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Monotone Empirical Bayes Test for Scale Parameter under Random Censorship*

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Abstract

We study the two-action problem in the scale-exponential family via the empirical Bayes (EB) approach and present a monotone EB test possessing a rate of convergence which can be arbitrarily close to $O(n^{-1})$ under the condition that the past samples are randomly censored from the right.

Keywords: Empirical Bayes, random censorship, scale exponential family.

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§ 1. Introduction

Consider the problem of testing the hypothesis

$$H_0 : \theta \le \theta_0 \longleftrightarrow H_1 : \theta > \theta_0 \tag{1.1}$$

in the scale-exponential family

$$f(x|\theta) = u(x)c(\theta)\exp(-x/\theta)I(x > 0), \qquad \theta > 0, \tag{1.2}$$

where θ_0 is a known constant and $f(x|\theta)$ denotes the conditional probability density function (pdf) of random variable (r.v.) X, given θ, $I(A)$ is the indicator of the set A and $u(x) > 0$ for $x > 0$. The distribution family (1.2) is very often and important, especially it includes the exponential distribution as a special case which can be used to describe many models appearing in survival analysis, reliability theory, engineering and environmental sciences.

To avoid the influence of measurement unit, we adopt the following weighted linear loss function:

$$L(\theta, d_j) = \frac{a(1-j)(\theta-\theta_0)}{\theta}I(\theta > \theta_0) + \frac{aj(\theta_0-\theta)}{\theta}I(\theta \le \theta_0), \tag{1.3}$$

where $j = 0, 1$ and $a > 0$ is a constant, and $\mathcal{D} = \{d_0, d_1\}$ denotes action space with d_j accepting H_j.

Assume that the parameter θ has an unknown non-degenerate prior $G(\theta)$ with support on $\Theta = \{\theta > 0 : c(\theta) > 0\}$. Hence the marginal pdf of r.v. X is

$$f(x) = \int_\Theta f(x|\theta)\mathrm{d}G(\theta) \mathrel{\widehat=} u(x)p(x), \tag{1.4}$$

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where $p(x) = \int_{\Theta} c(\theta)\exp(-x/\theta)I(x>0)\mathrm{d}G(\theta)$.

Put

$$\delta(x) = \mathsf{P}(\text{accepting } H_0|X=x). \tag{1.5}$$

Then the Bayes risk of the test $\delta(x)$ is

$$\begin{aligned} R(\delta(x), G(\theta)) &= \int_0^\infty \int_\Theta [L(\theta,d_0)\delta(x) + L(\theta,d_1)(1-\delta(x))]f(x|\theta)\mathrm{d}G(\theta)\mathrm{d}x \\ &\widehat{=} \int_0^\infty \alpha(x)\delta(x)\mathrm{d}x + \int_\Theta L(\theta,d_1)\mathrm{d}G(\theta) \end{aligned} \tag{1.6}$$

with

$$\begin{aligned} \alpha(x) &= a\int_\Theta \theta^{-1}(\theta-\theta_0)f(x|\theta)\mathrm{d}G(\theta) \\ &= af(x) - a\theta_0 u(x)\int_\Theta \theta^{-1}c(\theta)\exp(-x/\theta)I(x>0)\mathrm{d}G(\theta). \end{aligned} \tag{1.7}$$

Therefore, the best Bayes test minimizing $R(\delta(x), G(\theta))$ would have the form

$$\delta_G(x) = \begin{cases} 1 & \alpha(x) \le 0 \\ 0 & \alpha(x) > 0 \end{cases}. \tag{1.8}$$

The minimum Bayes risk is

$$R(\delta_G(x), G(\theta)) = \int_0^\infty \alpha(x)\delta_G(x)\mathrm{d}x + \int_\Theta L(\theta,d_1)\mathrm{d}G(\theta). \tag{1.9}$$

Under the assumption that $\mathsf{E}[c(\theta)] < \infty$ and $\mathsf{E}[\theta^{-1}c(\theta)] < \infty$, we can rewrite

$$\begin{aligned} \alpha(x) &= af(x) + a\theta_0 u(x)p^{(1)}(x) = \Big[a - a\theta_0\frac{u^{(1)}(x)}{u(x)}\Big]f(x) + a\theta_0 f^{(1)}(x) \\ &\widehat{=} g(x)f(x) + bf^{(1)}(x), \end{aligned} \tag{1.10}$$

where $a\theta_0 \widehat{=} b$ and $a - a\theta_0 u^{(1)}(x)/u(x) \widehat{=} g(x)$.

Let

$$\beta(x) = \frac{\alpha(x)}{f(x)}. \tag{1.11}$$

Also, assume that $\mathsf{E}[\theta^{-2}c(\theta)] < \infty$, by some simple algebra computation, we have

$$\beta^{(1)}(x) = b\frac{u^2(x)[p^{(2)}(x)p(x) - p^{(1)}(x)p^{(1)}(x)]}{f^2(x)}. \tag{1.12}$$

Then by Cauchy-Schwarz's inequality, we easily know

$$\beta^{(1)}(x) \ge 0, \tag{1.13}$$

which implies that $\beta(x)$ is strictly increasing under the condition that the prior $G(\theta)$ is non-degenerate. Therefore, there exists a unique point a_G such that $\beta(a_G) = 0$, which is usually regarded as the critical point corresponding to the prior $G(\theta)$.

Note that $\mathsf{E}[\theta^{-1}c(\theta)] < \infty$ can be derived from $\mathsf{E}[c(\theta)] < \infty$ and $\mathsf{E}[\theta^{-2}c(\theta)] < \infty$, hence, under the following condition

$$\mathsf{E}[c(\theta)] < \infty, \qquad \mathsf{E}[\theta^{-2}c(\theta)] < \infty, \tag{1.14}$$

we have

$$\delta_G(x) = \begin{cases} 1 & \alpha(x) \le 0 \\ 0 & \alpha(x) > 0 \end{cases} = \begin{cases} 1 & \beta(x) \le 0 \\ 0 & \beta(x) > 0 \end{cases} = \begin{cases} 1 & x \le a_G \\ 0 & x > a_G \end{cases}. \tag{1.15}$$

However, Bayes test $\delta_G(x)$ (1.15) is unavailable to use since the prior $G(\theta)$ is unknown. As an alternative we can use the EB approach to estimate $\alpha(x)$ so as to obtain an EB test.

Since Robbins' pioneering papers [9, 10], EB approach has generated considerable interest among the researchers, and the EB test problem has been studied extensively in the literature. For example, literature [13] and [12] discussed one-tail testing problem for the one-parameter continuous exponential family $c(\theta)u(x)\exp(-x\theta)$, while [11] considered nonparametric EB solutions to two-tail test $H_0 : \theta_1 \le \theta \le \theta_2 \longleftrightarrow H_1 : \theta < \theta_1$ or $\theta > \theta_2$ in the exponential family $f(x|\theta) = c(\theta)u(x)\exp(-x/\theta)$. Also, the readers are referred to literature [1] and [2] for some original details of EB test. Recently, employing Bernstein' inequality and following the lines of [2], [13] and [4], paper [6] has constructed an EB test for a normal mean and obtain a better rate of convergence under the assumption that the critical point a_G is within a known compact interval.

Usually, the EB approach assumes that there is a sequence of past data $X_1, X_2, \cdots, X_n$, which comes from the past n experiments, is available. Differing from the past many works, we suppose that the sequence is censored from the right by another sequence with an unknown distribution function. In fact, right censored data often arise in the study of survival analysis, medical follow-up and reliability. In past decades, statistical inference with censorship attracts considerable attention and has been widely applied to many fields.

By combining the censored data with the EB procedure, in this paper we propose an EB test for the scale parameter and exhibit its optimal rate of convergence without using the Bernstein' inequality.

The rest of this paper is organized as follows. Section 2 proposes an EB test under random censorship. In Section 3, we present some useful lemmas. Section 4 is devoted to obtaining the main result.

§ 2. Empirical Bayes Test

In the EB framework, we make the following assumptions: let $(X_1, \theta_1), \cdots, (X_n, \theta_n)$ and $(X_{n+1}, \theta_{n+1}) \hat{=} (X, \theta)$ be independent random vectors, the θ_i $(i = 1, \cdots, n)$ and θ are independently identically distributed (i.i.d.) and have the common prior distribution $G(\theta)$; $X_1, \cdots, X_n$ and X are i.i.d. and have the common marginal density $f(x)$. Usually, we call $X_1, \cdots, X_n$ the past samples, X denotes the present sample.

We suppose that the sequence $X_1, \cdots, X_n$ is censored from the right by $Y_1, \cdots, Y_n$ with unknown distribution function W. It is assumed that $X_1, \cdots, X_n$ are independent of $Y_1, \cdots, Y_n$. Let $Z_i = \min\{X_i, Y_i\}$ and $\Delta_i = I(X_i \le Y_i)$. Define $\overline{F} = 1 - F = 1 - \int_{-\infty}^{x} f(x)\mathrm{d}x$ and $\overline{W} = 1 - W$. Then Z_i are i.i.d. with the distribution function H, where $\overline{H} = 1 - H = \overline{F}\,\overline{W}$. A product limit (PL) estimator $\widehat{F}(x)$ of the distribution function F (see [4]) can be defined as

$$1 - \widehat{F}(x) = \begin{cases} \prod\limits_{i: Z_{(i)} \le t} \Big(\dfrac{n-i}{n-i+1}\Big)^{\Delta_{(i)}} & t < Z_{(n)} \\ 0 & t \ge Z_{(n)} \end{cases}, \tag{2.1}$$

where the $Z_{(1)}, \cdots, Z_{(n)}$ denote the ordered sample, and $\Delta_{(i)}$ being the concomitant of $Z_{(i)}$.

Estimation of the underlying density function $f(x)$ has been discussed by numerous authors (see [5] and [8], among others), and the usual procedure is to replace the empirical distribution function $F_n(y)$ in $f_n(x) = h_n^{-1} \int k((x-y)/h_n)\mathrm{d}F_n(y)$ by the PL estimator $\widehat{F}(y)$. Then we obtain an estimator of $f(x)$ given by

$$\widehat{f}_n(x) = \frac{1}{h_n} \int k_0\Big(\frac{x-y}{h_n}\Big)\mathrm{d}\widehat{F}(y), \tag{2.2}$$

where $0 < h_n \to 0$ as $n \to \infty$, and $k_0(\cdot)$ is a kernel function satisfying some certain conditions.

Similarly, we can define a kernel estimator for $f^{(1)}(x)$ as

$$\widehat{f}_n^{(1)}(x) = \frac{1}{h_n^2} \int k_1\Big(\frac{x-y}{h_n}\Big)\mathrm{d}\widehat{F}(y). \tag{2.3}$$

Obviously, to estimate $\alpha(x)$, we only need to estimate $f(x)$ and $f^{(1)}(x)$ in formula (1.10) because the $g(x)$ is known when the present sample $X = x$ is obtained.

Define

$$\widehat{\alpha}_n(x) = g(x)\widehat{f}_n(x) + b\widehat{f}_n^{(1)}(x). \tag{2.4}$$

In the following, we assume that the prior $G(\theta)$ belongs to the following class of distributions

$$\mathcal{F} = \{G(\theta) : 0 < A_1 \le a_G \le A_2 < \infty\}, \tag{2.5}$$

where A_1, A_2 are known constants. Then, it follows from the Bayes test $\delta_G(x)$ (1.15), we propose the EB test as follows

$$\delta_n(x) = \begin{cases} 1 & x < A_1 \quad \text{or} \quad (A_1 \le x \le A_2 \quad \text{and} \quad \widehat{\alpha}_n(x) \le 0) \\ 0 & x > A_2 \quad \text{or} \quad (A_1 \le x \le A_2 \quad \text{and} \quad \widehat{\alpha}_n(x) > 0) \end{cases}. \tag{2.6}$$

Hence, the Bayes risk of $\delta_n(x)$ is

$$R(\delta_n(x), G(\theta)) = \int_0^{\infty} \alpha(x)\mathsf{E}_n[\delta_n(x)]\mathrm{d}x + \int_{\Theta} L(\theta, d_1)\mathrm{d}G(\theta), \tag{2.7}$$

where E_n denotes the expectation with respect to the joint distribution of $(Z_1, \cdots, Z_n)$.

By definition, if for some $q > 0$, $R(\delta_n(x), G(\theta)) - R(\delta_G(x), G(\theta)) = O(n^{-q})$, then the rate of convergence of the EB test $\delta_n(x)$ is said to be the order $O(n^{-q})$.

§ 3. Several Lemmas

In this paper, $c, c_0, c_1, \cdots$ denote positive constants that do not depend on n. They can take different values while appearing even within the same expression.

Lemma 1 Let T be such that $1-H(T)>\delta$ with some $\delta>0$. Then the process $\widehat{F}(t)-F(t)$, $-\infty<t<\infty$, $1-H(t)>0$, can be represented as

$$\widehat{F}(t)-F(t)=\frac{1}{n}\sum_{j=1}^{n}[1-F(t)]M_j(t)+\frac{1}{n}R_n(t)$$

in such a way that

$$\mathsf{P}\Big(\sup_{t\le T}|R_n(t)|>\frac{2C}{\delta}\log^2 n+x\log n\Big)\le 2K\exp(-\lambda\delta^2 x),\qquad x>0,$$

where i.i.d. Gaussian processes $M_1(t), M_2(t),\cdots$, $\mathsf{E}M_n(t)=0$ with covariance function

$$\mathsf{E}M(s)M(t)=\mathsf{E}M(s)^2=\int_{-\infty}^{s}\frac{\mathrm{d}F(t)}{(1-W(t))[1-F(t)]^2},\qquad -\infty<s\le t<\infty.$$

Here $C>0$, $K>0$ and $\lambda>0$ are some universal constants.

Proof See [7]. #

Before establishing the next lemma, we first make the following assumptions about the kernel functions $k_i(\cdot)$ $(i=0,1)$:

(1) $k_i(x)$ are continuously differentiable with compact support $[0,1]$;

(2) $\displaystyle\int_0^1 x^j k_i(x)\mathrm{d}x=\begin{cases}(-1)^j & j=i\\ 0 & j\ne i\end{cases},\quad j=0,1,\cdots,s-1;$

(3) $\displaystyle\int_0^1 x^s k_i(x)\mathrm{d}x\ne 0,\ k_i(0)=k_i(1)=0,$

where $s\ge 2$ is an arbitrary but fixed integer.

Remark 1 In fact, for $s=2$, a possible choice of the kernel functions $k_i(x)$ $(i=0,1)$ satisfying the above assumption (1)-(3) is $k_0(x)=(60x^3-96x^2+36x)I(0\le x\le 1)$ and $k_1(x)=(120x^3-180x^2+60x)I(0\le x\le 1)$.

Lemma 2 Let $\widehat{f}_n(x)$ and $\widehat{f}_n^{(1)}(x)$ be defined in (2.2) and (2.3), respectively. If $f(x)$ is the s-th $(s\ge 2)$ continuously differentiable and the kernel function $k_i(x)$ $(i=0,1)$ satisfy the assumptions (1)-(3). Then for $x<T$, which is the same as in Lemma 1, taking $h_n=n^{-1/(2s+1)}$, we have

$$\mathsf{E}_n[\widehat{f}_n^{(i)}(x)-f^{(i)}(x)]^2\le\{c_{1i}[f^{(s)}(x)]^2+c_{2i}f(x)[1-W(x)]^{-1}\}n^{-(2s-2i)/(2s+1)}.$$

Proof Integrating by parts, we have

$$\begin{aligned}&\widehat{f}_n^{(i)}(x)-f^{(i)}(x)\\ =\ &\frac{1}{h_n^{1+i}}\int k_i\Big(\frac{x-t}{h_n}\Big)\mathrm{d}[\widehat{F}(t)-F(t)]+\frac{1}{h_n^{1+i}}\int k_i\Big(\frac{x-t}{h_n}\Big)\mathrm{d}F(t)-f^{(i)}(x)\\ =\ &\frac{1}{h_n^{2+i}}\int[\widehat{F}(t)-F(t)]k_i^{(1)}\Big(\frac{x-t}{h_n}\Big)\mathrm{d}t+\Big[\frac{1}{h_n^{i}}\int k_i(u)f(x-uh_n)\mathrm{d}u-f^{(i)}(x)\Big]\\ \widehat{=}\ &I_1+I_2.\end{aligned}\tag{3.1}$$

Since

$$\begin{aligned} I_1 &= \frac{1}{nh_n^{1+i}}\int_0^1 \sum_{j=1}^n [1-F(x-uh_n)]M_j(x-uh_n)k_i^{(1)}(u)\mathrm{d}u \\ &\quad + \frac{1}{nh_n^{1+i}}\int_0^1 R_n(x-uh_n)k_i^{(1)}(u)\mathrm{d}u \\ &\triangleq I_{11}+I_{12}, \end{aligned} \tag{3.2}$$

note that

$$\begin{aligned} I_{11}^2 &= \frac{1}{n^2h_n^{2+2i}}\int_0^1\int_0^1 \overline{F}(x-uh_n)\overline{F}(x-vh_n)\sum_{j=1}^n M_j(x-uh_n) \\ &\quad \cdot \sum_{l=1}^n M_l(x-vh_n)k_i^{(1)}(u)k_i^{(1)}(v)\mathrm{d}u\mathrm{d}v, \end{aligned} \tag{3.3}$$

by Lemma 1 and expanding $d(x-uh_n)$, $\overline{F}(x-uh_n)$ and $\overline{F}(x-vh_n)$ at point x, we can show

$$\begin{aligned} \mathsf{E}_n I_{11}^2 &= \frac{1}{nh_n^{2+2i}}\int_0^1\int_0^1 \mathsf{E}[M_1(x-uh_n)M_1(x-vh_n)] \\ &\quad \cdot \overline{F}(x-uh_n)\overline{F}(x-vh_n)k_i^{(1)}(u)k_i^{(1)}(v)\mathrm{d}u\mathrm{d}v \\ &= \frac{1}{nh_n^{2+2i}}\int_0^1\int_0^u d(x-uh_n)\overline{F}(x-uh_n)\overline{F}(x-vh_n)k_i^{(1)}(v)\mathrm{d}v k_i^{(1)}(u)\mathrm{d}u \\ &\quad + \frac{1}{nh_n^{2+2i}}\int_0^1\int_u^1 d(x-vh_n)\overline{F}(x-uh_n)\overline{F}(x-vh_n)k_i^{(1)}(v)\mathrm{d}v k_i^{(1)}(u)\mathrm{d}u \\ &\triangleq \frac{1}{nh_n^{2+2i}}\int_0^1 (Q_1(u)+Q_2(u))k_i^{(1)}(u)\mathrm{d}u, \end{aligned} \tag{3.4}$$

where

$$Q_1(u) = d(x)\overline{F}^2(x)\int_0^u \Big\{1+h_n\Big[u\frac{f(x)}{\overline{F}(x)}+v\frac{f(x)}{\overline{F}(x)}-u\frac{d^{(1)}(x)}{d(x)}\Big]+o(h_n)\Big\}k_i^{(1)}(v)\mathrm{d}v$$

and

$$Q_2(u) = d(x)\overline{F}^2(x)\int_u^1 \Big\{1+h_n\Big[u\frac{f(x)}{\overline{F}(x)}+v\frac{f(x)}{\overline{F}(x)}-v\frac{d^{(1)}(x)}{d(x)}\Big]+o(h_n)\Big\}k_i^{(1)}(v)\mathrm{d}v,$$

here $d^{(1)}(x)$ denotes the derivative of

$$d(x) = \int_{-\infty}^x \frac{\mathrm{d}F(t)}{[1-W(t)][1-F(t)]^2}. \tag{3.5}$$

Therefore, combing the assumptions of $k_i(x)$ with (3.4) together, it generates

$$\mathsf{E}_n I_{11}^2 = \frac{f(x)}{[1-W(x)]nh_n^{2i+1}}\int_0^1 k_i^2(u)\mathrm{d}u + o\Big(\frac{1}{nh_n^{2i+1}}\Big). \tag{3.6}$$

Also, using Lemma 1, it is easily seen that

$$\begin{aligned} \mathsf{E}_n I_{12}^2 &\le c_{1i}\Big(\frac{1}{nh_n^{1+i}}\Big)^2 \mathsf{E}\Big[\sup_{t\le T}|R_n(t)|\Big]^2 \\ &= c_{1i}\Big(\frac{1}{nh_n^{1+i}}\Big)^2\int_0^\infty x\,\mathsf{P}\Big(\sup_{t\le T}|R_n(t)|\ge x\Big)\mathrm{d}x \\ &= O\Big(\frac{\log^4 n}{n^2h_n^{2+2i}}\Big). \end{aligned} \tag{3.7}$$

On the other hand, expanding $f(x-uh_n)$ and using the assumption (1)-(3), we know

$$\begin{aligned} I_2 &= \frac{1}{h_n^i}\int_0^1 k_i(u)\Big[f(x)+\sum_{k=1}^{s-1}\frac{f^{(k)}(x)(-uh_n)^k}{k!}+\frac{f^{(s)}(x^*)(-uh_n)^s}{s!}\Big]\mathrm{d}u - f^{(i)}(x) \\ &= c_{2i}h_n^{s-i}f^{(s)}(x)+o(h_n^{s-i}), \qquad x^*\in(x-uh_n,x). \end{aligned} \tag{3.8}$$

Following from (3.1)-(3.8) and taking $h_n=n^{-1/(2s+1)}$, we conclude that the conclusion of the Lemma 2 is true. The proof of Lemma 2 is complete. #

Lemma 3 If $\mathsf{E}[c(\theta)]<\infty$ and $\mathsf{E}[\theta^{-s}c(\theta)]<\infty$, where s is a natural number, then

$$\sup_x|p^{(k)}(x)|<\infty, \qquad k=1,\cdots,s.$$

Proof By $\mathsf{E}[c(\theta)]<\infty$, we have

$$\begin{aligned} |p^{(k)}(x)| &= \Big|\int_\Theta(-\theta^{-1})^k c(\theta)\exp(-x/\theta)I(x>0)\mathrm{d}G(\theta)\Big| \le \mathsf{E}[\theta^{-k}c(\theta)] \\ &= \mathsf{E}[\theta^{-k}c(\theta)I(\theta>1)]+\mathsf{E}[\theta^{-k}c(\theta)I(\theta\le 1)] \\ &\le \mathsf{E}[c(\theta)]+\mathsf{E}[\theta^{-s}c(\theta)]<\infty. \end{aligned}$$

Hence, Lemma 3 holds. #

§ 4. Rate of Convergence

In this section, we exhibit the optimal rate of convergence of $\delta_n(x)$, which is defined in (2.6). By (1.9) and (2.7), we know

$$\begin{aligned} 0 &\le R(\delta_n(x),G(\theta))-R(\delta_G(x),G(\theta))=\int_0^\infty[\mathsf{E}_n\delta_n(x)-\delta_G(x)]\alpha(x)\mathrm{d}x \\ &= \int_{A_1}^{a_G}(\mathsf{P}(\widehat{\alpha}_n(x)\le 0)-1)\alpha(x)\mathrm{d}x+\int_{a_G}^{A_2}\mathsf{P}(\widehat{\alpha}_n(x)\le 0)\alpha(x)\mathrm{d}x \\ &= \int_{a_G}^{A_2}\mathsf{P}(\widehat{\alpha}_n(x)\le 0)\alpha(x)\mathrm{d}x-\int_{A_1}^{a_G}\mathsf{P}(\widehat{\alpha}_n(x)>0)\alpha(x)\mathrm{d}x \\ &\mathrel{\widehat{=}} I_1+I_2. \end{aligned} \tag{4.1}$$

For $1<\tau<2$, by Markov's inequality and Hölder's inequality, we have

$$\begin{aligned} I_1 &= \int_{a_G}^{A_2}\mathsf{P}(-\widehat{\alpha}_n(x)+\alpha(x)\ge\alpha(x))\alpha(x)\mathrm{d}x \\ &\le \int_{a_G}^{A_2}[\alpha(x)]^{1-\tau}[\mathsf{E}_n(\widehat{\alpha}_n(x)-\alpha(x))^2]^{\tau/2}\mathrm{d}x. \end{aligned} \tag{4.2}$$

Furthermore, it follows from (1.10) and (2.4) and Lemma 2,

$$\begin{aligned} &\mathsf{E}_n[\widehat{\alpha}_n(x)-\alpha(x)]^2 \\ \le\ & 2g^2(x)\mathsf{E}_n[\widehat{f}_n(x)-f(x)]^2+2b^2\mathsf{E}_n[\widehat{f}_n^{(1)}(x)-f^{(1)}(x)]^2 \\ \le\ & 2c_{10}g^2(x)[f^{(s)}(x)]^2n^{-2s/(2s+1)}+2c_{20}g^2(x)f(x)[1-W(x)]^{-1}n^{-2s/(2s+1)} \\ & +2c_{11}b^2[f^{(s)}(x)]^2n^{-(2s-2)/(2s+1)}+2c_{21}b^2f(x)[1-W(x)]^{-1}n^{-(2s-2)/(2s+1)}. \end{aligned} \tag{4.3}$$

Then, under the condition that

(I) $\int_{a_G}^{A_2} |g^\tau(x)|[\alpha(x)]^{1-\tau}|f^{(s)}(x)|^\tau \mathrm{d}x < \infty$;

(II) $\int_{a_G}^{A_2} |g^\tau(x)|[\alpha(x)]^{1-\tau}[f(x)(1-W(x))^{-1}]^{\tau/2}\mathrm{d}x < \infty$;

(III) $\int_{a_G}^{A_2} [\alpha(x)]^{1-\tau}|f^{(s)}(x)|^\tau \mathrm{d}x < \infty$;

(IV) $\int_{a_G}^{A_2} [\alpha(x)]^{1-\tau}[f(x)(1-W(x))^{-1}]^{\tau/2}\mathrm{d}x < \infty$, (4.4)

we have

$$I_1 = O(n^{-\tau(s-1)/(2s+1)}). \tag{4.5}$$

In fact, for $1<\tau<2$, note that a_G is a singular point, then by

$$\lim_{x\to a_G}(x-a_G)^{\tau-1}[\alpha(x)]^{1-\tau} = \Big(\frac{1}{f(a_G)}\lim_{x\to a_G}\frac{1}{\beta^{(1)}(x)}\Big)^{\tau-1} > 0, \tag{4.6}$$

we know that integration $\int_{a_G}^{A_2}[\alpha(x)]^{1-\tau}\mathrm{d}x$ is convergent. Furthermore, by Lemma 3, we find that if $g(x)$ is continuous and $u(x)$ is the s-th ($s\ge 2$) continuously differentiable, also, $\mathsf{E}[c(\theta)]<\infty$ and $\mathsf{E}[\theta^{-s}c(\theta)]<\infty$, then it is easy to see that the condition (4.4) is held.

Under the condition that $\int_{A_1}^{a_G}[-\alpha(x)]^{1-\tau}\mathrm{d}x<\infty$, by an similar discussion to I_2, we can show that

$$I_2 = O(n^{-\tau(s-1)/(2s+1)}). \tag{4.7}$$

Therefore, we state the following Theorem.

Theorem 1 Let $\delta_G(x)$ and $\delta_n(x)$ be defined in (1.8) and (2.6), respectively. If the following conditions are satisfied:

(i) $G(\theta)\in\mathcal{F}$, $\mathsf{E}[c(\theta)]<\infty$, $\mathsf{E}[\theta^{-s}c(\theta)]<\infty$;

(ii) $\int_{A_1}^{a_G}[-\alpha(x)]^{1-\tau}\mathrm{d}x<\infty$; $g(x)$ is continuous;

(iii) $u(x)$ is the s-th continuously differentiable.

Then for $1<\tau<2$, we have

$$0\le R(\delta_n(x),G(\theta)) - R(\delta_G(x),G(\theta)) = O(n^{-\tau(s-1)/(2s+1)}),$$

where $s\ge 2$ is an arbitrary but fixed integer.

Proof Following the preceding discussion, we can easily come to the conclusion of the Theorem. #

Remark 2 Especially, for the exponential distribution $f(x|\theta)=(1/\theta)\cdot\exp(-x/\theta)$, if we take the conjugate prior density $g(\theta)=(1/\theta)^2\exp(-1/\theta)$, then $\alpha(x)=af(x)+a\theta_0 f^{(1)}(x)$ with $f(x)=(x+1)^{-2}I(x>0)$. Hence $a_G=2\theta_0-1$. Let $A_1=0$, we can easily know the Theorem 1's conditions are satisfied and accordingly the rate of convergence can be arbitrarily close to $O(n^{-1})$ under the condition that τ is arbitrarily close to 2 and s is large enough.

References

[1] Johns, M.V.Jr. and Van Ryzin, J.R., Convergence rates for empirical Bayes two-action problem I: discrete case, *Ann. Math. Statist.*, **42**(1971), 1521–1539.

[2] Johns, M.V.Jr. and Van Ryzin, J.R., Convergence rates for empirical Bayes two-action problem II: continuous case, *Ann. Math. Statist.*, **43**(1972), 934–937.

[3] Kaplan, E.T. and Meier, P., Nonparametric estimation from incomplete observations, *J. Amer. Statist. Assoc.*, **53**(1958), 457–481.

[4] Karunamuni, R.J., Optimal rates of convergence of empirical Bayes tests for the continuous one-parameter exponential family, *Ann. Statist.*, **24**(1996), 212–231.

[5] Lemdani, M. and Ould-Saïd, E., Exact asymptotic $\mathcal{L}_1$-error of a kernel density estimator under censored data, *Statistics & Probability Letters*, **60**(2002), 59–68.

[6] Liang, T.C., On an empirical Bayes test for a normal mean, *Ann. Statist.*, **28**(2000), 648–655.

[7] Major, P. and Rejtö, L., Strong embedding of the estimator of the distribution function under random censorship, *Ann. Statist.*, **16**(1988), 1113–1132.

[8] Mielniczuk, J., Some asymptotic properties of kernel estimators of a density function in case of censored data, *Ann. Statist.*, **14**(1986), 766–773.

[9] Robbins, H., An empirical Bayes approach to statistics, *Proc. Third Berkeley Symp. Math. Statist. Prob.*, University of California Press, **1**(1955), 157–163.

[10] Robbins, H., The empirical Bayes approach to statistical decision problem, *Ann. Math. Statist.*, **35**(1964), 1–20.

[11] Singh, R.S. and Wei, L.S., Nonparametric empirical Bayes procedures, asymptotic optimality and rates of convergence for two-tail tests in exponential family, *Nonparametric Statistics*, **12**(2000), 475–501.

[12] Stijnen, TH., On the asymptotic behaviour of empirical Bayes tests for the continuous one-parameter exponential family, *Ann. Statist.*, **13**(1985), 403–412.

[13] Van Houwelingen, J.C., Monotone empirical Bayes test for the continuous one-parameter exponential family, *Ann. Statist.*, **4**(1976), 981–989.

随机删失下刻度参数的单调经验贝叶斯检验

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利用经验贝叶斯方法研究了刻度指数族的两行动问题，提出了一个在历史样本被随机右删失的条件下收敛速度可以任意接近 $O(n^{-1})$ 的单调经验贝叶斯检验.

关 键 词： 经验贝叶斯，随机删失，刻度指数族.

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