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GLOBAL DISCRETE INF SUP CONDITION FOR AN UNSTEADY INTERACTION PROBLEM

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Abstract

We consider a steady mixed strategy for computing the evolution of a viscous incompressible fluid inside an elastic Koiter shell in bending dominated state. We propose mixed finite element formulations of the different subproblems which we couple by mortar technics. The Koiter shell is approximated by a locking free finite element inspired by [1]. We deduce from the local discrete infsup conditions and the mortar coupling a global infsup condition in the same way that [2]. We derive then a convergence result and indicate that the different schemes preserve their accuracy and stability after coupling.

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Keywords: interaction, shell, locking, finite element, infsup condition.

1 Introduction

Because of its many applications to industrial [3], [4], and biological problems [5], [6], fluid structure interaction models have been extensively studied over the past few years. Many challenging applications involve an incompressible three dimensional fluid lying inside flexible thin shell as faced for example when studying medical flows in flexible pipes. In such situations, each component of the system has its own model and discretization requirements. Moreover, the structure can be very stiff along certain deformation modes. And when the finite element is used, the calculation faces major difficulties and the convergence is manifestly deteriorate as the shell thickness becomes small. This phenomena, referred to as locking is usually overcome by using mixed finite elements formulations treating some components of the strain tensor as independent variables [1], [7]. The challenge is then to match the different local finite elements while preserving their stability and robustness.

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In the present work, the model involves a time-dependent linearized interaction problem where an incompressible viscous fluid flows inside a flexible Koiter shell in bending dominated state. For simplicity, all change of configurations is neglected. Our space discretization is standard in the fluid part and is locking free in the shell part. The paper is organized as follows. In section 2, we introduce the shell Koiter dynamic model, we present a mixed formulation and define the interaction problem. In section3, we present our space discretization strategies. In section4, our attention is focused on the space discretization convergence. By using the local discrete infsup conditions, we prove a convergence result with constants independent of the shell thickness under some restrictive geometrical assumption.

2 Problem definition

We consider a system which occupies a fixed domain Ω made of a viscous incompressible fluid in motion in a fixed part Ω^f and a deformable Koiter shell on the complement Ω^s. We suppose that Ω^f is delimited by the fluid structure interface Γ and by external boundary Γ^f.

2.1 The Koiter shell model

Greek indices take their values in the set $\{1,2\}$ and the Latin indices take their values in $\{1,2,3\}$. Products containing repeated indices are summed.
We consider a shell with a thickness ε and a midsurface $S = \overrightarrow{\varphi}(\bar{\omega})$ where ω is a domain of R^2 and $\overrightarrow{\varphi} \in W^{2,\infty}(\omega, R^3)$ is an injective mapping. Let $\overrightarrow{a}_\alpha = \overrightarrow{\varphi}_{,\alpha}$ and $\overrightarrow{a}_3 = \frac{\overrightarrow{a}_1 \wedge \overrightarrow{a}_2}{\|\overrightarrow{a}_1 \wedge \overrightarrow{a}_2\|}$ be the covariant basis vectors, $a = \|\overrightarrow{a}_1 \wedge \overrightarrow{a}_2\|^2$ and $\underline{\underline{\underline{\underline{E}}}} = (E^{\alpha\beta\lambda\mu})_{\alpha\beta\lambda\mu}$ be the elasticity tensor assumed to be elliptic. For a displacement field $\overrightarrow{u_s}$, we define the linearized change of curvature tensor $\underline{\underline{\Upsilon}} = (\Upsilon_{\alpha\beta})_{\alpha,\beta}$ and the linearized membrane strain tensor $\underline{\underline{\Lambda}} = (\Lambda_{\alpha\beta})_{\alpha,\beta}$ by

$$\Upsilon_{\alpha\beta}(\overrightarrow{u_s}) = (\overrightarrow{u_s}_{,\alpha\beta} - \Gamma^{\rho}_{\alpha\beta}\overrightarrow{u_s}_{,\rho}).\overrightarrow{a}_3,$$
$$\Lambda_{\alpha\beta}(\overrightarrow{u}) = \frac{\overrightarrow{u_s}_{,\alpha}.\overrightarrow{a}_\beta + \overrightarrow{u_s}_{,\beta}.\overrightarrow{a}_\alpha}{2}.$$

We suppose the shell clamped on a part $\partial\omega_d$ of its boundary and set

$$H^1_{\partial\omega_d}(\omega) = \{u \in H^1(\omega), u = 0 \ \text{on} \partial\omega_d\},$$
$$H^2_{\partial\omega_d}(\omega) = \{u \in H^2(\omega), u = \frac{\partial u}{\partial n} = 0 \ \text{on} \partial\omega_d\}$$
$$V^s = \{\overrightarrow{v} = v_i a^i, v_\alpha \in H^1_{\partial\omega_d}(\omega), v_3 \in H^2_{\partial\omega_d}(\omega)\},$$

Note that V^s is a Hilbert space when endowed with the norm

$$||\overrightarrow{v}||_{V^s} = (\Sigma_\alpha ||v_\alpha||^2_{H^1} + ||v_3||^2_{H^2})^{1/2}.$$

Consider the dynamic bending dominated Koiter shell problem [1], [7], [8]

$$\text{find } \overrightarrow{u_s} \in L^2(0,T;V^s)$$
$$\widetilde{m}_s(\overrightarrow{\ddot{u}}_s;\overrightarrow{v}) + \tilde{A}_s(\overrightarrow{u_s};\overrightarrow{v}) = \tilde{L}_s(\overrightarrow{v}) \ \forall \overrightarrow{v} \in V^s \text{ a.e.in time},$$

where $\widetilde{m}_s$ is the inertia term, $\tilde{L}_s$ corresponds to the external energy and $\tilde{A}_s$ is the bilinear form associated to the internal energy given respectively by

$$\widetilde{m}_s(\overrightarrow{\ddot{u}_s};\overrightarrow{v}) = \varepsilon^3\int_\omega \rho_s \overrightarrow{\ddot{u}_s}.\overrightarrow{v}\sqrt{a}dx := \varepsilon^3 m_s(\overrightarrow{\ddot{u}};\overrightarrow{v}),$$
$$\tilde{L}_s(\overrightarrow{v}) = \varepsilon^3\int_\omega \overrightarrow{f_s}.\overrightarrow{v}\sqrt{a}dx := \varepsilon^3 L_s(\overrightarrow{v}),$$
$$\tilde{A}_s(\overrightarrow{u};\overrightarrow{v}) = \int_\omega E^{\alpha\sigma\lambda\mu}(\tfrac{\varepsilon^3}{12}\Upsilon_{\alpha\sigma}(\overrightarrow{u})\Upsilon_{\lambda\mu}(\overrightarrow{v}) + \tfrac{\varepsilon}{2}\Lambda_{\alpha\sigma}(\overrightarrow{u})\Lambda_{\lambda\mu}(\overrightarrow{v}))\sqrt{a}dx.$$

To overcome the locking phenomena, we introduce in the same way that [1] a new variable $\underline{\underline{\lambda}}$ by setting for a real $c_0 \in]0,\varepsilon^{-2}[$

$$\underline{\underline{\lambda}} = (\lambda_{\alpha\gamma})_{\alpha\gamma},\ \lambda_{\alpha\gamma} = (\tfrac{1}{\varepsilon^2} - c_o)E^{\alpha\gamma\sigma\mu}\Lambda_{\sigma\mu}(\overrightarrow{u_s}),$$

and seek pairs

$$(\overrightarrow{u_s},\underline{\underline{\lambda}}) \in L^2(0,T;V^s)\times L^\infty(0,T;W^s) \text{ such that}$$
$$m_s(\overrightarrow{\ddot{u}}_s(t);\overrightarrow{v}) + A_s(\overrightarrow{u}_s(t);\overrightarrow{v}) + B_s(\overrightarrow{v};\underline{\underline{\lambda}}(t)) = L_s(\overrightarrow{v})\ \forall\overrightarrow{v}\in V^s \text{ a.e in t}$$
$$B_s(\overrightarrow{u}^s(t);\hat{\underline{\underline{\lambda}}}) - \frac{\varepsilon^2}{1-c_o\varepsilon^2}C_s(\underline{\underline{\lambda}}(t);\hat{\underline{\underline{\lambda}}}) = 0, \forall\hat{\underline{\underline{\lambda}}}\in W^s = \{\underline{\underline{\varphi}}/\varphi^{\alpha\beta}\in L^2(\omega)\} \text{ a.e in t}$$

where

$$A_s(\overrightarrow{u};\overrightarrow{v}) = \int_\omega E^{\alpha\sigma\lambda\mu}(\tfrac{1}{12}\Upsilon_{\alpha\sigma}(\overrightarrow{u})\Upsilon_{\lambda\mu}(\overrightarrow{v}) + c_0\Lambda_{\alpha\sigma}(\overrightarrow{u})\Lambda_{\lambda\mu}(\overrightarrow{v}))\sqrt{a}dx,$$
$$B_s(\overrightarrow{v};\underline{\underline{\xi}}) = \int_\omega \Lambda_{\alpha\sigma}(\overrightarrow{v})\xi^{\alpha\sigma}\sqrt{a}dx,\ C_s(\underline{\underline{\lambda}};\underline{\underline{\xi}}) = \int_\omega (E^{-1})_{\alpha\sigma\delta\mu}\lambda^{\delta\mu}\xi^{\alpha\sigma}\sqrt{a}dx.$$

Note that the bilinear forms A_s, B_s C_s are continuous respectively on $V^s\times V^s$, $V^s\times W^s$ and $W^s\times W^s$, that A_s is V^s-elliptic and C_s is W^s-elliptic [1], [8].

2.2 The fluid-structure problem

We suppose that the fluid is viscous governed by the Stokes equations. Its evolution problem determines

$$\overrightarrow{v}_f \in L^2(0,T,H^1_{\Gamma^f}),\ p(t)\in L_0(\Omega^f) \text{ such that}$$
$$m_f(\overrightarrow{\dot{v}}_f,\overrightarrow{\hat{v}}_f) + A_f(\overrightarrow{v_f},\overrightarrow{\hat{v}_f}) + B_f(\overrightarrow{\hat{v}_f},p) = L_f(\overrightarrow{\hat{v}}_f), \forall\overrightarrow{\hat{v}_f}\in H^1_{\Gamma^f} \text{ a.e. in t},$$
$$B_f(\overrightarrow{v_f}(t),q) = \int_{\Omega^f} q\,div\overrightarrow{v_f}(t)dx = 0 \quad \forall q\in W^f = L_0(\Omega^f) \text{ a.e. in t.}$$

where

$$H^1_{\Gamma^f} = \left\{\overrightarrow{v}\in H^1(\Omega^f,\mathbb{R}^3),\ \overrightarrow{v} = 0 \text{ on } \Gamma^f\right\},$$
$$m_f(\overrightarrow{\dot{v}_f},\overrightarrow{\hat{v}}_f) = \int_{\Omega^f}\rho_f\overrightarrow{\dot{v}_f}.\overrightarrow{\hat{v}_f}dx,$$
$$A_f(\overrightarrow{v_f},\overrightarrow{\hat{v}_f}) = \frac{1}{4}\int_{\Omega^f}\mu(\nabla\overrightarrow{v_f} + \nabla^t\overrightarrow{v_f}) : (\nabla\overrightarrow{\hat{v}_f} + \nabla^t\overrightarrow{\hat{v}_f})dx,$$

$\mu > 0$ denotes the fluid viscosity, L_f is the resultant of the applied forces and ρ_f denotes the mass density of the fluid.

We suppose that Γ corresponds to the upper face of a part $\omega_\Gamma \subset \omega$:

$$\Gamma = \left\{\overrightarrow{x} = \overrightarrow{\varphi}(\xi_1,\xi_2) + \varepsilon\overrightarrow{a_3}(\xi_1,\xi_2),\ (\xi_1,\xi_2)\in\omega_\Gamma\right\}. \tag{2.1}$$

The global displacement field must be continuous at this interface Γ

$$\overrightarrow{u}_s(\xi_1,\xi_2;t) = \overrightarrow{u}_f(x,t), \forall \overrightarrow{x} \in \Gamma, a.e. \text{ in } t, \tag{2.2}$$

where $\overrightarrow{u}_f(.,t) = \overrightarrow{u}_f(.,0) + \int_0^t \overrightarrow{v}_f(.,\tau)d\tau$ is the fluid displacement field.
We define the global Hilbert space of stresses test functions by $W = W^s \times W^f$, the global kinematically admissible velocity test functions space by

$$V = \left\{ (\overrightarrow{v}_s, \overrightarrow{v}_f) \in V^s \times H^1_{\Gamma f},\ \overrightarrow{v}_f = \overrightarrow{v}_s \text{ on } \Gamma \right\} \tag{2.3}$$

and the space for unknown velocity fields by :

$$U = \left\{ \begin{array}{c} (\overrightarrow{v}_s, \overrightarrow{v}_f) \in L^2(0,T;L^2(\omega^s,R^3) \times L^2(\Omega^f,R^3)) \ ; \\ \overrightarrow{v}_f \in L^2(0,T;H^1_{\Gamma f}), \\ (\overrightarrow{u}_s, \overrightarrow{u}_f)(0) + \int_0^t (\overrightarrow{v}_s, \overrightarrow{v}_f)(\tau)d\tau \in L^2(0,T;V) \end{array} \right\}.$$

The mixed formulation describing the evolution of the global system is

$$(MF) \left\{ \begin{array}{l} \text{Find } \overrightarrow{v} = (\overrightarrow{v}_s, \overrightarrow{v}_f) \in U,\ (\underline{\underline{\lambda}}, p_f) \in W \text{ such that} \\ \\ m(\dot{\overrightarrow{v}}; \hat{\overrightarrow{v}}) + A_s(\overrightarrow{u}_s; \hat{\overrightarrow{v}}_s) + A_f(\overrightarrow{v}_f, \hat{\overrightarrow{v}}_f) \\ \\ +B_s(\hat{\overrightarrow{v}}_s; \underline{\underline{\lambda}}) + B_f(\hat{\overrightarrow{v}}_f; p_f) \\ \\ = L(\hat{\overrightarrow{v}}_s, \hat{\overrightarrow{v}}_f),\ \forall\ (\hat{\overrightarrow{v}}_s, \hat{\overrightarrow{v}}_f) \in V, \text{ a.e in time,} \\ \\ B_s(\overrightarrow{u}_s(t); \hat{\underline{\underline{\lambda}}}) - \frac{\varepsilon^2}{1-c_o\varepsilon^2} C_s(\underline{\underline{\lambda}}(t); \hat{\underline{\underline{\lambda}}}) = 0, \forall \hat{\underline{\underline{\lambda}}} \in W^s, \text{ a.e in time,} \\ \\ B_f(\overrightarrow{v}_f; q_f) = 0, \quad \forall q_f \in W^f, \text{ a.e in time,} \\ \\ \overrightarrow{u}_s(t) = \overrightarrow{u}_s(0) + \int_0^t \overrightarrow{v}_s(\tau)d\tau, \\ \\ (\overrightarrow{v}_s, \overrightarrow{v}_f)(0) \text{ is given in } V; \end{array} \right.$$

where $m(\dot{\overrightarrow{v}}; \hat{\overrightarrow{v}}) = m_s(\dot{\overrightarrow{v}}_s; \hat{\overrightarrow{v}}_s) + m_f(\dot{\overrightarrow{v}}_f, \hat{\overrightarrow{v}}_f)$ and $L(\hat{\overrightarrow{v}}_s, \hat{\overrightarrow{v}}_f) = L_s(\hat{\overrightarrow{v}}_s) + L_f(\hat{\overrightarrow{v}}_f)$.

Remark : The continuity requirement (2.2) between fluid and shell displacement is not completely exact. The displacement of the Koiter shell at its upper surface is not identical to the midsurface displacement $\overrightarrow{u}_s$. The true displacement is $\overrightarrow{u}_s + \varepsilon\delta\overrightarrow{a_3}$. The Kirchhoff kinematical assumption (i.e. zero shear strain) should be used to calculate the implicit rotations $\delta\overrightarrow{a_3}$ and gives [9]

$$\delta\overrightarrow{a_3} = (\overrightarrow{a}_3.\overrightarrow{u_s}_{,\alpha})\overrightarrow{a}^{\alpha}.$$

Assuming that the thickness is small, the simplification (2.2) is accepted as approximation at first order of the true continuity requirement.

3 Discretizations

We will concentrate hereafter on the space discretization and its convergence.The key issues for the time stepping is present in [10] for a similar interaction problem. The main difference compared to the earlier work [10] is that the convergence result should be independent of the small parameter ε. The shell discretisation strategy is then fundamentally different.

3.1 Shell discretization

We henceforth assume that the domain ω is a polygon triangulated by a regular triangulation τ_s^h and introduce the spaces

$$\begin{aligned}
L_h^1 &= \{v \in H^1_{\partial\omega_d}(\omega),\ v_{/T} \in P_3(T)\ \ \forall T \in \tau_s^h\}, && (3.1)\\
B_h^1 &= \{v \in H^1(\omega),\ v_{/T} = \lambda_1\lambda_2\lambda_3 p,\ \ p \in P_1(T)\ \ \forall T \in \tau_s^h\},\\
H_h^1 &= \{v = v_1 + v_2 \text{ such that } v_1 \in L_h^1, v_2 \in B_h^1\} = L_h^1 \oplus B_h^1,\\
L_h^2 &= \{v \in H^2_{\partial\omega_d}(\omega),\ v_{/T} \in P_5(T)\ \ \forall T \in \tau_s^h\}, && (3.2)\\
B_h^2 &= \{v \in H^2 \cap H_0^1(\omega),\ v_{/T} = \lambda_1^2\lambda_2^2\lambda_3^3 p,\ \ p \in P_1(T)\ \forall T \in \tau_s^h\},\\
H_h^2 &= \{v = v_1 + v_2 \text{ such that } v_1 \in L_h^2, v_2 \in B_h^2\} = L_h^2 \oplus B_h^2. && (3.3)
\end{aligned}$$

Above, λ_1, λ_2 and λ_3 denote the barycentric coordinates for each triangle T. We note that the space L_h^2 consists of the Argyris element and B_h^α are bubble function spaces that will be used for the local adjustment to achieve discrete stability. We introduce the discrete displacement and stress spaces by

$$V_h^s = \left\{ \overrightarrow{v}_h \in V^s,\ \overrightarrow{v}_h.\overrightarrow{a}_\alpha \in H_h^1,\ \overrightarrow{v}_h.\overrightarrow{a}_3 \in H_h^2 \right\},$$
$$W_h^s = \{\underline{\underline{\lambda}},\ \lambda_{\alpha\beta/T} \in P_1(T)\ \ \forall T\}$$

To prove uniform convergence with respect to the shell thickness, we need as in [1] an inf sup stability hypothesis where we assume that there exists a constant $\widetilde{C} > 0$ for which we have

$$\inf_{0\neq\lambda\in W_h^s}\ \sup_{0\neq\overrightarrow{v}\in V_h^s} \frac{B_s(\overrightarrow{v};\underline{\underline{\lambda}})}{||\overrightarrow{v}||.|||\underline{\underline{\lambda}}|||} \geq \widetilde{C}. \quad (3.4)$$

Below, $|||.|||$ is a semi norm on W^s defined by

$$|||\underline{\underline{\lambda}}||| = \sup_{0\neq\overrightarrow{v}\in V_h} \frac{B_s(\overrightarrow{v};\underline{\underline{\lambda}})}{||\overrightarrow{v}||}.$$

Lemma 1 : *If the first and second fundamental forms associated to he shell midsurface are piecewise constant then (3.4) is verified.*

Proof : The proof is based on the construction of an adequate projection operator $\pi : V^s \to V_h^s$ satisfying

$$\begin{aligned}
i) B_s(\pi\overrightarrow{v};\underline{\underline{\lambda}}) &= B_s(\overrightarrow{v};\underline{\underline{\lambda}})\ \forall\overrightarrow{v} \in V^s, \forall\underline{\underline{\lambda}} \in W_h^s, && (3.5)\\
ii) \|\pi\overrightarrow{v}\|_{V^s} &\leq C_o \|\overrightarrow{v}\|_{V^s}\ \forall\overrightarrow{v} \in V^s. && (3.6)
\end{aligned}$$

We consider the projection $\pi^1 : H^1 \to H_h^1$ constructed in [1] which satisfies

$$\begin{aligned} \left\|\pi^1 v\right\|_{H^1(T)} &\leq C\|v\|_{H^1(\widetilde{T})}; \forall T \in \tau_s^h, &(3.7)\\ \int_e (v-\pi^1 v)p &= 0\ \forall p \in P_1(T),\ \forall e \in \partial T,\ T \in \tau_s^h, &(3.8)\\ \int_T (v-\pi^1 v)p &= 0\ \forall p \in P_1(T),\ \forall T \in \tau_s^h, &(3.9)\end{aligned}$$

where $\widetilde{T}$ is the union of triangles in τ_s^h which meet T.
What then remains is to construct a projection $\pi^2 : H^2 \to H_h^2$ satisfying

$$\begin{aligned} \int_T (v-\pi^1 v)p &= 0\ \forall p \in P_1(T)\ \forall T \in \tau_s^h, &(3.10)\\ \left\|\pi^2 v\right\|_{H^2(T)} &\leq C\|v\|_{H^2(\widetilde{T})}. &(3.11)\end{aligned}$$

A constructive way to define a map $\pi_0^2 : H^2 \to L_h^2$ satisfying for any $T \in \tau_s^h$

$$\left\|v-\pi_0^2 v\right\|_{0,T} + h\left\|v-\pi_0^2 v\right\|_{1,T} + h^2\left\|v-\pi_0^2 v\right\|_{2,T} \leq Ch^2\|v\|_{2,\widetilde{T}}\ \ \forall v \in H^2$$

can be found in [11]. We define $\pi_1^2 : H_{\partial\omega}^2 \to B_h^2$ by the conditions

$$\int_T (v-\pi_1^2 v)p = 0\ \forall p \in P_1(T),\ \forall T \in \tau_s^h \tag{3.12}$$

and obtain by a scaling argument

$$\left\|v-\pi_1^2 v\right\|_{0,T} + h\left\|v-\pi_1^2 v\right\|_{1,T} + h^2\left\|v-\pi_1^2 v\right\|_{2,T} \leq C(h^2|v|_{2,T} + \|v\|_{0,T}).$$

Finally we set

$$\pi^2 v = \pi_0^2 v + \pi_1^2(v-\pi_0^2 v),$$

and obtain an operator π^2 which verifies *(3.10-3.11)* and an operator $\pi = (\pi^1,\pi^1,\pi^2)$ which verifies (3.5-3.6). Then given $\underline{\underline{\lambda}} \in W_h^s$, we may choose $\overrightarrow{v} \in V^s$ for which

$$\frac{B_s(\pi\overrightarrow{v};\underline{\underline{\lambda}})}{||\overrightarrow{v}||} \geq \frac{1}{2}|||\underline{\underline{\lambda}}|||$$

and then the lemma holds after writing

$$\frac{B_s(\pi\overrightarrow{v};\underline{\underline{\lambda}})}{||\pi\overrightarrow{v}||_{V^s}} = \frac{B_s(\overrightarrow{v};\underline{\underline{\lambda}})}{||\pi\overrightarrow{v}||_{V^s}} \geq \frac{B_s(\overrightarrow{v};\underline{\underline{\lambda}})}{C_o||\overrightarrow{v}||_{V^s}} \geq \frac{1}{2C_o}|||\underline{\underline{\lambda}}|||.$$

In this framework, we use the following theorem proved in [1] in an abstract cadre which proves an uniform convergence with respect to the thickness.

Theorem 1 : *If (3.4) is satisfied, there exists unique pairs* $(u_s,\underline{\underline{\lambda}}) \in V^s \times W^s$, $(u_{sh},\underline{\underline{\lambda}}_h) \in V_h^s \times W_h^s$ *solutions respectively of the continuous and discrete static mixed shell problems. Moreover, the errors are bounded by*

$$\begin{aligned} &\|\overrightarrow{u}_s - \overrightarrow{u}_{sh}\|_{V^s} + |||\underline{\underline{\lambda}} - \underline{\underline{\lambda}}_h||| + \tfrac{\varepsilon^2}{1-c_o\varepsilon^2}\left\|\underline{\underline{\lambda}} - \underline{\underline{\lambda}}_h\right\|_{W^s}\\ &\leq C\inf_{\hat{v}\in V_h^s,\hat{\lambda}\in W_h^s}\{\left\|\overrightarrow{u}_s - \overrightarrow{\hat{v}}\right\|_{V^s} + |||\underline{\underline{\lambda}} - \hat{\underline{\underline{\lambda}}}||| + \tfrac{\varepsilon^2}{1-c_o\varepsilon^2}\left\|\underline{\underline{\lambda}} - \hat{\underline{\underline{\lambda}}}\right\|_{W^s}\}\\ &\leq Ch^2\{\|\overrightarrow{u}_s\|_{H^3} + \|\overrightarrow{u}_s.\overrightarrow{a}_3\|_{H^4} + \left\|\underline{\underline{\lambda}}\right\|_{H^2}\}.\end{aligned}$$

where C is a constant independent of h and ε

3.2 Fluid and interface discretizations

Let τ_f^h be a regular triangulation of the three dimensional fluid domain, $\Omega^f = \cup_{T\in\tau_f^h} T$. We approximate the velocity and the pressure fields by [12] :

$$\begin{aligned} H_h^f &= \{\overrightarrow{v}_h \in H^1(\Omega^f,\mathbb{R}^3), v_{h/T} \in (P_1(T)\oplus\{\lambda_1\lambda_2\lambda_3\lambda_4\})^3, \overrightarrow{v}_h = 0 \text{ on } \Gamma^f\}, \quad (3.13)\\ W_h^f &= \{q_h \in H^1\left(\Omega^f\right); q_{h/T} \in P_1(T)\},\\ V_h^f &= \{\overrightarrow{v}_h \in H_h^f;\ \int_{\Omega^f} div\overrightarrow{v}_h.q_h dx = 0 \forall q_h W_h^f\}, \end{aligned}$$

$(\lambda_i)_{i=1,4}$ denote the barycentric coordinates for the tetrahedron T. This choice of spaces yields a convergence in $O(h)$ and a compatibility condition in the following sense [12]

$$Inf_{q_h\in W_h^f\cap L_0^2} Sup_{\overrightarrow{v}_h\in H_h^f} \frac{\int_{\Omega^f} q_h div\overrightarrow{v}_h}{\|q_h\|_{L^2}.\|\overrightarrow{v}_h\|_{H^1}} \geq \alpha > 0. \quad (3.14)$$

The triangulations used for the structure and for the fluid are incompatible. This prevents the discrete test functions from satisfying the continuity condition at the interface and we have $V^h \nsubseteq V$. A strategy for defining compatible traces on incompatible grids consists in introducing a mortar space $G_h \subset L^2(\Gamma)$ and imposing the continuity in the following weak sense

$$\int_\Gamma(\overrightarrow{v}_{sh} - \overrightarrow{v}_{fh}).\overrightarrow{g}_h = 0, \forall\overrightarrow{g}_h \in G_h, \forall(\overrightarrow{v}_{sh},\overrightarrow{v}_{fh}) \in V_h^s\times H_h^f.$$

A lot of flexibility can be introduced in the mortar space G_h, nevertheless we expect it to satisfy the compatibility condition [13]

$$\inf_{g_h\in G_h}\sup_{v_h\in H_h^f} \frac{\int_\Gamma \overrightarrow{v}_h.\overrightarrow{g}_h}{\|\overrightarrow{v}_h\|_{L^2(\Gamma)}\|\overrightarrow{g}_h\|_{L^2(\Gamma)}} \geq C > 0, \quad (3.15)$$

and refer to [2] and [13] for examples of such finite elements spaces. We note π_g the best approximation on the multiplier space G_h. When G_h uses finite element of order q, direct interpolation yields

$$\inf_{\overrightarrow{g_h}\in G_h} \|\overrightarrow{g} - \overrightarrow{g}_h\|_{(H^{1/2}(\Gamma))'} \leq Ch^q\|\overrightarrow{g}\|_{H^{q-1/2}(\Gamma)}, \forall\overrightarrow{g}\in H^{1/2}(\Gamma). \quad (3.16)$$

We define the global spaces W_h and V_h respectively by $W_h = W_h^s\times W_h^f$ and

$$V_h = \{\left(\overrightarrow{v}_{sh},\overrightarrow{v}_{fh}\right)\in V_h^s\times H_h^f;\ \int_\Gamma\overrightarrow{g}_h(\overrightarrow{v}_{fh} - \overrightarrow{v}_{sh}) = 0\ ,\ \forall\overrightarrow{g}_h\in G_h\}.$$

The semi discrete problem is then

$$(MF_h)\begin{cases}\text{Find } \overrightarrow{v}_h=(\overrightarrow{v}_{sh},\overrightarrow{v}_{fh}):t\in[0,T]\to V_h,\\[2ex] (\underline{\underline{\lambda}}_h,p_{fh}):t\in[0,T]\to W_h \text{ such that}\\[2ex] m(\overrightarrow{v}_h;\overrightarrow{\hat{v}}_h)+A_s(\overrightarrow{u}_{sh};\overrightarrow{\hat{v}}_{sh})+A_f(\overrightarrow{v}_{fh},\overrightarrow{\hat{v}}_{fh})\\[2ex] +B_s(\overrightarrow{\hat{v}}_{sh};\underline{\underline{\lambda}}_h)+B_f(\overrightarrow{\hat{v}}_{fh};p_{fh})\\[2ex] =L(\overrightarrow{\hat{v}}_{sh},\overrightarrow{\hat{v}}_{fh}),\ \forall\,(\overrightarrow{\hat{v}}_{sh},\overrightarrow{\hat{v}}_{fh})\in V_h,\ \text{a.e in time},\\[2ex] B_s(\overrightarrow{u}_{sh};\hat{\underline{\underline{\lambda}}}_h)-\frac{\varepsilon^2}{1-c_o\varepsilon^2}C_s(\underline{\underline{\lambda}}_h;\hat{\underline{\underline{\lambda}}}_h)=0,\forall\hat{\underline{\underline{\lambda}}}_h\in W_h^s,\ \text{a.e in time},\\[2ex] B_f(\overrightarrow{v}_{fh};q_{fh})=0,\quad\forall q_{fh}\in W_h^f,\ \text{a.e in time},\\ \overrightarrow{u}_{sh}(t)=\overrightarrow{u}_{sh}(0)+\int_0^t\overrightarrow{v}_{sh}(\tau)d\tau,\\ (\overrightarrow{v}_{sh},\overrightarrow{v}_{fh})(0)\text{ is given in }V_h.\end{cases}$$

(MF_h) can be written on the compact form

$$\begin{cases}\text{For a.e.}t\in[0,T],\ \text{find } \overrightarrow{v}_h(t)=(\overrightarrow{v}_{sh},\overrightarrow{v}_{fh})(t)\in V_h,\\[2ex] \text{and } \mu_h(t)=(\underline{\underline{\lambda}}_h,p_{fh})(t)\in W_h \text{ such that}\\[2ex] m(\overrightarrow{\hat{v}}_h,\overrightarrow{\hat{v}}_h)+a(\overrightarrow{v}_h,\overrightarrow{\hat{v}}_h)+b(\overrightarrow{\hat{v}_h};\mu_h)\\[2ex] =L(\overrightarrow{\hat{v}}_h)\ \forall\ \overrightarrow{\hat{v}}_h=(\overrightarrow{\hat{v}}_{sh},\overrightarrow{\hat{v}}_{fh})\in V_h,\\[2ex] b(\overrightarrow{v}_h,\hat{\mu}_h)-\varepsilon^2c(\mu_h;\hat{\mu}_h)=0,\forall\hat{\mu}_h\in W_h,\end{cases}$$

where

$$a(\overrightarrow{v}_h,\overrightarrow{\hat{v}}_h)=A_f(\overrightarrow{v}_{fh},\overrightarrow{\hat{v}}_{fh})+A_s(\overrightarrow{u}_{sh},\overrightarrow{\hat{v}}_{sh}),$$
$$b(\overrightarrow{\hat{v}_h};\mu_h)=B_f(\overrightarrow{\hat{v}_{fh}};p_{fh})+B_s(\overrightarrow{\hat{u}}_{sh};\underline{\underline{\lambda}}_h),$$
$$c(\mu_h,\widehat{\mu}_h)=C_s(\underline{\underline{\lambda}}_h;\hat{\underline{\underline{\lambda}}}_h),\ \varepsilon^2=\frac{\varepsilon^2}{1-c_o\varepsilon^2}.$$

4 Convergence

4.1 Truncation error and global inf sup condition

The following lemma proves that the truncation error on V_h is bounded by the interpolation error on the product space $V_h^s\times H_h^f$.

Lemma 2 : *For any $\overrightarrow{\hat{v}}=(\overrightarrow{\hat{v}}_s,\overrightarrow{\hat{v}}_f)\in V$, we note $I_h\overrightarrow{\hat{v}}$ the finite element interpolate of $\overrightarrow{\hat{v}}$ on the product space $V_h^s\times H_h^f$. We have*

$$\inf_{\overrightarrow{\tilde{v}}_h \in V_h} \left\| \overrightarrow{\tilde{v}} - \overrightarrow{\tilde{v}}_h \right\| \leq C \left\| \overrightarrow{\tilde{v}} - I_h \overrightarrow{\tilde{v}} \right\|_V .$$

Proof. Let $\overrightarrow{\hat{v}} = (\overrightarrow{\hat{v}}_s, \overrightarrow{\hat{v}}_f) \in V$ and let

$$\overrightarrow{\hat{v}}_h = (I_{hs}\overrightarrow{\hat{v}}_s, I_{hf}\overrightarrow{\hat{v}}_f)$$

the finite element interpolate of $\overrightarrow{\hat{v}}$ in the product space $V_h^s \times H_h^f$. We will build $\overrightarrow{\tilde{v}}_h \in V_h$ by correcting the interface jump

$$[[\overrightarrow{\hat{v}}_h]] = \overrightarrow{\hat{v}}_{hs/\Gamma} - \overrightarrow{\hat{v}}_{hf/\Gamma}$$

of $\overrightarrow{\hat{v}}_h$ on the slave side which is the fluid domain. Since (3.15) is verified, we can construct [13] a lifting operator $R_h^f : H^{\frac{1}{2}}(\Gamma) \to H_h^f$ which verifies for any $\overrightarrow{u}$ in $H^{\frac{1}{2}}(\Gamma)$

$$\begin{aligned} \left\| R_h^f \overrightarrow{u} \right\|_{H^1} &\leq C \|\overrightarrow{u}\|_{H^{\frac{1}{2}}(\Gamma)}, \\ Tr_h(R_h^f \overrightarrow{u}) &= Tr_h(\overrightarrow{u}). \end{aligned}$$

Above, Tr_h is defined in the following weak sense

$$\int_\Gamma Tr_h(\overrightarrow{u}).\overrightarrow{g}_h = \int_\Gamma \overrightarrow{u}.\overrightarrow{g}_h; \forall \overrightarrow{g}_h \in G_h; \forall \overrightarrow{u} \in L^2.$$

We set

$$\overrightarrow{\tilde{v}}_h = (I_{hs}\overrightarrow{\hat{v}}_s, I_{hf}\overrightarrow{\hat{v}}_f + R_h^f([[\overrightarrow{\hat{v}}_h]])$$

and obtain then $Tr_h(\overrightarrow{\tilde{v}}_{fh}) = Tr_h(I_{hs}\overrightarrow{\hat{v}}_s)$ and

$$\begin{aligned} \left\| \overrightarrow{\hat{v}}_f - \overrightarrow{\tilde{v}}_{fh} \right\|_{H^1} &\leq \left\| \overrightarrow{\hat{v}}_f - I_{hf}\overrightarrow{\hat{v}}_f \right\|_{H^1} + \left\| Tr_h(I_{hs}\overrightarrow{\hat{v}}_s - I_{hf}\overrightarrow{\hat{v}}_f) \right\|_{H^{\frac{1}{2}}(\Gamma)} \\ &\leq \left\| \overrightarrow{\hat{v}}_f - I_{hf}\overrightarrow{\hat{v}}_f \right\|_{H^1} + \left\| Tr_h(I_{hs}\overrightarrow{\hat{v}}_s) - I_{hs}\overrightarrow{\hat{v}}_s) \right\|_{H^{\frac{1}{2}}(\Gamma)} \\ &+ \left\| I_{hs}\overrightarrow{\hat{v}}_s - \overrightarrow{\hat{v}}_s \right\|_{H^{\frac{1}{2}}(\Gamma)} + \left\| I_{hf}\overrightarrow{\hat{v}}_s - \overrightarrow{\hat{v}}_f \right\|_{H^{\frac{1}{2}}(\Gamma)} \end{aligned}$$

which concludes our proof.

We now set $|||\chi_h||| = |||\underline{\underline{\lambda}}_h||| + ||q_h||_{L^2}$ for $\chi_h = (\underline{\underline{\lambda}}_h, q_h) \in W_h$ and have the following theorem.

Theorem 2 *The local inf sup conditions 3.4,3.14,3.15 imply the global condition*

$$\sup_{0 \neq v \in V^h} \frac{b(\overrightarrow{v}_h; \chi_h)}{||\overrightarrow{v}_h||_V |||\chi_h|||_W} \geq \widetilde{C}. \tag{4.1}$$

Proof : Let $\chi_h = (\underline{\underline{\lambda}}_h; \underline{q}_h) \in W_h$ be given, we associate $\overrightarrow{v}_{sh} \in V_h^s$ with bounded norm such that (3.4) is verified. Since (3.15) is verified, we can introduce an extension inside Ω^f by

$$\overrightarrow{v}^1_{fh} = R_h^f \circ tr_h(\overrightarrow{v}_{sh}).$$

Since (3.14) is verified, we can construct $(\overrightarrow{v}_{fh}, p_h) \in H_h^f \times W_h^f$ the bounded solution of the Stokes problem

$$\begin{array}{l} A_f(\overrightarrow{v}_{fh}, \overrightarrow{\hat{v}}_{fh}) + B_f(\overrightarrow{\hat{v}}_{fh}, p_h) = -A_f(\overrightarrow{v}^1_{fh}, \overrightarrow{\hat{v}}_{fh}) \; \forall \overrightarrow{\hat{v}}_{fh} \in H_h^f, \\ B_f(\overrightarrow{v}_{fh}, q_h) = -B_f(\overrightarrow{v}^1_{fh}, q_h) + \int_{\Omega^f} \frac{\underline{q}_h}{\|\underline{q}_h\|_{L^2}} q_h \; \forall q_h \in W_h^f, \\ \overrightarrow{v}_{fh/\Gamma} = 0. \end{array}$$

We finally introduce the bounded field $\overrightarrow{v}_h = (\overrightarrow{v}_{sh}, \overrightarrow{v}_{fh} + \overrightarrow{v}^1_{fh})$. By construction we have

$$\overrightarrow{v}_h \in V_h \text{ and } b(\overrightarrow{v}_h; \chi_h) \geq \gamma |||\underline{\underline{\lambda}}_h||| + ||\underline{q}_h||_{L^2}$$

which concludes our proof.

4.2 Convergence

Following the steps of [1] and [2] for the shell problem with the additional complexity coming from the weak interface continuity constraint and the time dependence, we can extend the discrete stability and convergence result to our fluid structure interaction problem.

Theorem 3 : *We suppose that the solution* $(\overrightarrow{v}, \chi)$, $\overrightarrow{v} = (\overrightarrow{v}_s, \overrightarrow{v}_f)$, $\chi = (\underline{\underline{\lambda}}, p_f)$ *of the continuous problem* (MF) *verifies* $\overrightarrow{v} \in C^1(0,T,V)$. *Let* $\overrightarrow{v}_h = (\overrightarrow{v}_{sh}, \overrightarrow{v}_{fh})$, $\chi_h = (\underline{\underline{\lambda}}_h, p_{fh})$ *be the solution of the semi discrete problem* (MF_h). *The errors are bounded by*

$$\begin{array}{c} \|\overrightarrow{v}_h - \overrightarrow{v}\|_{L^\infty(0,T,L^2)} + \left\|\overrightarrow{v}_{fh} - \overrightarrow{v}_f\right\|_{L^2(0,T,V^f)} + \|\overrightarrow{u}_{sh} - \overrightarrow{u}_s\|_{L^\infty(0,T,V^s)} \\ +\varepsilon \left\|\underline{\underline{\lambda}}_h - \underline{\underline{\lambda}}\right\|_{L^2(0,T,W^s)} + |||\chi_h - \chi|||_{L^2(0,T)} \\ \leq C\{\inf_{\overrightarrow{\tilde{g}}_h \in G_h} \left\|\overrightarrow{g} - \overrightarrow{\tilde{g}}_h\right\|_{L^2(0,T,(H^{1/2}(\Gamma))')} \\ + \inf_{\overrightarrow{\tilde{v}}_h \in V_h} \left\|\overrightarrow{\dot{\tilde{v}}}_h - \overrightarrow{\dot{v}}\right\|_{L^2(0,T,L^2)} + \inf_{\overrightarrow{\tilde{v}}_h \in V_h} \left\|\overrightarrow{v} - \overrightarrow{\tilde{v}}_h\right\|_{L^2(0,T,V)} \\ + \inf_{\tilde{\chi}_h \in W^h} \|\chi - \tilde{\chi}_h\|_{L^2(0,T,W)} + \inf_{\tilde{\chi}_h \in W^h} |||\chi - \tilde{\chi}_h|||_{L^2(0,T)} \\ + \|(\overrightarrow{v}_h - \overrightarrow{v})(0)\|_0 + \left\|(\overrightarrow{v} - \overrightarrow{\tilde{v}}_h)(0)\right\|_0 \\ + \left\|(\overrightarrow{u}_s - \overrightarrow{\tilde{u}}_{sh})(0)\right\|_1 + \|(\overrightarrow{u}_s - \overrightarrow{u}_{sh})(0)\|_1\} \end{array}$$

Proof : By substracting the continuous problem (MF) written with the multiplier lagrange $\overrightarrow{g}$ in $L^2(\Gamma)$ and the discrete problem (MF_h) we get

$$\begin{array}{c} m(\overrightarrow{\dot{v}}_h - \overrightarrow{\dot{v}}, \overrightarrow{\hat{v}}_h) + a(\overrightarrow{v}_h - \overrightarrow{v}, \overrightarrow{\hat{v}}_h) + b(\chi_h - \chi, \overrightarrow{\hat{v}}_h) + \int_\Gamma (\overrightarrow{g} - \overrightarrow{\tilde{g}}_h)[[\overrightarrow{\hat{v}}_h]] d\Gamma = 0 \\ \forall \overrightarrow{\tilde{g}}_h \in G_h, \forall \overrightarrow{\hat{v}}_h \in V_h, \\ b(\hat{\chi}_h, \overrightarrow{v}_h - \overrightarrow{v}) - \varepsilon^2 c(\chi_h - \chi, \hat{\chi}_h) = 0, \forall \hat{\chi}_h = (\hat{\underline{\underline{\lambda}}}_h, \hat{p}_{fh}) \in W_h. \end{array}$$

Using as test function $\overrightarrow{\hat{v}}_h = \overrightarrow{v}_h - \overrightarrow{\tilde{v}}_h$, $\hat{\chi}_h = \chi_h - \tilde{\chi}_h$ and substracting the second line from the first line yields

$$
\begin{gathered}
m(\vec{\dot{\hat{v}}}_h, \vec{\hat{v}}_h) + A_f(\vec{\hat{v}}_{fh}, \vec{\hat{v}}_{fh}) + A_s(\vec{\hat{u}}_{sh}, \vec{\hat{v}}_{sh}) + \varepsilon^2 C_s(\hat{\underline{\underline{\lambda}}}_h, \hat{\underline{\underline{\lambda}}}_h) \\
= \int_\Gamma (\vec{g} - \vec{\tilde{g}}_h)[[\vec{\hat{v}}_h]] d\Gamma + m(\vec{\dot{v}} - \vec{\dot{\tilde{v}}}_h, \vec{\hat{v}}_h) + A_f(\vec{v}_f - \vec{\tilde{v}}_{fh}, \vec{\hat{v}}_{fh}) \\
+ A_s(\vec{u}_s - \vec{\tilde{u}}_{sh}, \vec{\hat{v}}_{sh}) + b(\chi - \tilde{\chi}_h, \vec{\hat{v}}_h) - b(\hat{\chi}_h, \vec{v} - \vec{\hat{v}}_h) \\
+ \varepsilon^2 C_s(\underline{\underline{\lambda}} - \tilde{\underline{\underline{\lambda}}}_h, \hat{\underline{\underline{\lambda}}}_h) \quad \forall \vec{\hat{v}}_h \in V_h,\ \tilde{\chi}_h = (\tilde{\underline{\underline{\lambda}}}_h, \tilde{p}_{fh}) \in W_h,\ \vec{\tilde{g}} \in G_h.
\end{gathered}
$$

Using the continuity of a, b, c and the definition of the semi norm $|||.|||$, we get by Cauchy Schwartz inequality

$$
\begin{gathered}
m(\vec{\dot{\hat{v}}}_h, \vec{\hat{v}}_h) + A_f(\vec{\hat{v}}_{fh}, \vec{\hat{v}}_{fh}) + A_s(\vec{\hat{u}}_{sh}, \vec{\hat{v}}_{sh}) + \varepsilon^2 C_s(\hat{\underline{\underline{\lambda}}}_h, \hat{\underline{\underline{\lambda}}}_h) \\
\leq C \left\| \vec{\hat{v}}_h \right\|_V \{ \left\| \vec{g} - \vec{\tilde{g}}_h \right\|_{(H^{1/2}(\Gamma))'} + \left\| \vec{\dot{v}} - \vec{\dot{\tilde{v}}}_h \right\|_0 + \left\| \vec{v}_f - \vec{\tilde{v}}_{fh} \right\|_{V^f} \\
+ \left\| \vec{u}_s - \vec{\tilde{u}}_{sh} \right\|_{V^s} + \|\chi - \tilde{\chi}_h\|_W \} \\
+ |||\hat{\chi}_h|||(\left\| \vec{v}_f - \vec{\tilde{v}}_{fh} \right\|_{V^f} + \left\| \vec{u}_s - \vec{\tilde{u}}_{sh} \right\|_{V^s}) + \varepsilon^2 \left\| \underline{\underline{\lambda}} - \tilde{\underline{\underline{\lambda}}}_h \right\|_{W^s} \left\| \hat{\underline{\underline{\lambda}}}_h \right\|_{W^s}.
\end{gathered}
$$

By integrating in time from 0 to t we observe that we have

$$
\frac{1}{2} m(\vec{\hat{v}}_h, \vec{\hat{v}}_h) + \frac{1}{2} A_s(\vec{\hat{u}}_{sh}, \vec{\hat{u}}_{sh})(t) + \int_0^t A_f(\vec{\hat{v}}_{fh}, \vec{\hat{v}}_{fh})(s) ds \tag{4.2}
$$

$$
\begin{aligned}
&+ \varepsilon^2 \int_0^t C_s(\hat{\underline{\underline{\lambda}}}_h, \hat{\underline{\underline{\lambda}}}_h) ds \leq C \left\| \vec{\hat{v}}_h \right\|_{L^2(0,T,V)} \{ \left\| \vec{g} - \vec{\tilde{g}}_h \right\|_{L^2(0,T,(H^{1/2}(\Gamma))')} \\
&+ \left\| \vec{\dot{v}} - \vec{\dot{\tilde{v}}}_h \right\|_{L^2(0,T,L^2)} + \left\| \vec{v}_f - \vec{\tilde{v}}_{fh} \right\|_{L^2(0,T,V^f)} + \left\| \vec{u}_s - \vec{\tilde{u}}_{sh} \right\|_{L^2(0,T,V^s)} \\
&+ \|\chi - \tilde{\chi}_h\|_{L^2(0,T,W)} \} + |||\hat{\chi}_h|||_{L^2(0,T)} (\left\| \vec{v}_f - \vec{\tilde{v}}_{fh} \right\|_{L^2(0,T,V)} \\
&+ \left\| \vec{u}_s - \vec{\tilde{u}}_{sh} \right\|_{L^2(0,T,V^s)}) + \varepsilon^2 \left\| \underline{\underline{\lambda}} - \tilde{\underline{\underline{\lambda}}}_h \right\|_{L^2(0,T,W^s)} \left\| \hat{\underline{\underline{\lambda}}}_h \right\|_{L^2(0,T,W^s)} \\
&+ \frac{1}{2} m(\vec{\hat{v}}_h, \vec{\hat{v}}_h)(0) + \frac{1}{2} A_s(\vec{\hat{u}}_{sh}, \vec{\hat{u}}_{sh})(0).
\end{aligned}
$$

From the global inf sup condition (4.1) we have

$$
\begin{aligned}
|||\hat{\chi}_h||| &\leq |||\chi - \tilde{\chi}_h||| + |||\chi - \chi_h||| \leq |||\chi - \tilde{\chi}_h||| + C \sup_{0 \neq \vec{\hat{v}}_h \in V^h} \frac{b(\vec{\hat{v}}_h; \chi - \chi_h)}{||\vec{\hat{v}}_h||_V} \\
&\leq |||\chi - \tilde{\chi}_h||| + C \sup_{0 \neq \vec{\hat{v}}_h \in V^h} \frac{m(\vec{\dot{v}}_h - \vec{\dot{v}}, \vec{\hat{v}}_h) + a(\vec{v}_h - \vec{v}; \vec{\hat{v}}_h) - \int_\Gamma (\vec{g} - \vec{\tilde{g}}_h)[[\vec{\hat{v}}_h]] d\Gamma}{||\vec{\hat{v}}_h||_V} \\
&\leq |||\chi - \tilde{\chi}_h||| + C \{ \left\| \vec{v}_f - \vec{v}_{fh} \right\|_{V^f} + \|\vec{u}_s - \vec{u}_{sh}\|_{V^s} + \left\| \vec{g} - \vec{\tilde{g}}_h \right\|_{(H^{1/2}(\Gamma))'} \} \\
&\quad + C \sup_{0 \neq \vec{\hat{v}}_h \in V^h} \frac{m(\vec{\dot{v}}_h - \vec{\dot{\tilde{v}}}_h, \vec{v}_h - \vec{\tilde{v}}_h) + m(\vec{\dot{\tilde{v}}}_h - \vec{\dot{v}}, \vec{v}_h - \vec{\tilde{v}}_h)}{||\vec{v}_h - \vec{\tilde{v}}_h||_V}.
\end{aligned}
$$

We deduce after intergrating that

$$
\begin{aligned}
|||\hat{\chi}_h|||_{L^2(0,T)} \leq\ & |||\chi - \tilde{\chi}_h|||_{L^2(0,T)} + C \{ \left\| \vec{v}_f - \vec{v}_{fh} \right\|_{L^2(0,T,V^f)} \\
& + \|\vec{u}_s - \vec{u}_{sh}\|_{L^2(0,T,V^s)} + \left\| \vec{g} - \vec{\tilde{g}}_h \right\|_{L^2(0,T,(H^{1/2}(\Gamma))')} \\
& + \left\| \vec{\dot{\tilde{v}}}_h - \vec{\dot{v}} \right\|_{L^2(0,T,L^2)} + \left\| \vec{\hat{v}}_h \right\|_{L^2(0,T,L^2)} + \left\| \vec{\hat{v}}_h(0) \right\|_0 .
\end{aligned} \tag{4.3}
$$

Combining the estimates (4.2-4.3), and since A_s, A_f and C_s are coercive, we get for each $\overrightarrow{\tilde{v}}_h \in V^h$, $\tilde{\chi}_h \in W^h$ $\overrightarrow{\tilde{g}} \in G_h$

$$\begin{aligned}
&\left\|\overrightarrow{\tilde{v}}_h\right\|^2_{L^\infty(0,T,L^2)} + \left\|\overrightarrow{\tilde{v}}_{fh}\right\|^2_{L^2(0,T,V^f)} + \left\|\overrightarrow{\widehat{u}}_{sh}\right\|^2_{L^\infty(0,T,V^s)} + \varepsilon^2 \left\|\underline{\underline{\hat{\lambda}}}_h\right\|^2_{L^2(0,T,W^s)} \\
&\leq C \left\|\overrightarrow{\tilde{v}}_h\right\|_{L^2(0,T,V)} \{\left\|\overrightarrow{g} - \overrightarrow{\tilde{g}}_h\right\|_{L^2(0,T,(H^{1/2}(\Gamma))')} + \left\|\dot{\overrightarrow{\tilde{v}}}_h - \dot{\overrightarrow{v}}\right\|_{L^2(0,T,L^2)} \\
&+ \left\|\overrightarrow{v}_f - \overrightarrow{\tilde{v}}_{fh}\right\|_{L^2(0,T,V^f)} + \left\|\overrightarrow{u_s} - \overrightarrow{\tilde{u}}_{sh}\right\|_{L^2(0,T,V^s)} + \|\chi - \tilde{\chi}_h\|_{L^2(0,T,W)}\} \\
&+ \varepsilon^2 \left\|\underline{\underline{\lambda}} - \underline{\underline{\tilde{\lambda}}}_h\right\|_{L^2(0,T,W^s)} \left\|\underline{\underline{\hat{\lambda}}}_h\right\|_{L^2(0,T)} + \left\|\overrightarrow{v} - \overrightarrow{\tilde{v}}_h\right\|_{L^2(0,T,V)} \{|||\chi - \tilde{\chi}_h|||_{L^2(0,T)} \\
&+ \|\overrightarrow{v_f} - \overrightarrow{v}_{fh}\|_{L^2(0,T,V^f)} + \|\overrightarrow{u_s} - \overrightarrow{u}_{sh}\|_{L^2(0,T,V^s)} + \left\|\dot{\overrightarrow{v}} - \dot{\overrightarrow{\tilde{v}}}_h\right\|_{L^2(0,T,L^2)} \\
&+ \left\|\overrightarrow{\tilde{v}}_h\right\|_{L^2(0,T,V)} + \left\|\overrightarrow{g} - \overrightarrow{\tilde{g}}_h\right\|_{L^2(0,T,(H^{1/2}(\Gamma))')} + \left\|\overrightarrow{\tilde{v}}_h(0)\right\|_0\} \\
&+ \frac{1}{2} m(\overrightarrow{\tilde{v}}_h, \overrightarrow{\tilde{v}}_h)(0) + \frac{1}{2} A_s(\overrightarrow{\widehat{u}}_{sh}, \overrightarrow{\widehat{u}}_{sh})(0).
\end{aligned}$$

Writing this inequality as $x^2 - 2Kx - K^2 \leq 0$ with

$$x = \left\|\overrightarrow{\tilde{v}}_{fh}\right\|_{L^2(0,T,V^f)} + \left\|\overrightarrow{\widehat{u}}_{sh}\right\|_{L^\infty(0,T,V^s)} + \varepsilon^2 \left\|\underline{\underline{\hat{\lambda}}}_h\right\|_{L^2(0,T,W^s)}$$

implies that x is bounded by $(1 + \sqrt{2}K)$, which gives

$$\begin{aligned}
&\left\|\overrightarrow{\tilde{v}}_{fh}\right\|_{L^2(0,T,V^f)} + \left\|\overrightarrow{\widehat{u}}_{sh}\right\|_{L^\infty(0,T,V^s)} + \varepsilon^2 \left\|\underline{\underline{\hat{\lambda}}}_h\right\|_{L^2(0,T,W^s)} \\
&\leq C\{\left\|\overrightarrow{g} - \overrightarrow{\tilde{g}}_h\right\|_{L^2(0,T,(H^{1/2}(\Gamma))')} + \left\|\dot{\overrightarrow{\tilde{v}}}_h - \dot{\overrightarrow{v}}\right\|_{L^2(0,T,L^2)} \\
&+ \left\|\overrightarrow{v}_f - \overrightarrow{\tilde{v}}_{fh}\right\|_{L^2(0,T,V^f)} + \left\|\overrightarrow{u_s} - \overrightarrow{\tilde{u}}_{sh}\right\|_{L^2(0,T,V^s)} + \|\chi - \tilde{\chi}_h\|_{L^2(0,T,W)} \\
&+ |||\chi - \tilde{\chi}_h|||_{L^2(0,T)} + \left\|\underline{\underline{\lambda}} - \underline{\underline{\tilde{\lambda}}}_h\right\|_{L^2(0,T,W^s)} + \left\|\overrightarrow{\tilde{v}}_h(0)\right\|_0 + \left\|\overrightarrow{\tilde{u}}_{sh}(0)\right\|_1\}.
\end{aligned}$$

Taking into account the estimate (4.3), we finally get

$$\begin{aligned}
&\left\|\overrightarrow{\tilde{v}}_{fh}\right\|_{L^2(0,T,V^f)} + \left\|\overrightarrow{\widehat{u}}_{sh}\right\|_{L^\infty(0,T,V^s)} + \varepsilon^2 \left\|\underline{\underline{\hat{\lambda}}}_h\right\|_{L^2(0,T,W^s)} + |||\hat{\chi}_h|||_{L^2(0,T)} \\
&\leq C\{\left\|\overrightarrow{g} - \overrightarrow{\tilde{g}}_h\right\|_{L^2(0,T,(H^{1/2}(\Gamma))')} + \left\|\dot{\overrightarrow{\tilde{v}}}_h - \dot{\overrightarrow{v}}\right\|_{L^2(0,T,L^2)} \\
&+ \left\|\overrightarrow{v}_f - \overrightarrow{\tilde{v}}_{fh}\right\|_{L^2(0,T,V^f)} + \left\|\overrightarrow{u_s} - \overrightarrow{\tilde{u}}_{sh}\right\|_{L^2(0,T,V^s)} \\
&+ \|\chi - \tilde{\chi}_h\|_{L^2(0,T,W)} + |||\chi - \tilde{\chi}_h|||_{L^2(0,T)} + \left\|\underline{\underline{\lambda}} - \underline{\underline{\tilde{\lambda}}}_h\right\|_{L^2(0,T,W^s)} \\
&+ \left\|\overrightarrow{\tilde{v}}_h(0)\right\|_0 + \left\|\overrightarrow{\tilde{u}}_{sh}(0)\right\|_1\}.
\end{aligned}$$

from which the theorem follows.

Remark. Because of the lemma2 and the time regularity required on the continuous solution, we have

$$\inf_{\overrightarrow{v}_h \in V_h} \left\| \overrightarrow{v} - \overrightarrow{v}_h \right\|_V = \| \overrightarrow{v} - I_h \overrightarrow{v} \|_V$$

and

$$\inf_{\overrightarrow{v}_h \in V_h} \left\| \overrightarrow{\dot{v}}_h - \overrightarrow{\dot{v}} \right\|_0 = \left\| \overrightarrow{\dot{v}} - I_h \overrightarrow{\dot{v}} \right\|_0 .$$

Theorem3 shows then well a uniform convergence of our schemes with respect to the shell thickness. In view of the theorem1, the choice of the space approximation in the fluid part (3.13), the interface approximation hypothesis (3.16) and the lemma2, this convergence is of order $h_s^2 + h_f + h^q$.

5 Conclusion

We have shown herein how to efficiently couple discretization schemes while preserving the accuracy and the stability of the original elements. In other words, by combining locking free finite element approximation of the Koiter shell, the incompressibility of the Stokes fluid and the weak mortar coupling yield a locking free approximation of the coupled problem. Nevertheless, the problem studied is just an idealization of practical situations involving elastic shells interacting with viscous fluids. The system studied is linear and all changes of configurations are neglected which guarantees the ellipticity of the associated operators. On the other hand, the theory presented on the shell side is restricted to the bending dominated case and is subjected to the same geometrical restrictions of [1].

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