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STRUCTURE OF TRAJECTORIES OF DISCRETE DISPERSIVE DYNAMICAL SYSTEMS

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Abstract

In this paper we study global attractors of discrete dispersive dynamical systems generated by set-valued mappings and the structure of trajectories of these dynamical systems.

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1 Introduction

In the present paper we study a discrete-time dynamical system introduced in [2, 3] and studied in [4, 6]. This dynamical system is described by a compact metric space of states and a transition operator which is set-valued. Dynamical systems theory has been a rapidly growing area of research which has various applications to physics, engineering, biology and economics. In this theory one of the goals is to study the asymptotic behavior of the trajectories of a dynamical system. Usually in the dynamical systems theory a transition operator is single-valued. In the present paper we study dynamical systems with a set-valued transition operator. Such dynamical systems describe economical models [2, 3, 5].

Let (X,ρ) be a compact metric space and let $a : X \to 2^X \setminus \{\emptyset\}$ be a set-valued mapping whose graph

$$\mathrm{graph}(a) = \{(x,y) \in X \times X : \ y \in a(x)\}$$

is a closed subset of $X \times X$. For each nonempty subset $E \subset X$ set

$$a(E) = \cup\{a(x) : \ x \in E\} \text{ and } a^0(E) = E.$$

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By induction we define $a^n(E)$ for any natural number n and any nonempty subset $E \subset X$ as follows:

$$a^n(E) = a(a^{n-1}(E)).$$

In this paper we study convergence and structure of trajectories of the dynamical system generated by the set-valued mapping a. Following [2, 3] this system is called a discrete dispersive dynamical system. It should be mentioned that iterations of set-valued operators were also studied in [1].

First we define a trajectory of this system.

A sequence $\{x_t\}_{t=0}^{\infty} \subset X$ is called a trajectory of a (or just a trajectory if the mapping a is understood) if $x_{t+1} \in a(x_t)$ for all integers $t \geq 0$.

Put

$$\Omega(a) = \{z \in X : \text{ for each } \varepsilon > 0 \text{ there is a trajectory } \{x_t\}_{t=0}^{\infty} \text{ such that } \liminf_{t\to\infty} \rho(z, x_t) \leq \varepsilon\}. \tag{1.1}$$

Clearly, $\Omega(a)$ is a nonempty closed subset of (X, ρ). In the present paper the set $\Omega(a)$ will be called a global attractor of a. Note that in [2-4] $\Omega(a)$ was called a turnpike set of a. This terminology was motivated by mathematical economics [3].

For each $x \in X$ and each nonempty closed subset $E \subset X$ put

$$\rho(x, E) = \inf\{\rho(x, y) : y \in E\}.$$

It is clear that for each trajectory $\{x_t\}_{t=0}^{\infty}$ we have $\lim_{t\to\infty} \rho(x_t, \Omega(a)) = 0$.

Let $\phi : X \to R^1$ be a continuous function such that

$$\phi(z) \geq 0 \text{ for all } z \in X, \tag{1.2}$$

$$\phi(y) \leq \phi(x) \text{ for all } x \in X \text{ and all } y \in a(x). \tag{1.3}$$

It is clear that the function ϕ is a Lyapunov function for the dynamical system generated by the mapping a. Note that in [6] we consider a particular case of the dynamical system considered here with the function ϕ identically zero. In [7] we generalize the results of [6] for dynamical systems generated by mappings $a : K \to 2^X \setminus \{\emptyset\}$, where K is a closed subset of X. It should be mentioned that in mathematical economics usually X is a subset of the finite-dimensional Euclidean space and ϕ is a linear functional on this space [3, 5].

The following theorem is our first main result. It will be proved in Section 2.

Theorem 1.1. *The following properties are equivalent:*

(1) If a sequence $\{x_t\}_{t=-\infty}^{\infty} \subset X$ satisfies $x_{t+1} \in a(x_t)$ and $\phi(x_{t+1}) = \phi(x_t)$ for all integers t ,then

$$\{x_t\}_{t=-\infty}^{\infty} \subset \Omega(a).$$

(2) For each $\varepsilon > 0$ there exists a natural number $T(\varepsilon)$ such that for each trajectory $\{x_t\}_{t=0}^{\infty} \subset X$ satisfying $\phi(x_t) = \phi(x_{t+1})$ for all integers $t \geq 0$ the inequality $\rho(x_t, \Omega(a)) \leq \varepsilon$ holds for all integers $t \geq T(\varepsilon)$.

For each $x \in X$ set

$$\pi(x) = \sup\{\lim_{t\to\infty} \phi(x_t) : \{x_t\}_{t=0}^{\infty} \text{ is a trajctory and } x_0 = x\}. \tag{1.4}$$

The following two results will be proved in Section 3.

Proposition 1.2. *Let $x \in X$. Then there is a trajectory $\{x_t\}_{t=0}^{\infty}$ such that $x_0 = x$ and $\pi(x) = \lim_{t\to\infty} \phi(x_t)$.*

Proposition 1.3. *The function $\pi : X \to R^1$ is upper semicontinuous.*

It is clear that for each $x \in X$ and each $y \in a(x)$

$$\pi(y) \le \pi(x), \tag{1.5}$$

for each $x \in X$

$$\pi(x) \le \phi(x) \tag{1.6}$$

and that for each $x \in X$ and each natural number n

$$\pi(x) \le \sup\{\phi(y) : y \in a^n(x)\}. \tag{1.7}$$

It is easy to see that the following proposition holds.

Proposition 1.4. *Let $x \in X$ and $\{x_t\}_{t=0}^{\infty} \subset X$ be a trajectory such that $x_0 = x$. Then*

$$\lim_{t\to\infty} \phi(x_t) = \pi(x)$$

if and only if for each integer $t \ge 0$

$$\pi(x_{t+1}) = \max\{\pi(z) : z \in a(x_t)\}.$$

The following result will also be proved in Section 3.

Proposition 1.5. *Let $x \in X$. Then*

$$\pi(x) = \lim_{n\to\infty} \sup\{\phi(y) : \in a^n(x)\}.$$

The following theorem is our second main result which will be proved in Section 4.

Theorem 1.6. *Assume that the property (1) of Theorem 1.1 holds.*

Let $\varepsilon > 0$ and $x \in X$. Then there exist $\delta > 0$ and a natural number L such that for each integer $T > 2L$ and each trajectory $\{x_t\}_{t=0}^{T}$ satisfying

$$x_0 = x \text{ and } \phi(x_T) \ge \pi(x_0) - \delta$$

the following inequality holds:

$$\rho(x_t, \Omega(a)) \le \varepsilon,\ t = L, \dots, T - L.$$

In the paper we use the following property.

(P) If $x_1, x_2 \in \Omega(a)$ and $\phi(x_1) = \phi(x_2)$, then $x_1 = x_2$.

Note that the property (P) holds for many models of economic dynamics for which $\Omega(a)$ is a subinterval of a line [3, 5].

The following theorem will be proved in Section 5.

Theorem 1.7. *Assume that the property (P) holds. Then each trajectory of a converges to an element of* $\Omega(a)$.

It is not difficult to see that the following result holds.

Proposition 1.8. *Assume that the property (P) holds and that* $\{x_t\}_{t=0}^{\infty}$ *is a trajectory of a such that* $\lim_{t\to\infty}\phi(x_t)=\pi(x)$. *Then by Theorem 1.7 there exists*

$$F(x)=\lim_{t\to\infty} x_t,$$

the equality

$$\phi(F(x))=\lim_{t\to\infty}\phi(x_t)=\pi(x)$$

holds and moreover, $F(x)$ *is a unique element of* $\Omega(a)$ *belonging to* $\phi^{-1}(\pi(x))$.

In the sequel if the property (P) holds, then for each $x\in X$ we denote by $F(x)$ the unique element of $\Omega(a)\cap\phi^{-1}(\pi(x))$.

The following turnpike result [2, 4] describes the structure of optimal (with respect to the functional ϕ) trajectories of a. It will be proved in Section 6.

Theorem 1.9. *Assume that the property (P) and the property (1) of Theorem 1.1 hold. Let* $\epsilon>0$ *and* $x\in X$. *Then there exist* $\delta>0$ *and a natural number L such that for each integer* $T>2L$ *and each trajectory* $\{x_t\}_{t=0}^{T}$ *satisfying*

$$x_0=x \text{ and } \phi(x_T)\geq\pi(x)-\delta$$

the following inequality holds:

$$\rho(x_t,F(x))\leq\epsilon,\ t=L,\dots,T-L.$$

2 Proof of Theorem 1.1

It is clear that the property (2) implies the property (1).

Let us show that the property (1) implies the property (2). Assume that the property (1) holds and assume that the property (2) does not hold. Then there is $\epsilon>0$ such that for each natural number n there exist a trajectory $\{x_t^{(n)}\}_{t=0}^{\infty}$ and an integer $\tau_n\geq n$ such that

$$\rho(x_{\tau_n}^{(n)},\Omega(a))>\epsilon,$$

$$\phi(x_t^{(n)})=\phi(x_0^{(n)}),\ t=0,1,\dots. \tag{2.1}$$

Let $n\geq 1$ be an integer. Define

$$y_t^{(n)}=x_{t+\tau_n}^{(n)} \text{ for all integers } t\geq-\tau_n. \tag{2.2}$$

Clearly,

$$y_0^{(n)}=x_{\tau_n}^{(n)},\ n=1,2,\dots. \tag{2.3}$$

Extracting subsequences and using diagonalization process we obtain that there exists a strictly increasing sequence of natural numbers $\{n_j\}_{j=1}^{\infty}$ such that for each integer t there is

$$y_t = \lim_{j\to\infty} y_t^{(n_j)}. \tag{2.4}$$

It is not difficult to see that

$$y_{t+1} \in a(y_t) \text{ and } \phi(y_t) = \phi(y_{t+1}) \text{ for all integers } t. \tag{2.5}$$

In view of (2.1), (2.3) and (2.4),

$$\rho(y_0, \Omega(a)) \geq \varepsilon. \tag{2.6}$$

By (2.5) and property (1), $\{y_t\}_{t=-\infty}^{\infty} \subset \Omega(a)$. This contradicts (2.6). The contradiction we have reached proves that (1) implies (2). Theorem 1.1 is proved.

3 Proofs of Propositions 1.2, 1.3 and 1.5

Proof of Proposition 1.2. It is clear that for each integer $n \geq 1$ there is a trajectory $\{x_t^{(n)}\}_{t=0}^{\infty}$ such that

$$x_0^{(n)} = x,\ \lim_{t\to\infty} \phi(x_t^{(n)}) \geq \pi(x) - 1/n. \tag{3.1}$$

Extracting subsequences and using diagonalization process we obtain that there exists a strictly increasing sequence of natural numbers $\{n_j\}_{j=1}^{\infty}$ such that for each integer $t \geq 0$ there is

$$y_t = \lim_{j\to\infty} x_t^{(n_j)}. \tag{3.2}$$

Clearly, $\{y_t\}_{t=0}^{\infty}$ is a trajectory and $y_0 = x$. By (3.2) and (3.1) for each integer $s \geq 0$

$$\phi(y_s) = \lim_{j\to\infty} \phi(x_s^{(n_j)}) \geq \limsup_{j\to\infty} \lim_{t\to\infty} \phi(x_t^{(n_j)})$$

$$\geq \lim_{j\to\infty} \pi(x) - n_j^{-1} = \pi(x).$$

Proposition 1.2 is proved.

Proof of Proposition 1.3. Let

$$x \in X\ , \{x^{(n)}\}_{n=1}^{\infty} \subset X,\ \lim_{n\to\infty} x^{(n)} = x. \tag{3.3}$$

We show that

$$\pi(x) \geq \limsup_{n\to\infty} \pi(x^{(n)}).$$

We may assume without loss of generality that there is $\lim_{n\to\infty} \pi(x^{(n)})$.

By Proposition 1.2 for each integer $n \geq 1$ there is a trajectory $\{x_t^{(n)}\}_{t=0}^{\infty}$ such that for each integer $n \geq 1$

$$x_0^{(n)} = x^{(n)},\ \lim_{t\to\infty} \phi(x_t^{(n)}) = \pi(x_0^{(n)}) = \pi(x^{(n)}). \tag{3.4}$$

Extracting subsequences and using diagonalization process we obtain that there is a strictly increasing sequence of natural numbers $\{n_j\}_{j=1}^{\infty}$ such that for each integer $t \geq 0$ there is

$$x_t = \lim_{j\to\infty} x_t^{(n_j)}. \tag{3.5}$$

Clearly, $\{x_t\}_{t=0}^{\infty}$ is a trajectory and

$$x_0 = \lim_{j\to\infty} x_0^{(n_j)} = \lim_{j\to\infty} x^{(n_j)} = x. \tag{3.6}$$

By (3.5) and (3.4) for each integer $t \geq 0$

$$\phi(x_t) = \lim_{j\to\infty} \phi(x_t^{(n_j)}) \geq \limsup_{j\to\infty} \lim_{s\to\infty} \phi(x_s^{(n_j)}) = \limsup_{j\to\infty} \pi(x^{(n)}).$$

Together with (1.4) and (3.6) this implies

$$\pi(x) \geq \lim_{t\to\infty} \phi(x_t) \geq \lim_{n\to\infty} \pi(x^{(n)}).$$

Proposition 1.3 is proved.

Proof of Proposition 1.5. Clearly the sequence $\{\sup\{\phi(z) : z \in a^n(x)\}\}_{n=1}^{\infty}$ is monotone decreasing and its limit is larger or equal than $\pi(x)$.

Assume that the proposition does not hold. Then

$$l := \lim_{n\to\infty} \sup\{\phi(z) : z \in a^n(x)\} > \pi(x). \tag{3.7}$$

For each natural number n there is a trajectory $\{x_t^{(n)} : t = 0, \ldots, n\}$ such that

$$x_0^{(n)} = x, \ \phi(x_n^{(n)}) \geq l. \tag{3.8}$$

Extracting a subsequence and using diagonalization process we obtain that there exists a strictly increasing sequence of natural numbers $\{n_j\}_{j=1}^{\infty}$ such that for each integer $t \geq 0$ there is

$$x_t = \lim_{j\to\infty} x_t^{(n_j)}. \tag{3.9}$$

Clearly, $\{x_t\}_{t=0}^{\infty}$ is a trajectory and

$$x_0 = x. \tag{3.10}$$

By (3.7)-(3.9) for each integer $t \geq 0$,

$$\phi(x_t) = \lim_{j\to\infty} \phi(x_t^{(n_j)}) \geq l$$

and

$$\lim_{t\to\infty} \phi(x_t) \geq l > \pi(x).$$

This contradicts the definition of $\pi(x)$ (see (1.4)). The contradiction we have reached proves Proposition 1.5.

4 Proof of Theorem 1.6

Assume that the theorem does not hold. Then for each natural number n there exist an integer $T_n > 4n$, a trajectory $\{x_t^{(n)}\}_{t=0}^{T_n}$ satisfying

$$\phi(x_{T_n}^{(n)}) \geq \pi(x) - 1/n,\ x_0^{(n)} = x \tag{4.1}$$

and an integer

$$\tau_n \in [n, T_n - n] \tag{4.2}$$

such that

$$\rho(x_{\tau_n}^{(n)}, \Omega(a)) > \epsilon. \tag{4.3}$$

By (1.3) and (4.1) for each integer $n \geq 1$ and each each integer $t \in [0, T_n]$

$$\phi(x) \geq \phi(x_t^{(n)}) \geq \pi(x) - 1/n. \tag{4.4}$$

Let $n \geq 1$ be an integer. Set

$$y_t^{(n)} = x_{t+\tau_n}^{(n)} \text{ for all integers } t = -\tau_n, \dots, T_n - \tau_n. \tag{4.5}$$

In view of (4.5), (4.3) and (4.1)

$$\rho(y_0^{(n)}, \Omega(a)) > \epsilon, \tag{4.6}$$

$$\phi(y_{T_n - \tau_n}^{(n)}) \geq \pi(x) - 1/n. \tag{4.7}$$

By (4.5) and (4.4) for each integer $n \geq 1$ each integer $t \in [-\tau_n, T_n - \tau_n]$,

$$\phi(x) \geq \phi(y_t^{(n)}) \geq \pi(x) - 1/n. \tag{4.8}$$

Extracting subsequences and using diagonalization process we obtain that there exists a strictly increasing sequence of natural numbers $\{n_j\}_{j=1}^{\infty}$ such that for each integer t there exists

$$y_t = \lim_{j \to \infty} y_t^{(n_j)}. \tag{4.9}$$

By (4.9) and (4.6),

$$\rho(y_0, \Omega(a)) \geq \epsilon. \tag{4.10}$$

In view of (4.8) and (4.9) for each integer $t \geq 0$,

$$\phi(y_t) = \lim_{j \to \infty} \phi(y_t^{(n_j)}) \geq \pi(x). \tag{4.11}$$

Let t be an integer. By (4.1), (4.5), (4.9) and Proposition 1.5

$$\phi(y_t) = \lim_{j \to \infty} \phi(y_t^{(n_j)}) = \lim_{j \to \infty} \phi(x_{t+\tau_{n_j}}^{(n_j)})$$

$$\leq \limsup_{j \to \infty} \sup\{\phi(z) :\ z \in a^{t+\tau_{n_j}}(x)\} = \pi(x).$$

Together with (4.11) this implies that

$$\phi(y_t) = \pi(x) \text{ for all integers } t. \tag{4.12}$$

By (4.9) and (4.5), $y_{t+1} \in a(y_t)$ for all integers t. Combined with (4.12) and the property (1) this implies that

$$\{y_t\}_{t=-\infty}^{\infty} \subset \Omega(a).$$

This contradicts (4.10). The contradiction we have reached proves Theorem 1.6.

5 Proof of Theorem 1.7

Let $\{x_t\}_{t=0}^{\infty}$ be a trajectory of a and y be its limit point. Then $y \in \Omega(a)$ and

$$\phi(y) = \lim_{t\to\infty} \phi(x_t).$$

If z is also a limit point of $\{x_t\}_{t=0}^{\infty}$, then $z \in \Omega(a)$ and $\phi(z) = \lim_{t\to\infty} \phi(x_t)$. By the property (P) $y = z$. This implies that

$$y = \lim_{t\to\infty} x_t.$$

Theorem 1.7 is proved.

6 Proof of Theorem 1.9

Recall that $F(x)$ is as guaranteed by Proposition 1.8. Namely,

$$\{F(x)\} = \Omega(a) \cap \phi^{-1}(\pi(x)).$$

By the property (P) there is $\delta_1 > 0$ such that the following property holds:

(P1)

$$\text{if } z_1, z_2 \in \Omega(a) \text{ satisfy } |\phi(z_1) - \phi(z_2)| \leq 2\delta_1, \text{ then } \rho(z_1, z_2) \leq \varepsilon/4.$$

Since ϕ is uniformly continuous on X there is $\varepsilon_1 \in (0, \varepsilon/4)$ such that the following property holds:

(P2) For each $z_1, z_2 \in X$ satisfying $\rho(z_1, z_2) \leq 4\varepsilon_1$,

$$|\phi(z_1) - \phi(z_2)| \leq \delta_1/4.$$

By Proposition 1.5 there is a natural number L_0 such that

$$|\sup\{\phi(z) : \ z \in a^{L_0}(x)\} - \pi(x)| \leq \delta_1/2. \tag{6.1}$$

By Theorem 1.6 there exist $\delta \in (0, \delta_1)$ and a natural number $L > 2L_0$ such that the following property holds:

(P3) For each integer $T > 2L$ and each trajectory $\{x_t\}_{t=0}^{T}$ satisfying

$$\phi(x_T) \geq \pi(x_0) - \delta, \ x_0 = x$$

the inequality

$$\rho(x_t, \Omega(a)) \leq \varepsilon_1, \ t = L, \dots, T - L$$

holds.

Assume that an integer $T > 2L$ and that a trajectory $\{x_t\}_{t=0}^{T}$ satisfies

$$x_0 = x, \ \phi(x_T) \geq \pi(x) - \delta. \tag{6.2}$$

Then by (6.2) and (P3),

$$\rho(x_t, \Omega(a)) \leq \varepsilon_1, \ t = L, \dots, T - L. \tag{6.3}$$

Assume that an integer $t \in [L, T-L]$. By (6.3) and (6.2) there is z such that

$$z \in \Omega(a),\ \rho(x_t, z) \le \varepsilon_1. \tag{6.4}$$

In view of (6.1), (6.2) and the relation $L_0 < L \le t \le T$,

$$\phi(x_t) \ge \phi(x_T) \ge \pi(x) - \delta \ge \pi(x) - \delta_1,$$

$$\phi(x_t) \le \phi(x_{L_0}) \le \pi(x) + \delta_1/2$$

and

$$|\phi(x_t) - \pi(x)| \le \delta_1. \tag{6.5}$$

By (6.4) and (P2),

$$|\phi(z) - \phi(x_t)| \le \delta_1/4. \tag{6.6}$$

It follows from Proposition 1.8 and (6.6) that

$$|\phi(z) - \phi(F(x))| = |\phi(z) - \pi(x)| \le |\phi(z) - \phi(x_t)| + |\phi(x_t) - \pi(x)| \le (3/2)\delta_1.$$

Together with the definition of $F(x)$, (P1) and the inclusions z, $F(x) \in \Omega(a)$ this implies that

$$\rho(F(x), z) \le \varepsilon/4.$$

Together with (6.4) this implies that

$$\rho(x_t, F(x)) \le \rho(x_t, z) + \rho(z, F(x)) \le \varepsilon_1 + \varepsilon/4 < \varepsilon/2.$$

Theorem 1.9 is proved.

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