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Smooth Dependence by LAG of the Solution of a Delay Integro-Differential Equation from Biomathematics

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Abstract

Using the Perov's fixed point theorem and the theorem of fiber generalized contractions, we obtain the smooth dependence by lag of the positive periodic solution of a neutral delay integro-differential equation which arise in epidemiology. The smooth dependence is obtained also for the derivative of the solution.

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1. Introduction

The use of the Perov's fixed point theorem (see [9], [1], [6] and [13]) generates an efficient technique to approach systems of operatorial equations (see [15] and [13]) and operatorial (differential and integro-differential) equations of neutral type (see [1], [2], [3] and [4]). In the study of the smooth dependence by parameters of the solution of an operatorial equation is very useful the notion of Picard and weakly Picard operator (see [10] and [14]) and the theorem of fiber generalized contractions (see [11], [12] and [13]). Applications of the technique of Picard and weakly Picard operators can be viewed in [6] and [16]. Some applications of the fiber generalized contractions can be found in [11], [13], [8], [3], and [4]. In [2] we have obtained the existence and uniqueness of the positive periodic solution of equation (1.1) and in [3] we consider

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the initial value problem corresponding to this equation and obtain the conditions for existence, uniqueness and smooth dependence by parameter of the positive and smooth solution of this initial value problem. In [3] we obtain the Lipschitz property of the derivative of the solution and in the same conditions we construct a numerical method to approximate the solution and his derivative. In [4] we apply the technique generated by the Perov's fixed point theorem to the neutral delay differential equation,

$$x'(t) = f(t, x(t), x'(t-\tau)), \quad t \in [0, T].$$

Here, we continue the work from [2], in the study of the following neutral delay integro-differential equation:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s), x'(s))ds, \tag{1.1}$$

which is a model for the spread of certain infectious diseases with a contact rate that varies seasonally, generalizing the model governed by the delay integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds. \tag{1.2}$$

The equation (1.2) was studied in [5], [7] and [17]. In the equations (1.1) and (1.2) $x(t)$ is the proportion of infectives in the population at time moment t and $\tau > 0$ is the averaged length of time in which an individual remains infectious. In equation (1.1), $x'(t)$ is the speed of infection spreading at moment t. In this paper we will study the dependence by the lag τ, of the periodic positive solution of equation (1.1). We will use the following notions and results:

Definition 1.1 ([10], [13] and [14]): Let (X, d) be a metric space. An operator $A : X \to X$ is Picard operator if there exists $x^* \in X$ such that:
(a) x^* is the unique fixed point of A,
(b) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$, where $A^0 = Id(X)$, and $A^{n+1} = A \circ A^n, \ \forall n \in \mathbb{N}$.

Definition 1.2 ([10], [13] and [14]): Let (X, d) be a metric space. An operator $A : X \to X$ is weakly Picard operator if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.

Theorem 1.3 (A. I. Perov, [9], [1]): Let (X, d) a complete generalized metric space such that $d(x, y) \in \mathbb{R}^n$. Suppose that $A : X \to X$ is a map for which exists a matrix $Q \in \mathcal{M}_n(\mathbb{R})$ such that:

$$d(A(x), A(y)) \leq Qd(x, y), \forall x, y \in X.$$

If all the eigenvalues of Q lies in the open unit disc of $\mathbb{R}^2$, then A is a Q-contraction (that is $\lim_{m\to\infty} Q^m = 0$), has an unique fixed point x^* and the sequence of successive approximations, $x_m = A^m(x_0)$, converges to x^* for any $x_0 \in X$. Moreover, for any $m \in \mathbb{N}^*$ the following estimation holds

$$d(x_m, x^*) \le Q^m (I_n - Q)^{-1} d(x_0, x_1).$$

Theorem 1.4 (I. A. Rus, [11], [12] and [13]): Let (X, d) be a metric space (generalized or not) and (Y, ρ) be a complete generalized metric space ($\rho(x, y) \in \mathbb{R}^n_+$). Let $A : X \times Y \to X \times Y$ be a continuous operator and $C : X \times Y \to Y$ an operator. Suppose that:

(i) the operator $B : X \to X$ has an unique fixed point x^* and for any $x_0 \in X$ the sequence given by $x_{n+1} = B(x_n)$ converges in X to x^*

(ii) $A(x, y) = (B(x), C(x, y))$, for all $x \in X$, $y \in Y$

(iii) there exists a matrix $Q \in M_n(\mathbb{R}_+)$, with $Q^m \to 0$ as $m \to \infty$, such that

$$\rho(C(x, y_1), C(x, y_2)) \le Q \cdot \rho(y_1, y_2),$$

for all $x \in X$, and y_1 , $y_2 \in Y$.

Then, the operator A has an unique fixed point (x^*, y^*) and for any $(x_0, y_0) \in X \times Y$ the sequence given by $(x_{n+1}, y_{n+1}) = A((x_n, y_n))$ converge to (x^*, y^*) in $X \times Y$.

2. Main Result

Consider the integro-differential equation,

$$x(t,\tau) = \int_{t-\tau}^{t} f(s, x(s,\tau), x'_s(s,\tau), \tau) ds, \qquad t \in \mathbb{R}, \quad \tau \in [a, b] \tag{2.1}$$

where $a > 0$.

We will consider the conditions:

(i) (continuity): $f \in C(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times [a, b])$ and $f(\cdot, u, v, \cdot) \in C^1(\mathbb{R} \times [a, b])$, $\forall (u, v) \in \mathbb{R}_+ \times \mathbb{R}$, .

(ii) (boundedness): exists $m, M \ge 0$ such that

$$m \le f(t, u, v, \tau) \le M, \ \forall (t, u, v, \tau) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times [a, b]$$

(iii) (Lipschitz property) : exists $\alpha, \beta > 0$ such that

$$|f(t, u_1, v_1, \tau) - f(t, u_2, v_2, \tau)| \le \alpha |u_1 - u_2| + \beta |v_1 - v_2|,$$

$\forall t \in \mathbb{R}, \forall u_1, u_2 \in \mathbb{R}_+, \forall v_1, v_2 \in \mathbb{R}, \forall \tau \in [a, b]$

(iv) (periodicity): $\exists \varpi > 0$ such that

$$f(t+\varpi, u, v, \tau) = f(t, u, v, \tau),\ \forall (t, u, v, \tau) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times [a, b]$$

(v) (smoothness): $f \in C^1(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times [a, b])$, and

$$\frac{\partial f}{\partial t}(t+\varpi, u, v, \tau) = \frac{\partial f}{\partial t}(t, u, v, \tau)$$

$$\frac{\partial f}{\partial \tau}(t+\varpi, u, v, \tau) = \frac{\partial f}{\partial \tau}(t, u, v, \tau)$$

$$\frac{\partial f}{\partial x}(t+\varpi, u, v, \tau) = \frac{\partial f}{\partial x}(t, u, v, \tau)$$

$$\frac{\partial f}{\partial y}(t+\varpi, u, v, \tau) = \frac{\partial f}{\partial y}(t, u, v, \tau),\ \forall (t, u, v, \tau) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times [a, b].$$

If we derive the equation (2.1) by t and denoting $y(t, \tau) = x'_t(t, \tau)$, we obtain,

$$y(t, \tau) = f(t, x(t, \tau), y(t, \tau), \tau) - f(t-\tau, x(t-\tau, \tau), y(t-\tau, \tau), \tau). \tag{2.2}$$

Consider the following spaces,

$$X(\varpi) = \{x \in C(\mathbb{R} \times [a, b]) \mid x(t+\varpi, \tau) = x(t, \tau),\ \forall t \in \mathbb{R},\ \forall \tau \in [a, b]\}$$

$$X_+(\varpi) = \{x \in X(\varpi) \mid\ x(t, \tau) \geq 0,\ \forall t \in \mathbb{R},\ \forall \tau \in [a, b]\}$$

and denote $X = X_+(\varpi) \times X(\varpi),\ Y = X(\varpi) \times X(\varpi)$. On Y define the generalized metric, $\rho : Y \times Y \longrightarrow \mathbb{R}^2$

$$\rho((x_1, y_1), (x_2, y_2)) =$$

$$\left(\max_{t \in [0, \varpi], \tau \in [a, b]} |x_1(t, \tau) - x_2(t, \tau)|,\ \max_{t \in [0, \varpi], \tau \in [a, b]} |y_1(t, \tau) - y_2(t, \tau)| \right)$$

and consider $d : X \times X \longrightarrow \mathbb{R}^2,\ d = \rho\mid_{X \times X}$.

It is clear that (Y, ρ) and (X, d) are complete generalized metric spaces.

Since $C^1(\mathbb{R} \times [a, b])$ is not complete with the metric of uniform convergence we will study the equation (2.1) as in [2] and [4], considering the systems

$$\begin{pmatrix} x(t, \tau) \\ y(t, \tau) \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{t-\tau}^{t} f(s, x(s, \tau), y(s, \tau), \tau)\, ds \\ f(t, x(t, \tau), y(t, \tau), \tau) - f(t-\tau, x(t-\tau, \tau), y(t-\tau, \tau), \tau), \\ t \in \mathbb{R},\ \tau \in [a, b], \end{pmatrix} \tag{2.3}$$

and

$$\begin{pmatrix} u(t,\tau) \\ v(t,\tau) \end{pmatrix}$$

$$= \begin{pmatrix} \int\limits_{t-\tau}^{t} \left[\frac{\partial f}{\partial \tau}(s,x(s,\tau),y(s,\tau),\tau) + \frac{\partial f}{\partial x}(s,x(s,\tau),y(s,\tau),\tau)\cdot u(s,\tau) \right. \\ \left. +\frac{\partial f}{\partial y}(s,x(s,\tau),y(s,\tau),\tau)\cdot v(s,\tau) \right] ds, \\ \frac{\partial f}{\partial \tau}(t,x(t,\tau),y(t,\tau),\tau) + \frac{\partial f}{\partial x}(t,x(t,\tau),y(t,\tau),\tau)\cdot u(t,\tau) \\ +\frac{\partial f}{\partial y}(t,x(t,\tau),y(t,\tau),\tau)\cdot v(t,\tau) + \frac{\partial f}{\partial t}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau) \\ -\frac{\partial f}{\partial \tau}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau) - \frac{\partial f}{\partial x}(t-\tau,x(t-\tau,\tau), \\ y(t-\tau,\tau),\tau)\cdot[-y(t-\tau,\tau)+u(t-\tau,\tau)] \\ -\frac{\partial f}{\partial y}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau)\cdot \\ \cdot\left[-\frac{\partial}{\partial t}y(t-\tau,\tau)+(v(t-\tau,\tau))\right], \quad t\in\mathbb{R},\ \tau\in[a,b]. \end{pmatrix} \tag{2.4}$$

where we have denoted

$$u(t,\tau) = \frac{d}{d\tau}x(t,\tau), \qquad v(t,\tau) = \frac{d}{d\tau}y(t,\tau)$$

according to the condition (i). Define the operators, $B : X \longrightarrow X,\ C : X \times Y \longrightarrow Y,$ $A : X \times Y \longrightarrow X \times Y$ by,

$$B(x,y)(t,\tau) = (B_1(x,y)(t,\tau), B_2(x,y)(t,\tau))$$
$$= \begin{pmatrix} \int\limits_{t-\tau}^{t} f(s,x(s,\tau),y(s,\tau),\tau)\,ds \\ f(t,x(t,\tau),y(t,\tau),\tau) - f(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau), \\ ,\ t\in\mathbb{R},\ \tau\in[a,b], \end{pmatrix} \tag{2.5}$$

$$C((x,y),(u,v))(t,\tau) = (C_1((x,y),(u,v))(t,\tau), C_2((x,y),(u,v))(t,\tau))$$

$$= \begin{pmatrix} \displaystyle\int_{t-\tau}^{t} \left[\frac{\partial f}{\partial \tau}(s,x(s,\tau),y(s,\tau),\tau) + \frac{\partial f}{\partial x}(s,x(s,\tau),y(s,\tau),\tau)\cdot u(s,\tau) \right. \\ \left. + \frac{\partial f}{\partial y}(s,x(s,\tau),y(s,\tau),\tau)\cdot v(s,\tau)\right] ds, \\ \frac{\partial f}{\partial \tau}(t,x(t,\tau),y(t,\tau),\tau) + \frac{\partial f}{\partial x}(t,x(t,\tau),y(t,\tau),\tau)\cdot u(t,\tau) \\ + \frac{\partial f}{\partial y}(t,x(t,\tau),y(t,\tau),\tau)\cdot v(t,\tau) + \frac{\partial f}{\partial t}(t-\tau,x(t-\tau,\tau), \\ y(t-\tau,\tau),\tau) - \frac{\partial f}{\partial \tau}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau) \\ -\frac{\partial f}{\partial x}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau) \\ \cdot[-y(t-\tau,\tau)+u(t-\tau,\tau)] - \frac{\partial f}{\partial y}(t-\tau,x(t-\tau,\tau),y(t-\tau,\tau),\tau)\cdot \\ \cdot\left[-\frac{\partial}{\partial t}y(t-\tau,\tau) + (v(t-\tau,\tau))\right], \quad t\in\mathbb{R},\ \tau\in[a,b]. \end{pmatrix} \tag{2.6}$$

$$A((x,y),(u,v)) = (B(x,y), C((x,y),(u,v))),\ \forall x,y\in X, \forall u,v\in Y. \tag{2.7}$$

In the relation (2.6), the existence of $\frac{\partial}{\partial t}y(t-\tau,\tau)$ is ensured by the condition (i).

Remark: We can see that any $z\in X(\varpi)$ is bounded on $\mathbb{R}$, because of the continuity and periodicity of the elements of $X(\varpi)$. Therefore, exists $L_1\geq 0$ and $L_2\geq 0$ such that,

$$\left|\frac{\partial f}{\partial x}(s,u(t,\tau),v(t,\tau),\tau)\right| \leq L_1, \quad \forall t\in\mathbb{R}, \forall u\in X_+(\varpi), v\in X(\varpi), \forall\tau\in[a,b]$$

and

$$\left|\frac{\partial f}{\partial y}(s,u(t,\tau),v(t,\tau),\tau)\right| \leq L_2, \quad \forall t\in\mathbb{R}, \forall u\in X_+(\varpi), v\in X(\varpi), \forall\tau\in[a,b].$$

Theorem 2.1: a) In the conditions (i)-(iv) if $\alpha b + 2\beta < 1$ then the equation (2.3) has in X an unique solution (x^*, y^*). Moreover, for any $(x_0,y_0)\in X$, the sequence $((x_n,y_n))_{n\in\mathbb{N}}$ defined by

$$x_{n+1}(t,\tau) = \int_{t-\tau}^{t} f(s,x_n(s,\tau),y_n(s,\tau),\tau)\,ds$$

$$y_{n+1}(t,\tau) = f(t, x_n(t,\tau), y_n(t,\tau), \tau) - f(t-\tau, x_n(t-\tau,\tau), y_n(t-\tau,\tau), \tau)$$

uniformly converges on $\mathbb{R} \times [a,b]$ to (x^*, y^*) and $x^* \in C^1(\mathbb{R} \times [a,b])$,

$$y^*(t,\tau) = \frac{d}{dt} x^*(t,\tau), \; \forall t \in \mathbb{R}, \; \forall \tau \in [a,b].$$

b) In the conditions (i)-(v), if $bL_1 + 2L_2 < 1$ then $y^*(t,\cdot) \in C^1[a,b]$ and the pair $\left(\frac{\partial x^*}{\partial \lambda}, \frac{\partial y^*}{\partial \lambda}\right) \in Y$ is the unique solution of the equation (2.4).

Proof: a) The condition (ii) imply that

$$B_1(x,y)(t,\tau) \geq 0, \quad \forall (t,\tau) \in \mathbb{R} \times [a,b]$$
$$B_1(x,y)(t,\tau) \leq M\tau \leq Mb, \quad \forall (t,\tau) \in \mathbb{R} \times [a,b].$$

Using the transformation of variable $u = s + \varpi$, we have,

$$B_1(x,y)(t+\varpi,\tau) = \int_{t+\varpi-\tau}^{t+\varpi} f\left(s, x\left(s,\tau\right), y\left(s,\tau\right), \tau\right) ds$$

$$= \int_{t-\tau}^{t} f\left(u-\varpi, x\left(u-\varpi,\tau\right), y\left(u-\varpi,\tau\right), \tau\right) du =$$

$$= \int_{t-\tau}^{t} f\left(u-\varpi+\varpi, x\left(u-\varpi+\varpi,\tau\right), y\left(u-\varpi+\varpi,\tau\right), \tau\right) du$$

$$= B_1(x,y)(t,\tau), \quad \forall (t,\tau) \in \mathbb{R} \times [a,b], \forall (x,y) \in X,$$

Analogous,

$$B_2(x,y)(t+\varpi,\tau) = B_2(x,y)(t,\tau), \; \forall (t,\tau) \in \mathbb{R} \times [a,b], \forall (x,y) \in X.$$

Then, $B(X) \subset X$.
Let $(x_1,y_1),(x_2,y_2) \in X$. It is easy to prove the inequalities:

$$|B_1(x_1,y_1)(t,\tau) - B_1(x_2,y_2)(t,\tau)| \leq \alpha\tau \|x_1 - x_2\| + \beta\tau\|y_1 - y_2\|$$

$$\leq \alpha b\|x_1 - x_2\| + \beta b\|y_1 - y_2\|, \quad \forall (t,\tau) \in \mathbb{R} \times [a,b]$$

and

$$|B_2(x_1,y_1)(t,\tau) - B_2(x_2,y_2)(t,\tau)| \leq$$

$$\leq 2\alpha\|x_1 - x_2\| + 2\beta\|y_1 - y_2\|, \quad \forall (t,\tau) \in \mathbb{R} \times [a,b].$$

Then,

$$d(B(x_1,y_1),B(x_2,y_2)) \leq \begin{pmatrix} \alpha b & \beta b \\ 2\alpha & 2\beta \end{pmatrix} d((x_1,y_1),(x_2,y_2)),$$

$\forall (x_1,y_1),(x_2,y_2) \in X$. The eigenvalues of the matrix

$$Q = \begin{pmatrix} \alpha b & \beta b \\ 2\alpha & 2\beta \end{pmatrix}$$

are $\lambda_1 = 0$ and $\lambda_2 = 2\beta + \alpha b$. The condition $\alpha b + 2\beta < 1$ imply that $Q^m \to 0$ as $m \to \infty$ and according to the Perov's fixed point theorem, the operator B has in X an unique fixed point (x^*, y^*) and the sequence $((x_n, y_n))_{n\in\mathbb{N}}$ uniformly converges to (x^*, y^*) on $\mathbb{R} \times [a,b]$. Consequently,

$$x^*(t,\tau) = \int_{t-\tau}^{t} f\left(s, x^*\left(s,\tau\right), y^*\left(s,\tau\right), \tau\right) ds$$

and if we derive this equality with respect by t, we obtain that $x^* \in C^1(\mathbb{R} \times [a,b])$ and

$$y^*(t,\tau) = \frac{d}{dt} x^*(t,\tau),\ \forall t \in \mathbb{R},\ \forall \tau \in [a,b].$$

Moreover, since, $Q^n = \lambda_2^{n-1} \cdot Q$, $\forall n \in \mathbb{N}^*$ and

$$(I-Q)^{-1} = \frac{1}{1-\lambda_2} \begin{pmatrix} 1-2\beta & \beta b \\ 2\alpha & 1-\alpha b \end{pmatrix}$$

applying the Perov's theorem we obtain for $m \in \mathbb{N}^*$, the estimation:

$$d\left((x_m,y_m),(x^*,y^*)\right) \leq \frac{\lambda_2^{m-1}}{1-\lambda_2} \begin{pmatrix} \alpha b & \beta b \\ 2\alpha & 2\beta \end{pmatrix} d\left((x_1,y_1),(x_0,y_0)\right).$$

b) The condition (v) imply that $x_m, y_m \in C^1(\mathbb{R} \times [a,b])$, $\forall m \in \mathbb{N}^*$. For $(x,y) \in X$ arbitrary we consider $C((x,y),\cdot) : Y \longrightarrow Y$. After elementary calculus, using the condition (iv) and (v), we obtain,

$$C_1((x,y),(u,v))(t+\varpi,\tau) = C_1((x,y),(u,v))(t,\tau), \quad \forall (t,\tau) \in \mathbb{R} \times [a,b]$$

and

$$C_2((x,y),(u,v))(t+\varpi,\tau) = C_2((x,y),(u,v))(t,\tau), \quad \forall (t,\tau) \in \mathbb{R} \times [a,b].$$

Consequently, $C((x,y),Y) \subset Y, \forall (x,y) \in X$. For any $(u_1,v_1),(u_2,v_2) \in Y$ we have

$$\left| \int_{t-\tau}^{t} \left[\frac{\partial f}{\partial \tau}(s,x(s,\tau),y(s,\tau),\tau) + \frac{\partial f}{\partial x}(s,x(s,\tau),y(s,\tau),\tau)\cdot u_1(s,\tau) \right.\right.$$
$$\left. + \frac{\partial f}{\partial y}(s,x(s,\tau),y(s,\tau),\tau)\cdot v_1(s,\tau) \right] ds - \int_{t-\tau}^{t} \left[\frac{\partial f}{\partial \tau}(s,x(s,\tau),y(s,\tau),\tau) \right.$$
$$+ \frac{\partial f}{\partial x}(s,x(s,\tau),y(s,\tau),\tau)\cdot u_2(s,\tau)$$
$$\left.\left. + \frac{\partial f}{\partial y}(s,x(s,\tau),y(s,\tau),\tau)\cdot v_2(s,\tau)\right] ds \right|$$
$$\le \int_{t-\tau}^{t} \left| \frac{\partial f}{\partial x}(s,x(s,\tau),y(s,\tau),\tau)\right| \cdot |u_1(s,\tau) - u_2(s,\tau)|\, ds$$
$$+ \int_{t-\tau}^{t} \left| \frac{\partial f}{\partial y}(s,x(s,\tau),y(s,\tau),\tau)\right| \cdot |v_1(s,\tau) - v_2(s,\tau)|\, ds$$
$$\le L_1\tau \|u_1 - u_2\| + L_2\tau \|v_1 - v_2\|$$
$$\le b\cdot [L_1 \|u_1 - u_2\| + L_2 \|v_1 - v_2\|], \quad \forall (t,\tau) \in \mathbb{R}\times[a,b].$$

and then,

$$|C_1((x,y),(u_1,v_1))(t,\tau) - C_1((x,y),(u_2,v_2))(t,\tau)|$$
$$\le b\cdot [L_1 \|u_1 - u_2\| + L_2 \|v_1 - v_2\|], \quad \forall (t,\tau) \in \mathbb{R}\times[a,b].$$

Analogous, we obtain,

$$|C_2((x,y),(u_1,v_1))(t,\tau) - C_2((x,y),(u_2,v_2))(t,\tau)|$$
$$\le 2L_1 \|u_1 - u_2\| + 2L_2 \|v_1 - v_2\|, \quad \forall (t,\tau) \in \mathbb{R}\times[a,b].$$

We infer that,

$$\rho(C((x,y),(u_1,v_1)), C((x,y),(u_2,v_2))) \le$$
$$\le \begin{pmatrix} L_1 b & L_2 b \\ 2L_1 & 2L_2 \end{pmatrix} \cdot \rho((u_1,v_1),(u_2,v_2)),$$

$\forall (x,y) \in X$, $\forall (u_1,v_1),(u_2,v_2) \in Y$. Since, $bL_1 + 2L_2 < 1$, we infer that $C((x^*,y^*),\cdot)$ has an unique fixed point $(u^*,v^*) \in Y$, and therefore $((x^*,y^*),(u^*,v^*)) \in X\times Y$ is the unique fixed point of the operator A. From Theorem 1 follows that for any $x_0 \in X_+(\varpi)\cap C^2(\mathbb{R}\times[a,b])$, if we choose

$$y_0 = \frac{\partial x_0}{\partial t}, \quad u_0 = \frac{\partial x_0}{\partial \tau}, \quad v_0 = \frac{\partial y_0}{\partial \tau},$$

then the sequence

$$(A^n((x_0, y_0), (u_0, v_0)))_n = ((x_n, y_n), (u_n, v_n))$$

converges on $X \times Y$ to $((x^*, y^*), (u^*, v^*))$. So, $y^*(t, \cdot) \in C^1[a, b], \forall t \in \mathbb{R}$ and

$$x_n \rightrightarrows x^* \,, \quad y_n = \frac{\partial x_n}{\partial t} \rightrightarrows y^*$$

$$u_n = \frac{\partial x_n}{\partial \tau} \rightrightarrows u^* \,, \quad v_n = \frac{\partial y_n}{\partial \tau} \rightrightarrows v^*.$$

Then,

$$y^* = \frac{\partial x^*}{\partial t}, \quad u^* = \frac{\partial x^*}{\partial \tau}, \quad v^* = \frac{\partial y^*}{\partial \tau}.$$

and the pair $(\frac{\partial x^*}{\partial \tau}, \frac{\partial y^*}{\partial \tau}) \in Y$ is the unique solution of the equation (2.4). ■

Remark 3.2: From the above theorem we infer that the equation (2.1) has an unique positive, periodic and smooth solution which is smooth dependent by the lag τ together with his derivative (in respect by t). These means that the proportion of infectives in the population and the speed of infection spreading are smooth dependent by the length of time in which the individuals remain infectious.

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