in A$, $[\theta, x] \subset A$. To deal with the problem of the best approximation we need following two lemmas.

Lemma 2.1[6,9] *Suppose $0<\beta<1$ and $\phi \neq A \subset X$.*

(i) If A is β-convex and θ is an algebraic interior point of A, then A is a star shaped absorbing set.

(ii) If $(X, \|\cdot\|_\beta)$ is a β-normed space and $f \in X'_\beta$, then $f \in X^*_\beta$ if and only if $\|f\|<\infty$, and then we have $f(x) \leq \|f\|\|x\|_\beta, x \in X$.

Lemma 2.2 (Dominated extension theorem[2]) *Suppose Y is a non-trivial subspace of X, $f \in Y'_\beta$ and $p \in X'_\beta$ such that $f(x) \leq p(x)$, $\forall x \in Y$. Then there exists $g \in X'_\beta$ such that*

$$g(x)=f(x), \ \ \forall x \in Y; \quad g(x) \leq p(x), \ \ \forall x \in X. \tag{5}$$

When $f \in Y'_{a\beta}$ and $p \in X'_{a\beta}$ with $f(x) \leq p(x)$, $\forall x \in Y$, then there exists $g \in X'_{a\beta}$ satisfying (5), too.

To prove the main theorems of this paper, we need to improve the dominated extension theorem as follows:

Lemma 2.3 (Norm-preserving extension theorem) *Let Y be a non-trivial subspace of β-normed space $(X, \|\cdot\|_\beta)$, $f \in Y^*_\beta$. Then there exists a $g \in X^*_\beta$ such that*

$$g(x)=f(x), \forall x \in Y; \quad \|g\|=\|f\|. \tag{6}$$

When $f \in Y^*_{a\beta}$ there exists $g \in X^*_{a\beta}$ satisfying (6), too.

Proof Suppose $f \in Y^*_\beta$. Then by (ii) of Lemma 2.1 we have $\|f\|<\infty$. Take $p(x)=\|f\|\|x\|_\beta, x \in X$, then by $\|\cdot\|_\beta \in X^*_\beta$ we have $p \in X^*_\beta$ and $\|p\|=\|f\|$. Using (ii) of Lemma 2.1 once again, we have $f(x) \leq p(x)$, $\forall x \in Y$, namely, p is the dominant function of f. Thus by Lemma 2.2 there exists a dominated extension $g \in X'_\beta$ such that

$$g(x)=f(x), \ \ \forall x \in Y; \quad g(x) \leq p(x), \ \ \forall x \in X.$$

From $g(x) \leq p(x)$ and $p \in X^*_\beta$ we know g is continuous at θ. For every $x, y \in X$, if $\|x-y\|_\beta \to 0$, then by the subadditivity of g we have

$$|g(x)-g(y)| \leq \max\{g(x-y), g(y-x)\} \to 0,$$

so g is continuous on X or $g \in X^*_\beta$. By $g(x) \leq p(x)$, we have $\|g\| \leq \|p\| = \|f\|$. On the other hand, from

$$\|g\|=\sup_{x \in X, \|x\|_\beta \leq 1} g(x) \geq \sup_{x \in Y, \|x\|_\beta \leq 1} g(x) = \sup_{x \in Y, \|x\|_\beta \leq 1} f(x) = \|f\|$$

it follows $\|g\| \geq \|f\|$, thus $\|g\|=\|f\|$. If $f \in Y^*_{a\beta}$, then from $p \in Y^*_{a\beta}$ and the corresponding result of Lemma 2.2 we know that there exists norm-preserving extension $g \in X^*_{a\beta}$ satisfying (6). This completes the proof. □

The characteristics of best approximation elements in a closed subspace of a β-normed space are given by the following two theorems.

Theorem 2.1 *Let $(X, \|\cdot\|_\beta)$ be a β-normed space with $0 < \beta < 1$. Suppose E is a closed subspace of X and $x_0 \notin E$. Then y_0 is a best approximation to x_0 in E if and only if there is $f \in X^*_{a\beta}$ such that*

(i) $\|f\| = 1$;

(ii) $f(x_0) = \|y_0 - x_0\|_\beta$;

(iii) $f(y + x_0) = f(x_0)$, $\forall y \in E$.

Proof Let $f \in X^*_{a\beta}$ satisfy the conditions (i)–(iii). Then for every $y \in E$ we have

$$\|y_0 - x_0\|_\beta = f(x_0) = f(y - x_0) \leq \|f\|\|y - x_0\|_\beta = \|y - x_0\|_\beta,$$

thus y_0 is a best approximation to x_0. On the other hand, if $y_0 \in P_E(x_0)$, then by the closeness of E and $x_0 \notin E$ we have

$$d = \inf\{\|y - x_0\|_\beta : y \in E\} = \|y_0 - x_0\|_\beta > 0.$$

Denote by

$$Y = \text{span}\{x_0, E\} = \{tx_0 + y : t \in R, y \in E\}$$

the subspace generated by x_0 and E, and define

$$f_1(tx_0 + y) = |t|^\beta d, \quad tx_0 + y \in Y. \tag{7}$$

We can verify $f_1 \in Y^*_{a\beta}$ which satisfies the conditions (i)–(iii). It is clear by $0 < \beta \leq 1$ that f_1 has absolute β-homogeneity and subadditivity. Thus $f_1 \in Y'_{a\beta}$. In direct-sum space Y, $t_n x_0 + y_n \to tx_0 + y$ if and only if $t_n \to t$ and $y_n \to y$, then f_1 is continuous or $f_1 \in Y^*_{a\beta}$. By its structure, f_1 satisfies conditions (ii) and (iii). For every $t \neq 0$ and $y \in E$, since

$$\begin{aligned} f_1(tx_0 + y) &= |t|^\beta d = |t|^\beta \|y_0 - x_0\|_\beta \\ &= |t|^\beta \inf_{x \in E} \|x - x_0\|_\beta \leq |t|^\beta \|\frac{-y}{t} - x_0\|_\beta = \|tx_0 + y\|_\beta, \end{aligned}$$

$\|f_1\| \leq 1$. Otherwise for every $\varepsilon > 0$, by (4) there is $y_1 \in E$ such that $0 < \|y_1 - x_0\|_\beta < d + \varepsilon$. Then by the (2) and (7) we have

$$\|f_1\| \geq f_1(\frac{y_1 - x_0}{\|y_1 - x_0\|_\beta^{\frac{1}{\beta}}}) = \frac{d}{\|y_1 - x_0\|_\beta} \geq \frac{d}{d + \varepsilon}.$$

Let $\varepsilon \to 0$. We have $\|f_1\| = 1$, namely, condition (i) also holds. At last let $f \in X^*_{a\beta}$ be the norm-preserving extension of f_1 by Lemma 2.3. It is natural that f also satisfies conditions (i)–(iii), which completes the proof. □

Lemma 2.4 *Let E be a subspace of linear space X, $x_0 \notin E$ and $f \in X'_{a\beta}$. Then*

$$f(y + x_0) = f(x_0), \quad \forall y \in E \Leftrightarrow f(y) = 0, \forall y \in E. \tag{8}$$

Proof If $f(y+x_0)=f(x_0)$, $\forall y\in E$, then by the absolute β-homogeneity and the subadditivity of f we have

$$f(y)-f(x_0)=f(y)-f(-x_0)\le f(y+x_0)=f(x_0)$$

for every $y\in E$, i.e., $0\le f(y)\le 2f(x_0)$. If there is some $\theta\ne y_0\in E$ with $f(y_0)>0$, then by absolute β-homogeneity of f and above inequality we have

$$n^\beta f(y_0)=f(ny_0)\le 2f(x_0)$$

for every natural number $n\in N$. This is contrary to $f(y_0)>0$, so $f(y)=0$, $\forall y\in E$.

If $f(y)=0$, $\forall y\in E$, then for every $y\in E$, by subadditivity of f we have

$$f(y+x_0)\le f(y)+f(x_0)=f(x_0).$$

Otherwise for every $y\in E$, by $f(-y)=0$ and the subadditivity of f we have

$$f(x_0)=f(x_0)-f(-y)\le f(x_0+y),$$

thus $f(x_0)=f(x_0+y)$, $\forall y\in E$. □

By Lemma 2.4 we have immediately the improved form of Theorem 2.1

Theorem 2.2 *Let $(X,\|\cdot\|_\beta)$ be a β-normed space with $0<\beta<1$. Suppose E is a closed subspace of X and $x_0\notin E$. Then y_0 is a best approximation to x_0 in E if and only if there is $f\in X^*_{a\beta}$ such that*

(i) $\|f\|=1$;

(ii) $f(x_0)=\|y_0-x_0\|_\beta$;

(iii) $f(y)=0,\forall y\in E$.

3. The semi-Chebyshev problems

To deal with the semi-Chebyshev problems, we need to introduce the concept of strict convexity for β-normed spaces.

Definition 3.1 *A β-normed space $(X,\|\cdot\|_\beta)$ is called strictly convex if its unit ball $B=\{x\in X:\|x\|_\beta\le 1\}$ is strictly convex in the common sense, namely, for every $x\ne y$, $\|x\|_\beta=\|y\|_\beta=1$, there holds $\|\frac{1}{2}(x+y)\|_\beta<1$ or*

$$\|x+y\|_\beta<2^\beta. \tag{9}$$

Differing greatly from normed space, the unit ball of a β-normed space may not be convex if $0<\beta<1$. For example, $(R^2,\|\cdot\|_\beta)$ forms a β-normed space with

$$\|(x,y)\|_\beta=|x|^\beta+|y|^\beta\quad,0<\beta<1.$$

Its unit ball B is just the set surrounded by $|x|^\beta+|y|^\beta=1$, with the four β-curve segments bending toward origin with end points $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$. Thus B is not convex and $(R^2,\|\cdot\|_\beta)$ is not strictly convex. But with another β-norm

$$|||(x,y)|||_\beta=(|x|^2+|y|^2)^{\frac{\beta}{2}}$$

$(R^2, ||| \cdot |||_\beta)$ forms a strictly convex β-Banach space, and its unit ball is just the unit disk in the ordinary sense. The following two theorems imply that for a β-normed space with $0 < \beta < 1$, the strict convexity is of the characteristic that all convex sets or subspaces are semi-Chebyshev.

Theorem 3.1 *Suppose $(X, \|\cdot\|_\beta)$ is a β-normed space, where $0 < \beta < 1$. Then every nonempty convex set in X is semi-Chebyshev if and only if $(X, \|\cdot\|_\beta)$ is strictly convex.*

Proof Suppose $(X, \|\cdot\|_\beta)$ is a strictly convex β-normed space. If there is a nonempty convex set $M \subset X$ that is not semi-Chebyshev, then there exists some $x_0 \notin M$, $y_1, y_2 \in M$, and $y_1 \neq y_2$ such that $y_1, y_2 \in P_M(x_0)$. By the zero hypothesis $1')$ we have

$$d = d(M, x_0) = \|y_1 - x_0\|_\beta = \|y_2 - x_0\|_\beta > 0,$$

thus

$$\|\frac{y_1 - x_0}{d^{\frac{1}{\beta}}}\|_\beta = \|\frac{y_2 - x_0}{d^{\frac{1}{\beta}}}\|_\beta = 1.$$

By the strict convexity of $(X, \|\cdot\|_\beta)$ we have

$$\|\frac{\frac{y_1+y_2}{2} - x_0}{d^{\frac{1}{\beta}}}\|_\beta = \|\frac{1}{2}(\frac{y_1 - x_0}{d^{\frac{1}{\beta}}} + \frac{y_2 - x_0}{d^{\frac{1}{\beta}}})\|_\beta < 1,$$

so $\|\frac{y_1+y_2}{2} - x_0\|_\beta < d$. By the convexity of M we get $\frac{y_1+y_2}{2} \in M$, which contradicts the presupposition $d = d(M, x_0)$, and the sufficiency is proved.

Now let us show the necessity. Use S to denote the unit sphere of X, i.e., $S = \{x \in X : \|x\|_\beta = 1\}$. If X is not strictly convex, then there are $x_1, x_2 \in S$, $x_1 \neq x_2$ such that $\|\frac{1}{2}(x_1 + x_2)\|_\beta \geq 1$. Let $x_0 = \frac{1}{2}(x_1 + x_2)$. If $\|x_0\|_\beta > 1$ or $x_0 \notin B$, denote by

$$[x_1, x_2] = \{x_0 + t(x_2 - x_1) : t \in [-\frac{1}{2}, \frac{1}{2}]\}$$

the line segment with the end points x_1 and x_2. Then by $x_0 \notin B$, $x_1, x_2 \in B$ and the closeness of B there are

$$t_1 = \max\{t < 0 : x_0 + t(x_2 - x_1) \in B\} \in [-\frac{1}{2}, 0),$$

$$t_2 = \min\{t > 0 : x_0 + t(x_2 - x_1) \in B\} \in (0, \frac{1}{2}],$$

such that

$$x_1' = x_0 + t_1(x_2 - x_1) \in (x_1, x_0),$$

$$x_2' = x_0 + t_2(x_2 - x_1) \in (x_0, x_2),$$

and

$$\|x_1'\|_\beta = \|x_2'\|_\beta = 1; \quad (x_1', x_2') \bigcap B = \phi.$$

By $(x_1', x_2') \bigcap B = \phi$ and the star shaped property of B (see Lemma 2.1(i)) we know that $\|x\|_\beta > 1$ for every $x \in (x_1', x_2')$, thus we have $x_1', x_2' \in P_M(\theta)$ for convex set $M = [x_1', x_2']$. This is contrary to the hypotheses that every convex set is semi-Chebyshev. Improving above procedure a little, we can also prove that there is not any $x \in (x_1, x_2)$ such that $\|x\|_\beta > 1$.

Now it remains to consider $\|x_1\|_\beta=\|x_2\|_\beta=\|x_0\|_\beta=1$ and $\|x\|_\beta\le 1$, $\forall x\in(x_1,x_2)$. If $\|x\|_\beta=1$ for every $x\in(x_1,x_2)$, then we also have $x_1,x_2\in P_M(\theta)$ for closed convex set $M=[x_1,x_2]$. This is contrary to the hypothesis. Otherwise, assume without loss of generality that there is $y_2\in(x_0,x_2)$ with $\|y_2\|_\beta<1$ or $y_2\in \mathrm{int}B$. Thus there is $b>1$ such that $z_2=by_2\in S$. Assume $y_2=ax_1+(1-a)x_2$. Then by $y_2\in(x_0,x_2)$ we have $0<a<\frac{1}{2}$. Denote by $l^+(\theta,x_0)=\{tx_0:t\ge 0\}$ the ray starting from θ passing through x_0. Then we can prove that the open line segment (x_1,z_2) intersects $l^+(\theta,x_0)$ at a point outside x_0, i.e., there are some positive numbers $\lambda>1$ and $0<\mu<1$ such that $\mu x_1+(1-\mu)z_2=\lambda x_0$, i.e.,

$$\mu x_1+(1-\mu)b[ax_1+(1-a)x_2]=\frac{\lambda}{2}(x_1+x_2)$$

or

$$[\mu+(1-\mu)ba]x_1+(1-\mu)b(1-a)x_2=\frac{\lambda}{2}(x_1+x_2). \tag{10}$$

By the linear independence of x_1 and x_2 (10) is equivalent to

$$\begin{cases}\mu+(1-\mu)ba=(1-\mu)b(1-a);\\ \frac{\lambda}{2}=\mu+(1-\mu)ba>\frac{1}{2}.\end{cases} \tag{11}$$

From the first equality we have $\mu=\frac{b(1-2a)}{1+b(1-2a)}\in(0,1)$. Let $\lambda=2[\mu+(1-\mu)ba]$. Then by $b>1$ we have

$$\frac{\lambda}{2}=\mu+(1-\mu)ba=\frac{b(1-a)}{1+b(1-2a)}=\frac{1}{2}\cdot\frac{b+b(1-2a)}{1+b(1-2a)}>\frac{1}{2}$$

or $\lambda>1$, thus μ,λ are the solution of equalities (11). Thus we again find a pair of $x_1,z_2\in S$, $x_1\ne z_2$, and $z_0=\lambda x_0\in(x_1,z_2)$ such that $\|z_0\|_\beta=\lambda^\beta\|x_0\|_\beta=\lambda^\beta>1$, and obtain a negative result against above section. The contradiction implies that the necessity also holds and this completes the proof. □

Now let us discuss the relation between the semi-Chebyshev property of subspaces and the strict convexity of the space.

Theorem 3.2 *Let* $(X,\|\cdot\|_\beta)$ *be a β-normed space, where* $0<\beta<1$. *Then every nontrivial subspace of* X *is semi-Chebyshev if and only if* X *is strictly convex.*

Proof The sufficiency is from Theorem 3.1, so only necessity needs to be shown. When $x\ne y$, use $l(x,y)$ to denote the line determined by x and y, and use $l^+(x,y)$ to denote the ray starting from x passing through y. If X is not strictly convex, then there exist $x_1,x_2\in S$, $x_1\ne x_2$ such that $\|\frac{1}{2}(x_1+x_2)\|_\beta\ge 1$, namely, $x_0=\frac{1}{2}(x_1+x_2)$ satisfies $\|x_0\|_\beta\ge 1$. If $\|x_0\|_\beta>1$ or $x_0\notin B$, then assume without loss of generality that X is the two dimensional real space generated by x_1 and x_2. By the following proving process we can see that this assumption is permissible. Let $\mathcal{L}$ be the family of lines $l_{ax_0}(x,y)=l(x,y)$ in X that intersects $l^+(\theta,x_0)$ at ax_0, intersects S at x and y on x_1 side and x_2 side of $l^+(\theta,x_0)$ respectively. It is clear that $x\ne y$ and $l_{x_0}(x_1,x_2)=l(x_1,x_2)\in\mathcal{L}$. For each $a>0$, let

$$\mathcal{L}_a=\{l\in\mathcal{L}:l\bigcap l^+(\theta,x_0)=ax_0\}$$

be the family of concurrent lines in $\mathcal{L}$. Then $l(x_1,x_2)\in\mathcal{L}_1$. Let

$$A=\{a>0:\mathcal{L}_a\neq\phi\}.$$

Then by the boundedness and the compctness of B there exists the maximum $a'=\max A\in[1,+\infty)$. By the definition there are $x',y'\in S$, $x'\neq y'$, lying on x_1 side and x_2 side of $l^+(\theta,x_0)$ respectively, such that $l_{a'x_0}(x',y')=l(x',y')\in\mathcal{L}$. For subspace $M=l(x',y')-x'$, we assert that

(i) $l(x',y')\bigcap \text{int}B=\phi$;

(ii) $d(M,-x')=1$;

(iii) $\theta, y'-x'\in P_M(-x')$.

By assertion (ii) and (iii) we know that M is not a semi-Chebyshev subspace, which is contrary to the hypotheses. Now it remains only to verify above three assertions. If there is a $w\in l(x',y')\bigcap\text{int}B$, then there exists $\delta>1$ such that $\delta w\in S$. Assume without loss of generality that w and δw lie on the x_2 side of $l^+(\theta,x_0)$. Thus via the similar methods used in the proof of Theorem 3.1 we can find some $a''>a'$ such that $l(x',\delta w)\bigcap l^+(\theta,x_0)=a''x_0$. This is contrary to the assumption $a'=\max A$. Therefore (i) holds. From $x',y'\in B$ and (i) we have

$$d(M,-x')=\inf\{\|y+x'\|_\beta:y\in M\}=\inf\{\|y\|_\beta:y\in l(x',y')\}=1,$$

namely, (ii) holds. At last by

$$d(\theta,-x')=\|x'\|_\beta=1,\quad d(y'-x',-x')=\|y'\|_\beta=1$$

we also have assertion (iii). Improving above procedure a little, we can also prove that there is not any $x\in(x_1,x_2)$ such that $\|x\|_\beta>1$.

Thus it remains to consider the situation $\|x_1\|_\beta=\|x_2\|_\beta=\|x_0\|_\beta=1$ and $\|x\|_\beta\leq 1$, $\forall x\in(x_1,x_2)$. Now if $l(x_1,x_2)\bigcap\text{int}B=\phi$, then for subspace $M=l(x_1,x_2)-x_1$ we have the contrary result $\theta,x_2-x_1\in P_M(-x_1)$. If $l(x_1,x_2)\bigcap\text{int}B\neq\phi$, assume without loss of generality that there is a point $y_2\in\text{int}B\bigcap l^+(x_0,x_2)$. Then there are $b>1$ with $z_2=by_2\in S$ and an intersection point of ray $l^+(\theta,x_0)$ and segment (x_1,z_2), say, $z_0=\lambda x_0\in(x_1,z_2)(\lambda>1)$ such that $\|z_0\|_\beta>1$. We have obtained the negative result against above section. the contradiction implies that the necessity also holds and this completes2 the proof. □

Theorem 3.3 *Let $(X,\|\cdot\|_\beta)$ be a β-normed space, where $0<\beta<1$. Then X is strictly convex if and only if every finite dimensional subspace of X has Chebyshev property.*

Proof The sufficiency is from the proving procedure of Theorem 3.2. Let E be a finite dimensional subspace of a strictly convex β-normed X. Then by Theorems 3.1 and 3.2 we know that E is a semi-Chebyshev set. For every $x_0\notin E$, by the closeness of E we have $d=d(E,x_0)>0$. Let

$$D=\{x\in E:\|x-x_0\|_\beta\leq d+1\}.$$

Then by the fact that E has finite dimension we know that D is compact and $d=d(E,x_0)=d(D,x_0)>0$, $P_E(x_0)=P_D(x_0)$. By the compactness of D and the continuity of function $d(x)=\|x-x_0\|_\beta$, there is $y_0\in D\subset E$ such that $d(E,x_0)=d(D,x_0)=\|y_0-x_0\|_\beta$, thus

$P_E(x_0) \neq \phi$, namely, E is a Chebyshev subspace of X. □

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