# Stability Analysis of $M_4 \times S^1 \times S^2$ in the Seven-Dimensional Einstein-Maxwell Theory

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**Abstract** The compactification of the seven-dimensional Einstein-Maxwell theory into four-dimensional Minkowski space $M_4 \times S^1 \times S^2$ is found to be stable.

**Key words** compactification; monopole; graviton; tachyon; stability

## 0 Introduction

In recent years there has been attractiveness in superstring theories[1] to unify all four interactions with good quantum behavior. The low-energy limits[2] of type- I or heterotic string[3] are $d=10$ dimensional $N=1$ supergravity theories with additional terms given by superstring theories. The stability of compactifications of these higher dimensional theories is crucial for physics. The Einstein-Maxwell and Einstein-Yang-Mills theories[4] are found to be stable in Minkowski space-time with a monopole compactification in $d=6$ dimensions. The instanton compactification of $d=8$ dimensions into Minkowski space-time is also to be stable[5]. The compactifications of the eight-dimensional Einstein-Maxwell theory and Einstein- Maxwell- scalar theory into four-dimensional Minkowski space are found to be unstable[6]. Also, $M_4 \times S^2 \times S^2 \times S^2$ in the $d=10$ Einstein-Maxwell theory is unstable[7]. However, the stability for $M_4\times$ (product space) has not yet been fully analyzed to our knowledge. In this paper, we wish to study the stability of $M_4 \times S^1 \times S^2$ compactification in seven-dimensional Einstein-Maxwell theory.

## 1 Field equations

The action of seven-dimensional Einstein-Maxwell theory with cosmological constant is

$$s = -\int d^7x \sqrt{-g}\left[\frac{1}{k^2}R + \frac{1}{4}F_{MN}F^{MN} + \lambda\right], \tag{1.1}$$

where $R$ is the curvature scalar with $R_{MN} = R_{PMN}{}^{P}$, and the metric is $(-++\cdots+)$. The index

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$M$ is split into $\mu=0,1,2,3$, $m_1=4,5,m_2=6$ Here $k$ is the seven-dimensional gravitational coupling. The field equation for the graviton and Maxwell part are

$$R_{MN}=-\frac{1}{2}k^2[F_{MP}F_N{}^P-\frac{1}{10}F^2g_{MN}+\frac{2}{5}\lambda g_{MN}], \tag{1.2}$$

$$\nabla_M F^{MN}=0. \tag{1.3}$$

The 7-dimension space compactified into $M_4\times S^1\times S^2$ by giving the following vacuum expectation values $\dot{F}_{45}=\sqrt{2}/ka$. The cosmological constant is adjusted as $\lambda=\frac{1}{4}F^2$. The stability of this compactification[4,5] will be determined by examining the tachyonic modes in the linearized equations of motion. The field are then expanded the vacuum state as

$$g_{MN}=\dot{g}_{MN}+kh_{MN},\qquad V_M=\dot{V}_M+V_M, \tag{1.4}$$

$$F_{MN}=\dot{F}_{MN}+f_{MN},\qquad f_{MN}=2\partial_{[M}V_{N]}, \tag{1.5}$$

where the overdot over the field denotes the background value. The backgrounds for the Ricci tensor in tangent frames are read from Eq. (1.2) as

$$\dot{R}^2_{44}=\dot{R}^2_{55}=-\frac{1}{a^2},\qquad \dot{R}^2_{66}=0. \tag{1.6}$$

The gauge is fixed as

$$\nabla_P h^P_Q=\frac{1}{2}\nabla_Q h^P_P, \tag{1.7}$$

since this theory has the invariance of general coordinate and Abelian gauge transformations.

The linearized equations of (1.2) and (1.3) in tangent frames with external sources $T_{AB}$ and $J_A$ are

$$\Box(h_{AB}-\frac{1}{2}\eta_{AB}h^C_C)+2\dot{R}_{(ACh^c_B)}+2k\dot{F}_{(BCf_A)}{}^C-\frac{1}{2}k\eta_{AB}\dot{F}^{CD}f_{CD}=-T_{AB}, \tag{1.8}$$

$$\nabla^A f_{AB}+k\nabla^A(\frac{1}{2}h^C_C\dot{F}_{AB}-2h_{[A}{}^C\dot{F}_{CB]})=-J_B, \tag{1.9}$$

Here we introduced external source to obtain the fluctuation Green's functions. As a result of the Bianchi identity for the graviton, the sources satisfy the source conservation law

$$\nabla_A T^{AB}=K\dot{F}^{BC}J_C,\qquad \nabla_A J^A=0. \tag{1.10}$$

After the harmonic expansions on $S^1\times S^2$ are performed, the four-dimensional scalars are given by

$$\begin{aligned}
h_{a\pm}&=\frac{1}{2}[h_{a4}\mp ih_{a5}]\qquad a=0,1,2,3,\\
V_\pm&=\frac{1}{\sqrt{2}}[V_4\mp iV_5],\\
h_{+-}&=\frac{1}{2}[h_{44}+h_{55}],\\
h_{\pm1\pm1}&=\frac{1}{2}[h_{44}-h_{55}\mp 2ih_{45}],\\
h_{\pm1\pm2}&=\frac{1}{2}[h_{46}\mp ih_{56}],
\end{aligned} \tag{1.11}$$

The four-dimensional graviton $h_{ab}$, its trace $h_a^a$, vector $V_a$ and the above scalar satisfy the following linearized equations of motion

$$[P^2+\frac{L}{a^2}+n^2][h_{ab}+\eta_{ab}(h_{+-}+h_{66})]+\eta_{ab}\frac{\sqrt{L}}{a^2}[V_+-V_-]=T_{ab}-\frac{1}{2}\eta_{ab}T_C{}^C, \tag{1.12}$$

$$[P^2+\frac{L}{a^2}+n^2][h_a a+h_{66})]-\frac{4}{a^2}h_{+-}-\frac{2\sqrt{L}}{a^2}[V_+-V_-]=-2T_{+-}, \tag{1.13}$$

$$[P^2+\frac{L}{a^2}+n^2]h_{a\pm}\pm\frac{\sqrt{2}}{a}P_aV_\pm\mp\frac{\sqrt{L}}{a^2}V_a=T_{a\pm}, \tag{1.14}$$

$$[P^2+\frac{L}{a^2}+n^2]h_{a6}=T_{a6}, \tag{1.15}$$

$$[P^2+\frac{L}{a^2}+n^2]h_{\pm1\pm1}=T_{\pm1\pm1}, \tag{1.16}$$

$$[P^2+\frac{L}{a^2}+n^2][\frac{1}{2}h_a{}^a+h_{+-}-\frac{1}{2}h_{66}]+\frac{\sqrt{L}}{a^2}[V_+-V_-]=-T_{66}, \tag{1.17}$$

$$[P^2+\frac{L}{a^2}+n^2]h_{\pm1\pm2}\pm\frac{n}{a}V_\pm\mp\frac{\sqrt{L}}{\sqrt{2}\,a^2}V_6=T_{\pm1\pm2}, \tag{1.18}$$

$$[P^2+\frac{L}{a^2}+n^2]V_a-\frac{\sqrt{L}}{a^2}[h_{a+}-h_{a-}]=J_a, \tag{1.19}$$

$$[P^2+\frac{L}{a^2}+n^2]V_\pm\pm\frac{\sqrt{L}}{a^2}h_{+-}\pm\frac{\sqrt{L}}{2a^2}[h_a^a+h_{66}]\pm\frac{\sqrt{2}}{a}P_ah_{a\pm}\pm$$
$$\frac{2n}{a}h_{\pm1\pm2}=J_\pm, \tag{1.20}$$

$$[P^2+\frac{L}{a^2}+n^2]V_6-\frac{\sqrt{2L}}{a^2}[h_{+1+2}-h_{-1-2}]=J_b, \tag{1.21}$$

with $L=l(l+1)$. In a similar way the harmonic expansions of the source conservation laws are carried out

$$P_aT_b^a+\frac{1}{a\sqrt{2}}\sqrt{L}[T_{b+}+T_{b-}]+nT_{66}=0,$$

$$P_aT_\pm^a+\frac{1}{a\sqrt{2}}\sqrt{L}T_{+-}+\frac{1}{a\sqrt{2}}\sqrt{L-2}T_{\pm1\pm1}+\sqrt{2}nT_{+1+2}=\pm\frac{1}{a\sqrt{2}}J_\pm,$$
$$\tag{1.22}$$

$$P_aT_b^a+\frac{1}{a}\sqrt{L}[T_{+1+2}+T_{-1-2}]+nT_{66}=0,$$

$$P_aJ^a+\frac{1}{a\sqrt{2}}\sqrt{L}[J_++J_-]+nJ^6=0.$$

## 2 Results

To obtain the mass eigenstates, we follow the procedures of Randjbar-Daemi Salam[4] and strathdee. The fields are solved in terms of the sources as

$$h_{ab}-\frac{1}{4}\eta_{ab}h_{cc}=\frac{1}{P^2+M_2{}^2}[T_{ab}-\frac{1}{4}\eta_{ab}T_{cc}], \tag{1.23}$$

$$h_{aa}=\frac{[P^2+M_0{}^2]}{5[P^2+M_{0+}{}^2][P^2+M_{0-}{}^2]}T_{aa}+\frac{4}{5[P^2+M_2{}^2][P^2+M_{0+}{}^2][P^2+M_{0-}{}^2]}$$
$$\{-2[P^2+M_2{}^2]^2T_{+-}-[P^2+m_{2+}{}^2][P^2+m_{2-}{}^2]T_{66}+\frac{2\sqrt{L}[P^2+M_2{}^2]}{a^2}[J_+-J_-]-$$
$$\frac{2\sqrt{2L}}{a^3}P_a[T_{a+}+T_{a-}]-\frac{4n\sqrt{L}}{a^3}[T_{+1+2}+T_{-1-2}]\Big\}, \tag{1.24}$$

$$h_{+-}=\frac{-8\sqrt{L}}{5a^2[P^2+M_0{}^2][P^2+M_{0+}{}^2][P^2+M_{0-}{}^2]}\{[P^2+M_2{}^2][J_+-J_-]-$$
$$\frac{\sqrt{2}}{a}P_a[T_{a+}+T_{a-}]-\frac{2n}{a}[T_{+1+2}+T_{-1-2}]\Big\}-\frac{[P^2+M_2{}^2]}{5[P^2+M_{0+}{}^2][P^2+M_{0-}{}^2]}[T_{aa}+T_{66}]+$$
$$\frac{3[P^2+m_{4+}{}^2][P^2+m_{4-}{}^2]}{5[P^2+M_0{}^2][P^2+M_{0-}{}^2][P^2+M_{0-}{}^2]}T_{+-}, \tag{1.25}$$

$$h_{\pm1\pm1}=\frac{1}{[P^2+M_0{}^2]}T_{\pm1\pm1}, \tag{1.26}$$

$$h_{66}=-\frac{[P^2+m_{2+}{}^2][P^2+m_{2-}{}^2]]}{5[P^2+M_2{}^2][P^2+M_{0+}{}^2][P^2+M_{0-}{}^2]}T_{aa}+$$
$$\frac{1}{5[P^2+M_2{}^2][P^2+M_{0+}{}^2][P^2+M_{0-}{}^2]}\{-2[P^2+M_2{}^2]T_{+-}+$$
$$4[P^2+m_{1+}{}^2][P^2+m_{1-}{}^2]T_{66}+\frac{2\sqrt{L}[P^2+M_2{}^2]}{a^2}[J_+-J_-]-\frac{2\sqrt{2L}}{a^3}P_a[T_{a+}+T_{a-}]-$$
$$\frac{4n\sqrt{L}}{a^3}[T_{+1+2}+T_{-1-2}]\Big\}, \tag{1.27}$$

$$V_a=\frac{[P^2+M_2{}^2]}{[P^2+M_{1+}{}^2][P^2+M_{1-}{}^2]}J_a+\frac{\sqrt{L}}{a^2[P^2+m_{1+}{}^2][P^2+m_{1-}{}^2]}[T_{a+}-T_{a-}]-$$
$$\frac{\sqrt{2L}P_a}{[P^2+M_2{}^2][P^2+M_0{}^2][P^2+M_{1+}{}^2][P^2+M_{1-}{}^2]}\{[P^2+m_{1+}{}^2][P^2+m_{1-}{}^2]$$
$$[J_+-J_-]-\frac{\sqrt{2}[P^2+M_2{}^2]}{a}P_b[T_{b+}-T_{b-}]-\frac{2\sqrt{2L}}{a^3}P_bJ_b-\frac{2n\sqrt{2L}}{a^3}J_6-$$
$$\frac{2n[P^2+M_2{}^2]}{a}[T_{+1+2}-T_{-1-2}]\Big\}, \tag{1.28}$$

$$2V_\pm=\frac{1}{[P^2+M_2{}^2]^2[P^2+M_0{}^2]}\{[P^2+m_{1+}{}^2][P^2+m_{1-}{}^2][J_++J_-]-$$
$$\frac{\sqrt{2}[P^2+M_2{}^2]}{a}P_a[T_{a+}-T_{a-}]-\frac{2\sqrt{2L}}{a^3}P_aJ_a-\frac{2n[P^2+M_2{}^2]}{a}[T_{+1+2}-T_{-1-2}]-$$

$$\left.\frac{2n\sqrt{2L}}{a^3}J_6\right\} \pm \frac{1}{5[P^2+M_0{}^2][P^2+M_{0+}{}^2][P^2+M_{0-}{}^2]}\left\{-\frac{16\sqrt{L}[P^2+M_2{}^2]}{a^2}T_{+-}+\right.$$

$$5[P^2+M_2{}^2][P^2+M_0{}^2][J_+-J_-]+\frac{2\sqrt{L}}{a^2}[P^2+M_0{}^2][T_{aa}+T_{bb}]-$$

$$\left.\frac{5\sqrt{2}}{a}[P^2+M_0{}^2]P_a[T_{a+}+T_{a-}]-\frac{10n}{a}[P^2+m_0{}^2][T_{+1+2}+T_{-1-2}]\right\}, \qquad (1.29)$$

$$2h_a\pm=\frac{1}{[P^2+M_2{}^2]}[T_{a+}+T_{a-}]\pm\frac{1}{[P^2+M_{1+}{}^2][P^2+M_{1-}{}^2]}\{[P^2+M_2{}^2]$$

$$\left.[T_{a+}-T_{a-}]\}+\frac{2\sqrt{L}}{a^2}J_a\right\}\mp\frac{\sqrt{2}P_a/a}{[P^2+M_2{}^2][P^2+M_0{}^2][P^2+M_{1+}{}^2][P^2+M_{1-}{}^2]}$$

$$\left\{[P^2+m_{1+}{}^2][P^2+m_{1-}{}^2][J_++J_-]-\frac{2[P^2+M_2{}^2]}{a}P_b[T_{b+}-T_{b-}]-\right.$$

$$\left.\frac{2\sqrt{2L}}{a^3}P_bJ_b-\frac{2n[P^2+M_2{}^2]}{a}[T_{+1+2}-T_{-1-2}]-\frac{2n\sqrt{2L}}{a^3}J_6\right\}-$$

$$\frac{\sqrt{2}P_a/a}{5[P^2+M_2{}^2][P^2+M_0{}^2][P^2+M_{0+}{}^2][P^2+M_{0-}{}^2]}\{5[P^2+M_2{}^2][P^2+m_0{}^2][J_+-J_-]+$$

$$\frac{2\sqrt{L}}{a^2}[P^2+M_0{}^2][T_{aa}+T_{66}]-\frac{16\sqrt{L}}{a^2}[P^2+M_2{}^2]T_{+-}-$$

$$\left.\frac{5\sqrt{2}}{a}[P^2+m_0{}^2]P_b[T_{b+}+T_{b-}]-\frac{10n}{a}[P^2+m_0{}^2][T_{+1+2}+T_{-1-2}]\right\}, \qquad (1.30)$$

$$V_6=\frac{1}{[P^2+M_2{}^2][P^2+M_{1+}{}^2][P^2+M_{1-}{}^2][P^2+M_0{}^2]}\{[(P^2+M_2{}^2)^3(P^2+M_0{}^2)+$$

$$\frac{4Ln^2}{a^6}]J_6+\frac{2n\sqrt{L}}{a^4}[P^2+M_2{}^2]P_b[T_{b+}-T_{b-}]+\frac{\sqrt{2L}}{a^2}[P^2+M_2{}^2][P^2+m_{3+}{}^2]$$

$$[P^2+m_{3-}{}^2][T_{+1+2}-T_{-1-2}]-\frac{n\sqrt{2L}}{a^3}[P^2+M_{1+}{}^2][P^2+M_{1-}{}^2][J_++J_-]+$$

$$\frac{2\sqrt{2n}L}{a^5}P_bJ_b, \qquad (1.31)$$

$$2h_{\pm1\pm2}=\frac{1}{[P^2+M_2{}^2]}[1+\frac{2n^2[P^2+m_0{}^2]/a}{[P^2+M_0{}^2][P^2+M_{0+}{}^2][P^2+M_{0-}{}^2]}]$$

$$[T_{+1+2}+T_{-1-2}]-\frac{n/a}{5[P^2+M_2{}^2][P^2+M_0{}^2][P^2+M_{0+}{}^2][P^2+M_{0-}{}^2]}$$

$$\left\{5[P^2+M_2{}^2][P^2+M_0{}^2][J_+-J_-]+\frac{2\sqrt{L}}{a^2}[P^2+M_0{}^2\right][T_{aa}+T_{66}]-$$

$$\left.\frac{16\sqrt{L}}{a^2}[P^2+M_2{}^2]T_{+-}-\frac{5\sqrt{2}}{a}[P^2+m_0{}^2]P_b[T_{b+}+T_{b-}]\right\}\pm$$

$$\frac{[P^2+m_{3+}{}^2][P^2+m_{3-}{}^2]}{[P^2+M_0{}^2][P^2+M_{1+}{}^2][P^2+M_{1-}{}^2]}[T_{+1+2}-T_{-1-2}]\pm$$

$$\frac{1}{[P^2+M_0{}^2][P^2+M_2{}^2][P^2+M_{1+}{}^2][P^2+M_{1-}{}^2]}\left\{\frac{\sqrt{2}n[P^2+M_2{}^2]}{a^3}\right.$$

$$P_b[T_{b+}-T_{b-}]+\frac{2n\sqrt{2L}}{a^4}P_bJ_b-\frac{n}{a}[P^2+m_{1+}{}^2][P^2+m_{1-}{}^2][J_++J_-]+$$
$$\left.\frac{\sqrt{2L}}{a^2}[P^2+m_{3+}{}^2][P^2+m_{3-}{}^2]J_6\right\},\qquad(1.32)$$

$$h_{a6}=\frac{1}{[P^2+M_2{}^2]}T_{a6},\qquad(1.33)$$

Here the following symbols are used

$$M_0{}^2=[l(l+1)+n^2a^2-2]/a^2\qquad(1\geqslant 2),\qquad(1.34)$$

$$M_2{}^2=[l(l+1)+n^2a^2]/a^2\qquad(1\geqslant 0),\qquad(1.35)$$

$$M_0{}^2\pm=[5l(l+1)+5n^2a^2+3\pm\sqrt{9+80l(l+1)}]/5a^2\quad(l\geqslant 0),\qquad(1.36)$$

$$M_{1\pm}{}^2=[l(l+1)\pm\sqrt{2l(l+1)}+n^2a^2]/a^2\qquad(l\geqslant 0),\qquad(1.37)$$

$$m_0{}^2=[5l(l+1)+5n^2a^2+6]/5a^2,\qquad(1.38)$$

$$m_1{}^2\pm=[2l(l+1)+2n^2a^2+1\pm\sqrt{1+12l(l+1)}]/2a^2,\qquad(1.39)$$

$$m_2{}^2\pm=[l(l+1)+n^2a^2+1\pm\sqrt{1+4l(l+1)}]/a^2,\qquad(1.40)$$

$$m_3{}^2\pm=[l(l+1)+n^2a^2-1\pm\sqrt{1-2n^2a^2}]/a^2,\qquad(1.41)$$

$$m_4{}^2\pm=[3l(l+1)+3n^2a^2-3\pm\sqrt{9-48l(l+1)}]/3a^2,\qquad(1.42)$$

These solutions are to be substituted into the expression

$$I(T,J)=\frac{1}{2}\int d^7x\ \sqrt{-g}\left(\frac{1}{2}T_{AB}h_{AB}+J_AV_A\right).\qquad(1.43)$$

To see the massive states it is much better to use the rest frame $P_a=(p_0,0,0,0)$. For massless states we use the light-cone frame $P_a=(p_0,0,0,p_3)$. Here we notice that $M_0{}^2, M_2{}^2, M_0^2\pm$, $M_1^2\pm$ denotes the physical masses and $m_0{}^2, m_1^2\pm, m_2^2\pm$, $m_3^2\pm$, $m_4^2\pm$ do not appear in the denominator of Eq. (1.23)~(1.33) and thus do not correspond to the physical masses.

To extract the massless particle, we insert Eqs. (1.23)~(1.33) into Eq. (1.43) following the procedure of Ref. 4. When this is done one finds the terms

$$I(M=0)=\frac{1}{P^2}\Big[\frac{1}{4}|T_{22}-T_{66}|^2+\frac{1}{3}|T_{11}-\frac{1}{2}T_{22}-\frac{1}{2}T_{66}|^2+|T_{12}|^2+$$
$$|J_1|^2+|J_2|^2+|J_6|^2+\frac{1}{2}|T_{+1+2}-T_{-1-2}|^2\Big]_{n=0}^{l=0}+$$
$$\Big[|T_{l+}-T_{l-}+\sqrt{2}J_l|^2+|T_{2+}-T_{2-}+\sqrt{2}J_2|^2\Big]_{n=0}^{l=1}.\qquad(1.44)$$

Massive particles should be reanalyzed in the rest frame $P_a=(p_0,0,0,0)$similar analyses are performed and finally we observe that there are no tachyons and, since the residues all have the same sign, no negative metric state. This means that the ground state is at least perturbatively stable.

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# 7维爱因斯坦-麦克斯韦理论紧致到$M_4\times S^1\times S^2$稳定性分析

周昺路

**提　要**　7维爱因斯坦-麦克斯韦理论紧致到4维闵可夫斯基空间$(M_4)\times S^1\times S^2$，没有发现有快子，因此可认为这是稳定的．

**关键词**　紧致；磁单极；引力微子；快子；稳定

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