第27卷第2期 数 学 研 究 与 评 论 Vol.27, No.2
2007年5月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION May, 2007

Article ID: 1000-341X(2007)02-0298-09 **Document code**: A

Modified Ishikawa Iterative Process with Errors in Normed Linear Spaces

YAO Yong-hong[1], *CHEN Ru-dong*[1], *ZHOU Hai-yun*[2]
(1. Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China;
2. Dept. of Math., Shijiazhuang Mechanical Engineering College, Hebei 050003, China)
(E-mail: yuyanrong@tjpu.edu.cn)

Abstract: Ishikawa iterative sequences with errors different from the iterative sequences introduced by Liu and Xu are given. Moreover, the problem of approximating the fixed points of φ-hemicontractive mapping in normed linear spaces by the modified Ishikawa iterative sequences with errors is investigated. The results presented in this paper improve and extend the results of the others.

Key words: φ-hemicontractive mapping; modified Ishikawa iterative sequences with errors; fixed point; normed linear spaces.
MSC(2000): 47H05; 47H10; 47H14
CLC number: O177.91

1. Introduction and preliminaries

Let X be a real normed linear space with norm $\|\cdot\|$ and X^* be the dual space of X. The normalized duality mapping $J: X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

An operator T with the domain $D(T)$ and range $R(T)$ is said to be *Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|$$

holds for all $x, y \in D(T)$. Without loss of generality, we may assume that $L \ge 1$.

Recall that an operator T is said to be accretive if, for every $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \ge 0. \tag{1.1}$$

An operator T is said to be strongly accretive if there exists a positive constant k such that, for every $x, y \in D(T)$,

$$\langle Tx - Ty, j(x-y) \rangle \ge k\|x - y\|^2. \tag{1.2}$$

Received date: 2005-03-22; **Accepted date**: 2006-07-02

T is said to be $\varphi-$*strongly accretive* if there exists a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that the inequality

$$\langle Tx - Ty, j(x - y)\rangle \geq \varphi(\|x - y\|)\|x - y\| \tag{1.3}$$

holds for all $x, y \in D(T)$. Let $N(T) = \{x \in X : Tx = 0\}$. If $N(T) \neq \emptyset$ and Inequalities (1.1)–(1.3) hold for any $x \in D(T)$ and $y \in N(T)$, then the corresponding operator T is called quasi-accretive, strongly quasi-accretive, and φ-strongly quasi-accretive, respectively.

Let $F(T) = \{x \in D(T) : Tx = x\}$. A mapping $T : D(T) \subset X \to X$ is called a φ-hemicontractive mapping if $(I - T)$ is φ-strongly quasi-accretive, where I is the identity mapping on X. It is very clear that, if T is $\varphi-$hemicontractive, then $F(T) \neq \emptyset$ and there exists a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that the inequality

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \varphi(\|x - y\|)\|x - y\| \tag{1.4}$$

holds for all $x \in D(T)$ and $y \in F(T)$. Such operators have been studied and used extensively by several researchers[1−10]. There are errors always occurring in the iterative process because the manipulation is inaccurate. It is no doubt that researching the convergent problems of iterative methods with errors members is a significant job.

Recently, Liu[11] and Xu[12] introduced the Ishikawa iterative schemes with errors as follows, respectively.

Let E be a nonempty subset of X, $T : E \to X$ be a mapping, the Ishikawa iterative sequences with errors are defined as follows:

1) The iterative method introduced by Liu[11]

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n + u_n, \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n + v_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ be two real sequences in $[0, 1]$ satisfying some conditions, $\{u_n\}, \{v_n\}$ be two sequences in E with some restrictions (for example $\sum_{n=0}^{\infty} \|u_n\| < +\infty$, $v_n \to 0 (n \to \infty)$).

The errors members $\{u_n\}$ and $\{v_n\}$ are difficult to be given beforehand, so Xu[12] introduced a new Ishikawa iterative process with errors.

2) The iterative method introduced by Xu[12]

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n Ty_n + \gamma_n u_n, \\ y_n = (1 - \beta_n - \delta_n)x_n + \beta_n Tx_n + \delta_n v_n, \quad \forall n \geq 0, \end{cases} \tag{ISE}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}$ and $\{\gamma_n\}$ are four real sequences in $[0, 1]$ satisfying some conditions, $\{u_n\}, \{v_n\}$ are two bounded sequences in E.

This is compatible with the randomness of the occurrence of errors, so it is satisfactory. But we know the boundedness of the errors is difficult to measure directly due to the randomness. In addition, the errors are closely related to the initial value and the iterative sequences. Inspired by this idea, let $\{u_n - x_n\}, \{v_n - x_n\}$ be two bounded sequences in E, i.e., there exists some constant $M > 0$ such that $\|u_n - x_n\| \leq M, \|v_n - x_n\| \leq M$, then we obtain the following immediately.

(1) If the iterative sequence $\{x_n\}$ is bounded, so are the errors $\{u_n\}$ and $\{v_n\}$. Then we obtain the iterative sequences introduced by Xu.

(2) It follows from (ISE), we obtain

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + \gamma_n(u_n - x_n), \\ y_n = (1-\beta_n)x_n + \beta_n T x_n + \delta_n(v_n - x_n), \quad \forall n \geq 0. \end{cases} \tag{ISEM}$$

Then we obtain the iterative sequences introduced by Liu. Unlike Liu's iterative method with errors, the (ISEM) iterative method with errors is always well defined. The (ISEM) iterative method with errors takes the form given by Xu, therefore there is a guarantee that the schemes are in E.

(3) It follows from (ISEM), we obtain

$$\begin{cases} x_{n+1} = (1-\alpha'_n)x_n + \alpha'_n T y_n + \gamma_n(u_n - x_n), \\ y_n = (1-\beta'_n)x_n + \beta'_n T x_n + \delta_n(v_n - x_n), \quad \forall n \geq 0, \end{cases} \tag{ISEMM}$$

where $\alpha'_n = \frac{\alpha_n}{1+\|x_n\|+\|Ty_n\|}$, $\beta'_n = \frac{\beta_n}{1+\|x_n\|+\|Tx_n\|}$.

The purpose of this paper is to investigate the problem of approximating the fixed points of φ-hemicontractive mapping by the modified Ishikawa iterative sequences with errors different from the iterative sequences introduced by Liu[11] and Xu[12] in normed linear spaces. The results presented in this paper improve and extend the results of others.

Lemma 1.1[12] *Let $\{\rho_n\}, \{\sigma_n\}$ be two nonnegative real sequences, if there exists some positive integer n_0, when $n \geq n_0$ such that*

$$\rho_{n+1} \leq (1-t_n)\rho_n + \sigma_n t_n,$$

where $t_n \in [0,1]$, $\sum_{n=0}^{\infty} t_n = +\infty$, $\sigma_n \to 0$, then $\lim_{n\to\infty} \rho_n = 0$.

Lemma 1.2 *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences, if there exists some positive integer n_1, when $n \geq n_1$ such that*

$$a_{n+1}^2 \leq (1+b_n)a_n^2 + c_n a_{n+1}, \tag{1.5}$$

where $\sum_{n=1}^{\infty} b_n < +\infty$, $\sum_{n=1}^{\infty} c_n < +\infty$, then $\{a_n\}$ is bounded.

Proof From (1.5) and inequality $a^2 + b^2 \geq 2ab$, we have

$$a_{n+1}^2 \leq (1+b_n)a_n^2 + \frac{1+a_{n+1}^2}{2} c_n. \tag{1.6}$$

Notice that $c_n \to 0 (n \to \infty)$, there exists a positive integer $n_2 \geq n_1$ such that $\frac{1}{2} \leq 1 - \frac{c_n}{2} \leq 1$ as $n \geq n_2$. It follows from (1.6) that

$$a_{n+1}^2 \leq (1 + 2b_n + c_n)a_n^2 + c_n,$$

when $n \geq n_2$, which implies that $\{a_n^2\}$ is bounded. This completes the proof.

2. Main results

Theorem 2.1 *Let X be a real normed linear space, E be a nonempty closed convex subset of X, and $T : E \to E$ be a Lipschitz φ-hemicontractive mapping. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ be four sequences in $[0, 1]$ satisfying the following conditions*

(i) $\alpha_n + \gamma_n \leq 1, \beta_n + \delta_n \leq 1$;

(ii) $\alpha_n \to 0, \beta_n \to 0, \delta_n \to 0 (n \to \infty)$;

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty, \gamma_n = o(\alpha_n)$.

For an arbitrary $x_0 \in E$, the sequence $\{x_n\}$ is defined by (ISEM). If there exists some constant $M > 0$ such that $\|u_n - x_n\| \leq M, \|v_n - x_n\| \leq M$, then $\{x_n\}$ converges strongly to the unique fixed point q of T.

Proof We observe that, for all $x \in E$ and $y \in F(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \varphi(\|x - y\|)\|x - y\|.$$

It follows from the definition of T that, if $F(T) \neq \emptyset$, then $F(T)$ must be a singleton. Let q denote the unique fixed point of T.

By the definition of (ISEM), we obtain

$$\begin{aligned} x_{n+1} - q =& (1 - \alpha_n)(x_n - q) + \alpha_n(Ty_n - Tx_{n+1}) + \\ & \alpha_n(Tx_{n+1} - q) + \gamma_n(u_n - x_n). \end{aligned}$$

Evaluating $j(x_{n+1} - q) \in J(x_{n+1} - q)$ on this equality, we obtain

$$\begin{aligned} \|x_{n+1} - q\|^2 =& (1 - \alpha_n)\langle x_n - q, j(x_{n+1} - q)\rangle + \alpha_n\langle Ty_n - Tx_{n+1}, j(x_{n+1} - q)\rangle + \\ & \alpha_n\langle Tx_{n+1} - q, j(x_{n+1} - q)\rangle + \gamma_n\langle u_n - x_n, j(x_{n+1} - q)\rangle \\ \leq& (1 - \alpha_n)\|x_n - q\|\|x_{n+1} - q\| + \alpha_n L\|y_n - x_{n+1}\|\|x_{n+1} - q\| + \\ & \alpha_n[\|x_{n+1} - q\|^2 - \varphi(\|x_{n+1} - q\|)\|x_{n+1} - q\|] + M\gamma_n\|x_{n+1} - q\|. \end{aligned} \tag{2.1}$$

Now we estimate $\|x_{n+1} - y_n\|$ which follows from (ISEM)

$$\begin{aligned} \|x_{n+1} - y_n\| &= \|(1 - \alpha_n)(x_n - y_n) + \alpha_n(Ty_n - y_n) + \gamma_n(u_n - x_n)\| \\ &\leq \|\beta_n(Tx_n - x_n) + \delta_n(v_n - x_n)\| + \alpha_n\|Ty_n - y_n\| + M\gamma_n \\ &\leq \beta_n(L + 1)\|x_n - q\| + \alpha_n(L + 1)\|y_n - q\| + M(\gamma_n + \delta_n). \end{aligned}$$

Since

$$\|y_n - q\| \leq \|x_n - q\| + \beta_n L\|x_n - q\| + M\delta_n \leq (L + 1)\|x_n - q\| + M\delta_n,$$

so

$$\begin{aligned} \|x_{n+1} - y_n\| &\leq [\alpha_n(L + 1)^2 + \beta_n(L + 1)]\|x_n - q\| + [1 + \alpha_n(L + 1)]M\delta_n + M\gamma_n \\ &= d_n\|x_n - q\| + e_n, \end{aligned}$$

where $d_n = \alpha_n(L+1)^2 + \beta_n(L+1) \to 0$, $e_n = [1+\alpha_n(L+1)]M\delta_n + M\gamma_n \to 0 (n\to\infty)$.

Substituting the above inequality into (2.1), we get

$$\begin{aligned}\|x_{n+1}-q\|^2 \le &(1-\alpha_n)\|x_n-q\|\|x_{n+1}-q\| + \alpha_n L d_n\|x_n-q\|\|x_{n+1}-q\| + \\ &\alpha_n L e_n\|x_{n+1}-q\| + M\gamma_n\|x_{n+1}-q\| + \\ &\alpha_n[\|x_{n+1}-q\|^2 - \varphi(\|x_{n+1}-q\|)\|x_{n+1}-q\|].\end{aligned}$$

Since $\gamma_n = o(\alpha_n)$, we let $\gamma_n = \varepsilon_n\alpha_n$, where $\varepsilon_n \to 0$ $(n\to\infty)$, thus the above inequality becomes

$$\begin{aligned}\|x_{n+1}-q\|^2 \le &(1-\alpha_n+\alpha_n L d_n)\|x_n-q\|\|x_{n+1}-q\| + \\ &\alpha_n L e_n\|x_{n+1}-q\| + M\varepsilon_n\alpha_n\|x_{n+1}-q\| + \\ &\alpha_n[\|x_{n+1}-q\|^2 - \varphi(\|x_{n+1}-q\|)\|x_{n+1}-q\|].\end{aligned}$$

By using inequality $x^2+y^2 \ge 2xy$ it yields

$$\begin{aligned}\|x_{n+1}-q\|^2 \le &(1-\alpha_n+\alpha_n L d_n)\frac{1}{2}[\|x_n-q\|^2 + \|x_{n+1}-q\|^2] + \\ &\alpha_n(Le_n + M\varepsilon_n)\frac{1}{2}(1+\|x_{n+1}-q\|^2) + \\ &\alpha_n[\|x_{n+1}-q\|^2 - \varphi(\|x_{n+1}-q\|)\|x_{n+1}-q\|].\end{aligned}$$

Let $\varepsilon_n' = Le_n + M\varepsilon_n$, hence $\varepsilon_n' \to 0 (n\to\infty)$. Notice that $\alpha_n \to 0$, there exists a positive integer n_0 such that $0 < 1-\alpha_n(1+Ld_n+\varepsilon_n') < 1$ as $n \ge n_0$. By transposing and simplifying the above inequality, we have

$$\begin{aligned}\|x_{n+1}-q\|^2 \le &\frac{1-\alpha_n+\alpha_n L d_n}{1-\alpha_n(1+Ld_n+\varepsilon_n')}\|x_n-q\|^2 + \\ &\frac{\alpha_n(\varepsilon_n' - 2\varphi(\|x_{n+1}-q\|)\|x_{n+1}-q\|)}{1-\alpha_n(1+Ld_n+\varepsilon_n')}.\end{aligned} \tag{2.2}$$

Set $\sigma = \liminf_{n\to\infty} \frac{\varphi(\|x_{n+1}-q\|)}{1+\|x_{n+1}-q\|}$.

We now consider the following two possible cases.

Case 1 If $\sigma > 0$, by taking $\gamma \in (0, \min\{1,\sigma\})$, then there exists a positive integer $n_1 \ge n_0$ such that $\varphi(\|x_{n+1}-q\|) > \gamma\|x_{n+1}-q\|$, $\alpha_n < \frac{1}{4}$, $2Ld_n < \frac{\gamma}{4}, \varepsilon_n' < \frac{\gamma}{4}$ from the definition of σ and that $\alpha_n \to 0, d_n \to 0, \varepsilon_n' \to 0 (n\to\infty)$. Therefore from (2.2) it follows that

$$\|x_{n+1}-q\|^2 \le \frac{1-\alpha_n+\alpha_n L d_n}{1-\alpha_n(1+Ld_n+\varepsilon_n')}\|x_n-q\|^2 + \frac{\alpha_n\varepsilon_n' - 2\alpha_n\gamma\|x_{n+1}-q\|^2}{1-\alpha_n(1+Ld_n+\varepsilon_n')}. \tag{2.3}$$

Transposing and operating (2.3), we have

$$\|x_{n+1}-q\|^2 \le \frac{1-\alpha_n+\alpha_n L d_n}{1-\alpha_n(1+Ld_n+\varepsilon_n'-2\gamma)}\|x_n-q\|^2 + \frac{\alpha_n\varepsilon_n'}{1-\alpha_n(1+Ld_n+\varepsilon_n'-2\gamma)}, \tag{2.4}$$

and

$$\begin{aligned}
&(1-\alpha_n+\alpha_n Ld_n)-(1-\frac{\gamma}{2}\alpha_n)[1-\alpha_n(1+Ld_n+\varepsilon'_n-2\gamma)]\\
&=\alpha_n[2Ld_n+\varepsilon'_n-2\gamma-\frac{\gamma}{2}\alpha_n(1+Ld_n+\varepsilon'_n-2\gamma)]\\
&\le\alpha_n[2Ld_n+\varepsilon'_n-2\gamma+\alpha_n\gamma^2]\\
&\le\alpha_n[\frac{\gamma}{4}+\frac{\gamma}{4}-2\gamma+\frac{1}{4}\gamma^2]<0,
\end{aligned}$$

which implies that

$$\|x_{n+1}-q\|^2\le(1-\frac{\gamma}{2}\alpha_n)\|x_n-q\|^2+\frac{\alpha_n\varepsilon'_n}{1-\alpha_n(1+Ld_n+\varepsilon'_n-2\gamma)}, \tag{2.5}$$

for all $n_1\ge n_0$.

From Lemma 1.1 and Inequality (2.5), we have $x_n\to q(n\to\infty)$.

Case 2 If $\sigma=0$, then there exists a subsequence $\{x_{n_j+1}\}\subset\{x_n\}$ which converges strongly to q. For an arbitrary number $\varepsilon\in(0,1)$, we can prove that there exists some positive integer j_0 such that for any natural number k, $\|x_{n_j+k}-q\|\le\varepsilon$ holds as $j\ge j_0$. Since $\alpha_n\to 0, d_n\to 0, \varepsilon'_n\to 0$, there exists a positive integer $n_2\ge n_1$ such that $0<1-\alpha_n(1+Ld_n+\varepsilon'_n)<1$, $2Ld_n<\frac{1}{2}\varphi(\varepsilon)$, $\varepsilon'_n<\frac{1}{4}\varphi(\varepsilon)\varepsilon$ as $n\ge n_2$. Now we use the induction method. Because $\|x_{n_j+1}-q\|\to 0$ as $j\to\infty$, hence there exists j_1 with $n_{j_1}\ge n_2$ such that

$$\|x_{n_j+1}-q\|\le\varepsilon, \tag{2.6}$$

when $j\ge j_1$. Suppose that

$$\|x_{n_j+m}-q\|\le\varepsilon \tag{2.7}$$

holds for $k=m$, we assert $\|x_{n_j+m+1}-q\|\le\varepsilon$. If this is not the case, then $\|x_{n_j+m+1}-q\|>\varepsilon$. Noting the strictly increasing property of φ, we have $\varphi(\|x_{n_j+m+1}-q\|)>\varphi(\varepsilon)$. From Inequality (2.2) we obtain the following estimates

$$\begin{aligned}
\varepsilon^2<&\|x_{n_j+m+1}-q\|^2\\
\le&\|x_{n_j+m}-q\|^2+\frac{\alpha_{n_j+m}(2Ld_{n_j+m}+\varepsilon'_{n_j+m})}{1-\alpha_{n_j+m}(1+Ld_{n_j+m}+\varepsilon'_{n_j+m})}\|x_{n_j+m}-q\|^2+\\
&\frac{\alpha_{n_j+m}\varepsilon'_{n_j+m}-2\alpha_{n_j+m}\varphi(\|x_{n_j+m+1}-q\|)\|x_{n_j+m+1}-q\|}{1-\alpha_{n_j+m}(1+Ld_{n_j+m}+\varepsilon'_{n_j+m})}\\
\le&\|x_{n_j+m}-q\|^2+\frac{\alpha_{n_j+m}[(2Ld_{n_j+m}+\varepsilon'_{n_j+m})\|x_{n_j+m}-q\|^2+\varepsilon'_{n_j+m}-2\varphi(\varepsilon)\varepsilon]}{1-\alpha_{n_j+m}(1+Ld_{n_j+m}+\varepsilon'_{n_j+m})}\\
\le&\varepsilon^2+\frac{\alpha_{n_j+m}}{1-\alpha_{n_j+m}(1+Ld_{n_j+m}+\varepsilon'_{n_j+m})}\cdot[(\frac{1}{2}\varphi(\varepsilon)+\frac{1}{4}\varphi(\varepsilon)\varepsilon)\varepsilon^2+\frac{1}{4}\varphi(\varepsilon)\varepsilon-2\varphi(\varepsilon)\varepsilon]\\
\le&\varepsilon^2,
\end{aligned}$$

which is a contradiction. Hence we have $\|x_{n_j+m+1}-q\|\le\varepsilon$, the induction is completed. Consequently, $\{x_n\}$ converges strongly to q.

Theorem 2.2 *Let X be a real normed linear space, E be a nonempty closed convex subset of X and $T : E \to E$ be a uniformly continuous and φ-hemicontractive mapping. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, and $\{\delta_n\}$ be four sequences in $[0,1]$ satisfying the following conditions*

(i) $\alpha_n + \gamma_n \le 1, \beta_n + \delta_n \le 1$;

(ii) $\beta_n \to 0, \delta_n \to 0 (n \to \infty)$;

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \alpha_n^2 < \infty, \sum_{n=0}^{\infty} \gamma_n < \infty$.

For an arbitrary $x_0 \in E$, the sequence $\{x_n\}$ is defined by (ISEMM). If there exists some constant $M > 0$ such that $\|u_n - x_n\| \le M, \|v_n - x_n\| \le M$, then $\{x_n\}$ converges strongly to the unique fixed point q of T.

Proof From Theorem 2.1, we assume q denote the unique fixed point of T.

By the conditions of Theorem 2.2, we obtain

$$\|x_{n+1} - y_n\| \le 2(\alpha_n + \beta_n) + M(\gamma_n + \delta_n) \to 0 (n \to \infty). \tag{2.8}$$

From (ISEMM), there exists $j(x_{n+1} - q) \in J(x_{n+1} - q)$ such that

$$\begin{aligned}\|x_{n+1} - q\|^2 \le & (1 - \alpha_n^{'})^2 \|x_n - q\|^2 + 2\alpha_n^{'} \|Ty_n - Tx_{n+1}\| \|x_{n+1} - q\| + \\ & 2\alpha_n^{'} [\|x_{n+1} - q\|^2 - \varphi(\|x_{n+1} - q\|)\|x_{n+1} - q\|] + 2M\gamma_n \|x_{n+1} - q\|.\end{aligned}$$

Notice that $\alpha'_n \to 0$, there exists a positive integer n_0 such that $0 < 1 - 2\alpha'_n < 1$ and by the above inequality, we have

$$\begin{aligned}\|x_{n+1} - q\|^2 \le & \|x_n - q\|^2 + \frac{\alpha_n^{'2}}{1 - 2\alpha_n^{'}} \|x_n - q\|^2 - \frac{2\alpha_n^{'}}{1 - 2\alpha_n^{'}} \|x_{n+1} - q\| \times \\ & [\varphi(\|x_{n+1} - q\|) - \rho_n] + \frac{2M\gamma_n}{1 - 2\alpha_n^{'}} \|x_{n+1} - q\|,\end{aligned} \tag{2.9}$$

where $\rho_n = \|Tx_{n+1} - Ty_n\| \to 0 (n \to \infty)$.

Set $\liminf_{n\to\infty} \frac{\varphi(\|x_{n+1} - q\|)}{1 + \|x_{n+1} - q\|} = \sigma$.

We now consider the following two possible cases.

Case 1 If $\sigma = 0$, then there exists a subsequence $\{x_{n_j+1}\} \subset \{x_{n+1}\}$ which converges strongly to q. For arbitrary number $\varepsilon \in (0,1)$, we can prove that there exists some positive integer i_0 such that $\|x_{n_i+m} - q\| < \varepsilon$ holds as $i \ge i_0, m \in N$. Since $\rho_n \to 0, \sum_{n=0}^{\infty} \alpha_n^2 < \infty, \sum_{n=0}^{\infty} \gamma_n < \infty$, there exists a positive integer n_1 such that $\rho_n \le \frac{1}{2}\varphi(\frac{\varepsilon}{2}), \sum_{n=n_0}^{\infty} \alpha_n^2 < \frac{\varepsilon}{16}, \sum_{n=n_0}^{\infty} \gamma_n < \frac{\varepsilon}{32M}, \alpha'_n \le \alpha_n < \frac{1}{4}$ as $n \ge n_1$. Now we use the induction method. Since $\|x_{n_i+1} - q\| \to 0$ as $i \to \infty$, there exists a positive integer i with $n_i \ge n_1$ such that

$$\|x_{n_i+1} - q\| < \frac{\varepsilon}{2}. \tag{2.10}$$

Suppose that

$$\|x_{n_i+k} - q\| < \varepsilon \tag{2.11}$$

holds for $k = 1, 2, \ldots, m$.

Set $\rho = \max\{r \in N : \|x_{n_i+r} - q\| < \frac{\varepsilon}{2}, 1 \le r \le m+1\}$.

If $\rho = m+1$, then we have $\|x_{n_i+m+1} - q\| < \frac{\varepsilon}{2} < \varepsilon$, the induction is completed.

If $1 \le \rho \le m$, we have

$$\|x_{n_i+r} - q\| \ge \frac{\varepsilon}{2}, \quad p+1 \le r \le m+1. \tag{2.12}$$

By (2.9), (2.12) and the strictly increasing property of φ, we have

$$\|x_{n_i+r+1} - q\|^2 \le \|x_{n_i+r} - q\|^2 + \frac{\alpha_{n_i+r}^{'2}}{1-2\alpha_{n_i+r}}\|x_{n_i+r} - q\|^2 + \frac{2M\gamma_{n_i+r}}{1-2\alpha'_{n_i+r}}\|x_{n_i+r+1} - q\|$$

holds for $r = p, p+1, \ldots, m$.

From (2.11) and the above inequality, we have

$$\begin{aligned}\|x_{n_i+m+1} - q\|^2 \le& \|x_{n_i+p} - q\|^2 + \sum_{r=p}^{m} 2\alpha_{n_i+r}^{'2}\|x_{n_i+r} - q\|^2 + \\ & 4M\gamma_{n_i+m}\|x_{n_i+m+1} - q\| + \sum_{r=p}^{m-1} 4M\gamma_{n_i+r}\|x_{n_i+r+1} - q\| \\ \le& \frac{\varepsilon^2}{2} + \frac{\varepsilon}{8}\|x_{n_i+m+1} - q\|,\end{aligned}$$

which implies that

$$\|x_{n_i+m+1} - q\| \le \frac{1}{2}[\frac{\varepsilon}{8} + \sqrt{(\frac{\varepsilon}{8})^2 + 4(\frac{\varepsilon^2}{2})}] \le \varepsilon.$$

The induction is completed, and consequently, $\{x_n\}$ converges strongly to q.

Case 2 If $\sigma > 0$, there exists a positive integer $n_2 \ge n_1$ such that $\varphi(\|x_{n+1} - q\|) \ge \sigma(1 + \|x_{n+1} - q\|) \ge \sigma, \rho_n \le \frac{\sigma}{2}, \alpha_n' \le \alpha_n \le \frac{1}{4}$. From (2.9), when $n \ge n_2$, we get

$$\|x_{n+1} - q\|^2 \le \|x_n - q\|^2 + 2\alpha_n^{'2}\|x_n - q\|^2 - \sigma\alpha_n'\|x_{n+1} - q\| + 4M\gamma_n\|x_{n+1} - q\|. \tag{2.13}$$

Letting $a_n = \|x_n - q\|$, from (2.13), we have

$$a_{n+1}^2 \le a_n^2 + 2\alpha_n^{'2}a_n^2 + 4M\gamma_n a_{n+1}. \tag{2.14}$$

So it follows from Lemma 1.2 that $\{a_n\}$ is bounded. Let $a_n \le M_1, \forall n \ge n_2$. By the strictly increasing property of φ and (2.13), we have

$$a_{r+1}^2 \le a_{n_2}^2 + 2M_1^2\sum_{n=n_2}^{r}\alpha_n^{'2} - \sigma\varphi^{-1}(\sigma)\sum_{n=n_2}^{r}\alpha_n' + 4MM_1\sum_{n=n_2}^{r}\gamma_n,$$

when $n \ge n_2$, which implies that

$$\sigma\varphi^{-1}(\sigma)\sum_{n=n_2}^{r}\alpha_n' \le a_{n_2}^2 + 2M_1^2\sum_{n=n_2}^{r}\alpha_n^{'2} + 4MM_1\sum_{n=n_2}^{r}\gamma_n,$$

let $r \to \infty$, we have $\sum_{n=n_2}^{\infty} \alpha'_n < \infty$, which is a contradiction. This shows that $\sigma > 0$ is impossible. This completes the proof.

Remark 2.3 The results presented in this paper improve and extend the corresponding results in [8, 10] the following aspects:

(i) Theorems 2.1 and 2.2 extend the results in [8, 10] to the modified Ishikawa iterative sequences with errors different from the iterative sequences introduced by Liu and Xu.

(ii) Theorem 2.2 drops the boundedness of $D(T)$ and Lipschitzian continuity of T.

(iii) The methods of Theorem 2.2 are different.

References:

[1] CHANG Shi-sheng, CHO Y J, LEE B S et.al. *Iterative approximations of fixed points and solutions for strongly accretive and strongly pseudo-contractive mappings in Banach spaces* [J]. J. Math. Anal. Appl., 1998, **224**: 149-165.

[2] HUANG Zhen-yu. *Approximating fixed points of φ-hemicontractive mappings by the Ishikawa iteration process with errors in uniformly smooth Banach spaces* [J]. Comput. Math. Appl., 1998, **36**: 13-21.

[3] OSLIKE M O. *Iterative solution of nonlinear equations of the φ-strongly accretive type* [J]. J. Math. Anal. Appl., 1996, **200**: 259-271.

[4] REICH S. *An iterative procedure for constructing zeros of accretive operators in Banach spaces* [J]. Nonlinear Anal., 1978, **2**: 85-92.

[5] ZHOU Hai-yun, JIA Yu-ti. *Approximating the zeros of accretive operators by the Ishikawa iteration process* [J]. Abstr. Appl. Anal.. 1996, **1**(2): 153-167.

[6] ZHOU Hai-yun, JIA Yu-ti. *Approximation of fixed points of strongly pseudo-contractive maps without Lipschitz assumption* [J]. Proc. Amer. Math. Soc., 1997, **125**: 1705-1709.

[7] ZHOU Hai-yun. *Iterative solution of nonlinear equations involving strongly accretive operators without Lipschitz assumption* [J]. J. Math. Anal. Appl., 1997, **213**: 296-307.

[8] ZHOU Hai-yun. *Approximating zeros of $\varphi-$strongly accretive operators by the Ishikawa iterative process with errors* [J]. Acta. Math. Sinica., 1998, **41**(5): 1091-1100.

[9] ZHOU Hai-yun, CHO Y J. *Ishikawa and Mann iterative processes with errors for nonlinear φ-strongly quasi-accretive mappings in normed linear spaces* [J]. J. Korean Math. Soc., 1999, 36: 1067-1073.

[10] ZHOU Hai-yun, CHO Y J, CHANG Shi-sheng. *Approximating the fixed points of φ-hemicontractions by the Ishikawa iterative process with mixed errors in normed linear spaces* [J]. Nonlinear Anal., 2001, **47**: 4819-4826.

[11] LIU Li-shan. *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces* [J]. J. Math. Anal. Appl., 1995, **194**: 114-125.

[12] XU Yu-guang. *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive operator equations* [J]. J. Math. Anal. Appl., 1998, **224**(1): 91-101.

赋范线性空间中修改的具误差的 Ishikawa 迭代程序

姚永红[1]， 陈汝栋[1]，周海云[2]

(1. 天津工业大学数学系，天津 300160; 2. 石家庄军械工程学院数学系，河北 石家庄 050003)

摘要: 本文给出了有别于刘立山以及徐洪坤的意义下的具误差的 Ishikawa 迭代程序. 进一步，还研究了赋范线性空间中 φ- 半压缩映象不动点的具误差的 Ishikawa 迭代逼近问题. 所得的结果改进和推广了许多相应的结果.

关键词: φ- 半压缩映象；修改的具误差的 Ishikawa 迭代程序；不动点；赋范线性空间.