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Representation of Simple $(n+1)$-Dimensional n-Lie Algebras

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Abstract: This paper gives the sufficient and necessary conditions when an irreducible $L(A)$-module is an A-module, and gets the classification of finite dimensional irreducible representations of simple $(n+1)$-dimensional n-Lie algebras.

Key words: n-Lie algebra; representation; semi-simple; completely reducible.
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1. Introduction

In 1985, Filippov[1] introduced the concept of n-Lie algebra, which is a natural generalization of the concept of Lie algebra to the case where the fundamental multiplication is n-ary, $n > 2$. The n-Lie algebra has its background in geometry and physics. In 1973, Yoichiro Nambu generalized the Poisson bracket-a "binary" operation on classical observation on the phase space to the "multiple" operation of higher order $n > 2$, and proposed the Nambu mechanics[2]. In 1993, Leno Takhtajan formulated the fundamental identity FI-generalization of Jacobi identity, which is a consistency condition for the Nambu's dynamics[3]. Based on FI, Leno Takhtajan introduced the notion of Nambu Poisson manifolds, which play the same role in Nambu mechanics that Poisson manifolds play in Hamiltonian mechanics. The linear Poisson bracket structure is equivalent to the Lie algebra structure on the dual space. The linear Nambu structures of order n are in one-to-one corresponding with Nambu-Lie algebras of order n on the dual space.

Because of the multiplication in n-Lie algebras, the study on it is more difficult than one on Lie algebras. In recent years, the local structure of n-Poisson and n-Jacobi manifolds[4], the decomposition of n-Lie algebras[7] and Cartan subalgebras[8] were studied. But the systematic study on n-Lie algebras is far from being completed. We know that representation theory is very important in the theory on Lie algebras and in its applications. Especially representations of simple 3-dimensional Lie algebra is a powerful tool in studying Lie algebras. In order to study n-Lie algebras by means of complete theory on Lie algebras, we try to find more structural properties on n-Lie algebras, and more relations between them.

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In this paper, we study finite dimensional representations of $(n+1)$-dimensional n-Lie algebras over an algebraically closed field F with characteristic zero.

2. Preliminary

An n-Lie algebra is a vector space A over a field F on which an n-ary multilinear operation $[,\cdots,]$ is defined, satisfying the identities

$$[x_1,\cdots,x_n]=(-1)^{\tau(\sigma)}[x_{\sigma(1)},\cdots,x_{\sigma(n)}] \tag{2.1}$$

$$[[x_1,\cdots,x_n],y_2,\cdots,y_n]=\sum_{i=1}^{n}[x_1,\cdots,[x_i,y_2,\cdots,y_n],\cdots,x_n], \tag{2.2}$$

where σ runs over the symmetric group S_n and the number $\tau(\sigma)$ is equal to 0 or 1 depending on the parity of the permutation σ.

The map $R(x_2,\cdots,x_n):A\to A$, $(x_1)R(x_2,\cdots,x_n)=[x_1,\cdots,x_n]$ for $x_i\in A$ is called the right multiplication defined by elements $x_2,\cdots,x_n\in A$.

A derivation of an n-Lie algebra is a linear transformation D of A into itself satisfying the condition

$$([x_1,\cdots,x_n])D=\sum_{i=1}^{n}[x_1,\cdots,(x_i)D,\cdots,x_n], \tag{2.3}$$

for $x_1,\cdots,x_n\in A$. By virtue of (2.2), the right multiplications are derivations which are said to be inner. All the derivations of A generate a subalgebra of Lie algebra $\mathrm{gl}(A)$ which is called the derivation algebra of A, and denoted by $\mathrm{Der}A$. Let $L(A)$ denote the Lie algebra generated by operators $R(a_1,\cdots,a_{n-1})$, where $a_i\in A$, $1\le i\le n-1$.

By virtue of (2.2) and (2.3), we have

$$\begin{aligned}&[R(a_1,\cdots,a_{n-1}),R(b_1,\cdots,b_{n-1})]\\&\quad=R(a_1,\cdots,a_{n-1})R(b_1,\cdots,b_{n-1})-R(b_1,\cdots,b_{n-1})R(a_1,\cdots,a_{n-1})\\&\quad=\sum_{i=1}^{n-1}R(a_1,\cdots,(a_i)R(b_1,\cdots,b_{n-1}),\cdots,a_{n-1}),\end{aligned} \tag{2.4}$$

$$R([a_1,\cdots,a_n],b_1,\cdots,b_{n-2})=\sum_{i=1}^{n}(-1)^{i+1}R(a_i,b_1,\cdots,b_{n-2})R(a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n).$$

We call

$$Z(A)=\{x\in A,[x,A,\cdots,A]=0\} \tag{2.5}$$

the center of A. Clearly $Z(A)$ is an ideal of A.

Let A be an $n-$Lie algebra and V be a linear space over F, then a polynomial mapping $\rho: A^{n-1}\to \mathrm{End}V$ is said to be a representation of A in V denoted by (V,ρ) if the operators $\rho(x_1,\cdots,x_{n-1}),x_i\in A$, are skew-symetric functions of their arguments and satisfy the identities

$$[\rho(a),\rho(b)]=\sum_{i=1}^{n-1}\rho(a_1,\cdots,a_iR(b),\cdots,a_{n-1}), \tag{2.6}$$

$$\rho([a_1,\cdots,a_n],b_2,\cdots,b_{n-1})=\sum_{i=1}^{n}(-1)^{i+1}\rho(a_i,b_2,\cdots,b_{n-1})\rho(a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n), \quad (2.7)$$

where $a,b\in A^{n-1}, a=(a_1,\cdots,a_{n-1}), b=(b_1,\ldots,b_{n-1})$.

The kernel of ρ is the subspace

$$\mathrm{Ker}\rho=\{x\in A,\rho(x,A,\cdots,A)=0\}. \quad (2.8)$$

By virtue of (2.6) and (2.7), Kerρ is an ideal of A. If Ker$\rho=0$, the representation is said to be faithful. A particular instance of the representation is the regular representation $a\to R(a), a=(a_1,\cdots a_{n-1})\in A^{n-1}$. The kernel of the regular representation is the center $Z(A)$ of A.

Denote $L_\rho(A)$ the Lie algebra generated by the operators $\rho(x), x\in A^{n-1}$, and for any $a=(a_1,\cdots,a_{n-1}), b=(b_1,\cdots,b_{n-1}), c=(c_1,\cdots,c_{n-1})\in A^{n-1}$,

$$[\rho(a)\rho(b),\rho(c)]=[\rho(a),\rho(c)]\rho(b)+\rho(a)[\rho(b),\rho(c)]. \quad (2.9)$$

Let (V,ρ) be an arbitrary representation of A, and $B=V\dot{+}A$ the linear space of direct sum of V and A. From identities (2.6) and (2.7), B is endowed with an $n-$ary skew-symetric operation:

$$[V,V,B,\cdots,B]_B=0, \quad (2.10)$$

$$[v,a_1,\cdots,a_{n-1}]_B=(v)\rho(a_1,\cdots,a_{n-1}), \quad (2.11)$$

$$[a_1,\cdots,a_n]_B=[a_1,\cdots,a_n], \quad (2.12)$$

where $v\in V, a_i\in A$, such that B is an n-Lie algebra, V is an Abelian ideal of B and n-Lie algebra A is a subalgebra of n-Lie algebra B. It is often convenient to use the language of modules along with the (equivalent) language of representation. So the representation (V,ρ) often is called A-module. The following lemma is easily proved by the defination:

Lemma 2.1 *Let (V,ρ) be a representation of A , then V is $L_\rho(A)$- module under the natural action:*

$$v.\rho(a_1,\cdots,a_{n-1})=(v)\rho(a_1,\cdots,a_{n-1}). \quad (\star)$$

And a subspace V_1 of V is an A-submodule if and only if V_1 is a $L_\rho(A)$-submodule. Hence (V,ρ) is irreducible if and only if V is an irreducible $L_\rho(A)$-module under the action $(\star)$. Denote by $L^\rho(B)$ a subalgebra of $L(B)$ generated by the right multiplications $R_B(a_1,\cdots,a_{n-1}), a_i\in A$. Then $L(B)=L^\rho(B)\dot{+}Z_\rho(V)$, where $Z_\rho(V)$ is a Lie algebra generated by the right multiplications $R_B(v,a_1,\cdots,a_{n-2}), v\in V, a_i\in A$, and $Z_\rho(V)$ is an Abelian ideal of $L(B)$, and $R_B^2(v,a_1,\cdots,a_{n-2})=0$. Set

$$\omega_1: L^\rho(B)\to gl(A), \quad D\mapsto D|_A, D\in L^\rho(B). \quad (2.13)$$

Then for any $a_1,\cdots,a_{n-1},b_1,\cdots,b_{n-1}\in A$, $(R_B(a_1,\cdots,a_{n-1}))\omega_1=R(a_1,\cdots,a_{n-1})$,

$$\mathrm{Im}(\omega_1)=L(A), \quad (2.14)$$

$$[(R_B(a_1,\cdots,a_{n-1}))\omega_1,(R_B(b_1,\cdots,b_{n-1}))\omega_1]=[R(a_1,\cdots,a_{n-1}),R(b_1,\cdots,b_{n-1})]$$

$$= ([R_B(a_1,\cdots,a_{n-1}),R_B(b_1,\cdots,b_{n-1})])\omega_1.$$

This leads to the following result

Lemma 2.2 *ω_1 is a Lie homomorphism from $L^\rho(B)$ onto $L(A)$ and*

$$L(A)\cong\frac{L^\rho(B)}{\mathrm{Ker}\omega_1},\tag{2.15}$$

$$\mathrm{Ker}\omega_1=\{D\in L^\rho(B),\quad D|_A=0\}\subseteq Z(L^\rho(B)).\tag{2.16}$$

Set

$$\omega_2: L^\rho(B)\to \mathrm{gl}(V),\quad D\mapsto D|_V,\quad D\in L^\rho(B).\tag{2.17}$$

Lemma 2.3 *ω_2 is a Lie homomorphism from $L^\rho(B)$ into $L_\rho(A)$, and*

$$\mathrm{Im}(\omega_2)=L_\rho(A),\tag{2.18}$$

$$\mathrm{Ker}\omega_2=\{D\in L^\rho(B),D|_V=0\},\tag{2.19}$$

$$L_\rho(A)\cong\frac{L^\rho(B)}{\mathrm{Ker}\omega_2},\tag{2.20}$$

$$\mathrm{Ker}\omega_1\cap\mathrm{Ker}\omega_2=0.\tag{2.21}$$

Theorem 2.4 *Let A be a simple $(n+1)$-dimensional n-Lie algebra $(n\geq 3)$, and (V,ρ) be a finite dimensional representation of A, then V is completely reducible; and if $\rho\neq 0$, then we have: When $n=2m+1>3$,*

$$L_\rho(A)\cong L^\rho(B)\cong L(A)=D_{m+1},\tag{2.22}$$

and V is a representation of $L(A)$ under the natural action

$$v.R(a_1,\cdots,a_{n-1})=(v)\rho(a_1,\cdots,a_{n-1})=[v,a_1,\cdots,a_{n-1}]_B;\tag{2.23}$$

When $n=2m$,

$$L_\rho(A)\cong L^\rho(B)\cong L(A)=B_m,\tag{2.24}$$

and V is a representation of $L(A)$ under the natural action (2.24);
When $n=3$,

$$L_\rho(A)\cong L^\rho(B)\cong L(A)\cong \mathrm{sl}(2,F)\oplus\mathrm{sl}(2,F),\tag{2.25}$$

and V is a representation of $L(A)$ under the natural action (2.23). So V is completely reducible, that is, for any A-submodule V_1, there exists a A-submodule V_2 such that $V=V_1\dot{+}V_2$.

Proof If $\rho=0$, it is obviously V is a completely reducible. When $\rho\neq 0$, then $L_\rho(A)\neq 0$, we divide the proof into three cases:

Case 1. When $n=2m+1>3$. Let $e_1,\cdots,e_{n+1}$ be a basis of A, and the multiplication table as follows:

$$[e_1,\cdots,\hat{e}_{2t-1},e_{2t},\cdots,e_{n+1}]=-e_{2t},\quad [e_1,\cdots,e_{2t-1},\hat{e}_{2t},\cdots,e_{n+1}]=e_{2t-1},\tag{2.26}$$

where the symbol $\hat{e}_t$ means that e_t is omitted, $1 \le t \le m+1$. By virtue of Table (2.26) and Theorem 1 in [6], the inner derivation algebra $L(A)$ is a simple Lie algebra $L(A) = D_{m+1}$ and has a basis : $R(\hat{e}_{2i-1}, \hat{e}_{2j}) = E_{2i-1,2j-1} - E_{2j,2i}, 2i-1 < 2j$; $R(\hat{e}_{2i}, \hat{e}_{2j-1}) = E_{2i,2j} - E_{2j-1,2i-1}, 2i < 2j-1$; $R(\hat{e}_{2i-1}, \hat{e}_{2j-1}) = -E_{2i-1,2j} + E_{2j-1,2i}, R(\hat{e}_{2i}, \hat{e}_{2j}) = -E_{2i,2j-1} + E_{2j,2i-1}, i < j$; where $1 \le i, j \le m+1$, E_{ij} are $(n+1)$-matrix units and as an endomorphism of A, satisfying$(e_i)E_{ij} = e_j$, and $R(\hat{e}_i, \hat{e}_j) = R(e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_{j-1}, e_{j+1}, \cdots, e_{n+1}), 1 \le i, j \le n+1, i \ne j$. From the definition of $L^\rho(B)$ and since $e_1, \cdots, e_{n+1}$ is a basis of A, $L^\rho(B)$ is linearly spanned by $R_B(\hat{e}_{2i-1}, \hat{e}_{2j}), 2i-1 < 2j$; $R_B(\hat{e}_{2i}, \hat{e}_{2j-1}), 2i < 2j-1$; $R_B(\hat{e}_{2i-1}, \hat{e}_{2j-1}), i < j$; $R_B(\hat{e}_{2i}, \hat{e}_{2j}), i < j$, where $1 \le i, j \le m+1$. This implies $\dim L^\rho(B) \le \dim L(A)$. By virtue of Theorem 2.2, $L(A) \cong \frac{L^\rho(B)}{\mathrm{Ker}\omega_1}$, then $\dim A \le \dim L^\rho(B)$, so we get

$$L^\rho(B) \cong L(A) = D_{m+1}, \tag{2.27}$$

and $\mathrm{Ker}\omega_1 = 0$. Then $L^\rho(B)$ is simple.

From Lemma 2.3, $L_\rho(A) \cong \frac{L^\rho(B)}{\mathrm{Ker}\omega_2} \ne 0$, then $\mathrm{Ker}\omega_2 = 0$, and

$$L_\rho(A) \cong L^\rho(B) \cong D_{m+1}. \tag{2.28}$$

$$\pi : L_\rho(A) \to L(A), \rho(a_1, \cdots, a_{n-1}) \mapsto R(a_1, \cdots, a_{n-1}) \tag{2.29}$$

is a Lie isomorphism. By Lemma 2.1, V is $L(A)$-module under the action (2.23), and V is a completely reducible A-module.

Case 2. When $n = 2m$, let $e_1, \cdots, e_{n+1}$ be a basis of A, and the multiplication table as follows:

$$[e_1, \cdots, \hat{e}_{2t-1}, e_{2t}, \cdots, e_{n+1}] = -e_{2t}, \quad [e_1, \cdots, e_{2t-1}, \hat{e}_{2t}, \cdots, e_{n+1}] = e_{2t-1},$$

$$[e_1, \cdots, e_n, \hat{e}_{n+1}] = -e_{n+1}, \tag{2.30}$$

where the symbol $\hat{e}_t$ means that e_t is omitted, $1 \le t \le m$. By virtue of Table (2.30) and Theorem 1 in [6], the inner derivation algebra $L(A)$ is a simple Lie algebra $L(A) = B_m$ and has a basis:

$$R(\hat{e}_{2i-1}, \hat{e}_{2j}) = E_{2i-1,2j-1} - E_{2j,2i}, 1 \le 2i-1 < 2j \le n;$$

$$R(\hat{e}_{2i}, \hat{e}_{2j-1}) = E_{2i,2j} - E_{2j-1,2i-1}, 2 \le 2i < 2j-1 \le n-1;$$

$$R(\hat{e}_{2i-1}, \hat{e}_{2j-1}) = -E_{2i-1,2j} + E_{2j-1,2i}, 1 \le 2i-1 < 2j-1 \le n-1;$$

$$R(\hat{e}_{2i}, \hat{e}_{2j}) = -E_{2i,2j-1} + E_{2j,2i-1}, 2 \le 2i < 2j \le n;$$

$$R(\hat{e}_{2i-1}, \hat{e}_{n+1}) = E_{n+1,2i} - E_{2i-1,n+1}, 1 \le 2i-1 \le n;$$

$$R(\hat{e}_{2i}, \hat{e}_{n+1}) = E_{2i,n+1} - E_{n+1,2i-1}, 1 \le i \le m.$$

The discussion is similar to Case 1,

$$L_\rho(A) \cong L^\rho(B) \cong L(A) = B_m. \tag{2.31}$$

And V is a $L(A)$-module under the action (2.23), thus V is a completely reducible A-module.

Case 3. When $n=3$, let $e_i, i=1,2,3,4$, be a basis of A, and multiplication table as follows

$$[e_1,e_2,e_3]=e_3, \quad [e_1,e_2,e_4]=-e_4, \quad [e_1,e_3,e_4]=e_1, \quad [e_2,e_3,e_4]=-e_2. \tag{2.32}$$

By direct computation we get $R(e_1,e_2)=E_{33}-E_{44}$, $R(e_3,e_4)=E_{11}-E_{22}$, $R(e_1,e_3)=-E_{23}+E_{41}$, $R(e_2,e_3)=E_{13}-E_{42}$, $R(e_1,e_4)=E_{24}-E_{31}$, $R(e_2,e_4)=-E_{14}+E_{32}$. Then $R(e_i,e_j), 1\le i<j\le 4$ is a basis of $L(A)$, and

$$L(A)=L_1\oplus L_2, \tag{2.33}$$

$$L_1=FR(e_3,e_1)+FR(e_4,e_2)+F(R(e_1,e_2)+R(e_3,e_4))\cong sl(2,F), \tag{2.34}$$

$$\begin{aligned}&[R(e_3,e_1),R(e_4,e_2)]=R(e_1,e_2)+R(e_3,e_4),\\ &[R(e_1,e_2)+R(e_3,e_4),R(e_3,e_1)]=-2R(e_3,e_1),\\ &[R(e_1,e_2)+R(e_3,e_4),R(e_4,e_2)]=+2R(e_4,e_2).\end{aligned} \tag{2.35}$$

$$L_2=FR(e_3,e_2)+FR(e_4,e_1)+F(R(e_1,e_2)-R(e_3,e_4))\cong sl(2,F), \tag{2.36}$$

$$\begin{aligned}&[R(e_3,e_2),R(e_4,e_1)]=R(e_1,e_2)-R(e_3,e_4),\\ &[R(e_1,e_2)-R(e_3,e_4),R(e_3,e_2)]=-2R(e_3,e_2),\\ &[R(e_1,e_2)-R(e_3,e_4),R(e_4,e_1)]=+2R(e_4,e_1).\end{aligned} \tag{2.37}$$

By a similar discussion as in Case 1, we get

$$\mathrm{Ker}\omega_1=0, \quad L^\rho(B)\cong L(A)\cong \mathrm{sl}(2,F)\oplus\mathrm{sl}(2,F), \tag{2.38}$$

$$L^\rho(B)=L_1(B)\oplus L_2(B), \quad L_1=(L_1(B))\omega_1, \quad L_2=(L_2(B))\omega_1, \tag{2.39}$$

$$L_1(B)=FR_B(e_3,e_1)+FR_B(e_4,e_2)+F(R_B(e_1,e_2)+R_B(e_3,e_4))\cong sl(2,F), \tag{2.40}$$

$$L_2(B)=FR_B(e_3,e_2)+FR_B(e_4,e_1)+F(R_B(e_1,e_2)-R_B(e_3,e_4))\cong sl(2,F). \tag{2.41}$$

We now prove $\mathrm{Ker}\omega_2=0$, that is $L_\rho(A)\cong L^\rho(B)$.

If $\mathrm{Ker}\omega_2\ne 0$, because $L_\rho(A)\ne 0$, then $\mathrm{Ker}\omega_2=L_1$, or $\mathrm{Ker}\omega_2=L_2$. Suppose $\mathrm{Ker}\omega_2=L_1$, then $L_\rho(A)\cong L_2$. From Theorem 3.1 there exists an irreducible $L_\rho(A)$-submodule of V_1, and $\dim V_1>1$. (In fact, if for any A-submodule $V_1\ne 0$, $\dim V_1=1$. Since V is a module of semi-simple Lie algebra $L^\rho(B)$, then $(V_1)R^\rho(B)=0$, thus $(V)R^\rho(B)=0$, and $(V)\rho=0$, this contradicts to $\rho\ne 0$.) Suppose v_0 is a maximal vector of the highest weight λ of $(\rho(e_1,e_2)-\rho(e_3,e_4))$ in V_1.

By the properties of representations of $\mathrm{sl}(2,F)$, $\lambda=\dim V_1-1$ is a positive integer and

$$(v_0)\rho(e_3,e_2)=0, \quad (v_0)(\rho((e_1,e_2)-\rho(e_3,e_4))=\lambda V_0. \tag{2.42}$$

From (2.7) and (2.42),

$$(v_0)R_B(e_3,e_1)=(v_0)\rho(e_3,e_1)=0, \quad (v_0)R_B(e_4,e_2)=(v_0)\rho(e_4,e_2)=0,$$

$$(v_0)(R_B(e_1,e_2)+R_B(e_3,e_4))=(v_0)(\rho(e_1,e_2)+\rho(e_3,e_4))=0. \tag{2.43}$$

From equation (2.42) and (2.43),

$$(v_0)\rho(e_1,e_2)=\frac{\lambda}{2}v_0, \quad (v_0)\rho(e_3,e_4)=-\frac{\lambda}{2}v_0. \tag{2.44}$$

By the equation (2.7), $\frac{\lambda}{2}v_0=(v_0)\rho([e_1,e_3,e_4],e_2)=(v_0)[\rho(e_1,e_2)\rho(e_3,e_4)-\rho(e_3,e_2)\rho(e_1,e_4)+\rho(e_4,e_2)\rho(e_1,e_3)]=-\frac{\lambda^2}{4}v_0$, then we get $\lambda=-2$, which contradicts that λ is positive. This proves $\mathrm{Ker}\omega_2\neq L_1$. Similarly $\mathrm{Ker}\omega_2\neq L_2$. Therefore, $\mathrm{Ker}\omega_2=0$,

$$L_\rho(A)\cong L^\rho(B)\cong \mathrm{sl}(2,F)\oplus\mathrm{sl}(2,F), \tag{2.45}$$

and V is a completely reducible A-module.

Set $\pi: L_\rho(A)\to L(A), \quad \rho(e_i,e_j)\mapsto R(e_i,e_j)$. From the discussion above, π is a Lie isomorphism, and V is a $L(A)$-module under the action (2.23). The proof is completed.

Let (V_1,ρ_1), (V_2,ρ_2) be two A-modules, the $(V_1,\rho_1),(V_2,\rho_2)$ are called isomorphic and denoted by $(V_1,\rho_1)\cong(V_2,\rho_2)$ if there exists a bijection $\sigma: V_1\to V_2$, such that $((v)\rho_1(a))\sigma=((v)\sigma)\rho_2(a)$, $a\in A^{n-1}$.

Theorem 2.5 *Let $(V_1,\rho_1),(V_2,\rho_2)$ be A-modules, then $(V_1,\rho_1)\cong(V_2,\rho_2)$ if and only if $V_1\cong V_2$ as $L_\rho(A)$-module.*

For any irreducible $A-$module (V,ρ), if λ is the highest weight of irreducible $L_\rho(A)$-module under the nature action $(\star)$, then we call (V,ρ) is an irreducible $A-$module with the highest weight λ. From Lemma 2.1, $(V_1,\rho_1)\cong(V_2,\rho_2)$, if and only if the highest weights of them are equal.

3. The main results

Theorem 3.1 *Let A be a simple 4-dimensional 3-Lie algebra over the field F, then for any irreducible A-module (V,ρ), there exists a positive integer m such that*

$$\lambda(h_1)=\lambda(h_2)=m, \tag{3.7}$$

is the highest weight of $L_\rho(A)$-module V under the nature action $(\star)$. Conversely, for any positive integer m, there exists an irreducible A-module (V,ρ) with the highest weight λ, satisfing (3.7), where $\lambda\in H^*=Fh_1+Fh_2$, and H is a Cartan subalgebra of $L_\rho(A)$.

Proof Let (V,ρ) be an irreducible A-module. From Lemma 2.1, V is an irreducible $L_\rho(A)$-module under the natural action $(\star)$. From Theorem 2.4 and Equations (2.32-45), if we set

$$h_1=\rho(e_1,e_2)+\rho(e_3,e_4), h_2=\rho(e_1,e_2)-\rho(e_3,e_4),$$

and λ is the highest weight of $H=Fh_1+Fh_2$, then $\lambda_1=\lambda(\rho(e_1,e_2)+\rho(e_3,e_4))=\lambda(h_1)\neq 0$, $\lambda_2=\lambda(\rho(e_1,e_2)-\rho(e_3,e_4))=\lambda(h_2)\neq 0$. Thus λ_1,λ_2 are positive integers, and

$$\lambda(\rho(e_1,e_2))=\frac{\lambda_1+\lambda_2}{2}, \quad \lambda(\rho(e_3,e_4))=\frac{\lambda_1-\lambda_2}{2}. \tag{3.8}$$

Let v_0 be the maximal vector in V of the highest weight λ, then $(v_0)\rho(e_3,e_1)=(v_0)\rho(e_3,e_2)=0$,

$$\begin{aligned}\frac{\lambda_1+\lambda_2}{2}(v_0) &= (V_0)\rho(e_1,e_2)=(v_0)\rho([e_1,e_3,e_4],e_2)\\ &= (v_0)(\rho(e_1,e_2)\rho(e_3,e_4)-\rho(e_3,e_2)\rho(e_1,e_4)+\rho(e_4,e_2)\rho(e_1,e_3))\\ &= \frac{\lambda_1^2-\lambda_2^2}{4}v_0+(v_0)[\rho(e_4,e_2),\rho(e_1,e_3)]=(\frac{\lambda_1^2-\lambda_2^2}{4}+\lambda_1)v_0,\end{aligned}$$

$\frac{\lambda_1+\lambda_2}{2}=(\frac{\lambda_1^2-\lambda_2^2}{4}+\lambda_1)$, $(\lambda_1-1)^2=(\lambda_2-1)^2$. Since λ_1,λ_2 are positive integers, we get $\lambda_1=\lambda_2$. Therefore, $\lambda(h_1)=\lambda(h_2)$.

Conversely, for any positive integer m, let V be a $(m+1)^2$-dimensional vector space, and $v_{00},v_{01},\cdots,v_{0m};v_{10},\cdots,v_{1m};\cdots;v_{m0},\cdots,v_{mm}$ be a basis of V. Define skew-symetric function $\rho:A^2\to \mathrm{End}V$ as follows

$$(v_{ij})\rho(e_1,e_2)=(m-i-j)v_{ij},\quad (v_{ij})\rho(e_3,e_4)=(i-j)v_{ij},$$

$$(v_{ij})\rho(e_3,e_1)=(-jm+j(j-1))v_{i(j-1)},$$

$$(v_{ij})\rho(e_3,e_2)=(-im+i(i-1))v_{(i-1)j},$$

$$(v_{ij})\rho(e_4,e_2)=v_{i(j+1)},(v_{ij})\rho(e_4,e_1)=v_{(i+1)j},\tag{3.9}$$

$$(v_{ij})(\rho(e_1,e_2)+\rho(e_3,e_4))=(m-2j)v_{ij},$$

$$(v_{ij})(\rho(e_1,e_2)-\rho(e_3,e_4))=(m-2i)v_{ij},$$

$$v_{i(m+1)}=v_{(m+1)j}=v(-1)j=v_{i(-1)}=0,\quad (v_{00})\rho(e_3,e_1)=(v_{00})\rho(e_3,e_2)=0.$$

By direct computation, (V,ρ) is a A-module, and $L_\rho(A)=L_1\oplus L_2$,

$$L_1=F\rho(e_3,e_1)+F\rho(e_4,e_2)+F(\rho(e_1,e_2)+\rho(e_3,e_4))\cong sl(2,F),\tag{3.10}$$

$$[\rho(e_3,e_1),\rho(e_4,e_2)]=\rho(e_1,e_2)+\rho(e_3,e_4),$$

$$[\rho(e_3,e_1),\rho(e_1,e_2)+\rho(e_3,e_4)]=2\rho(e_3,e_1),$$

$$[\rho(e_4,e_2),\rho(e_1,e_2)+\rho(e_3,e_4)]=-2\rho(e_4,e_2).$$

$$L_2=F\rho(e_3,e_2)+F\rho(e_4,e_1)+F(\rho(e_1,e_2)-\rho(e_3,e_4))\cong sl(2,F),\tag{3.11}$$

$$[\rho(e_3,e_2),\rho(e_4,e_1)]=\rho(e_1,e_2)-\rho(e_3,e_4),$$

$$[\rho(e_3,e_2),\rho(e_1,e_2)-\rho(e_3,e_4)]=2\rho(e_3,e_2),$$

$$[\rho(e_4,e_1),\rho(e_1,e_2)-\rho(e_3,e_4)]=-2\rho(e_4,e_1).$$

From equation table (3.9), V is an irreducible $L_\rho(A)$-module under the natural action $(\star)$ and $\lambda(\rho(e_1,e_2)+\rho(e_3,e_4))=\lambda(\rho(e_1,e_2)-\rho(e_3,e_4))=m$ is the highest weight of $H=F(\rho(e_1,e_2)+\rho(e_3,e_4))+F(\rho(e_1,e_2)-\rho(e_3,e_4))$. □

Corollary 3.2 *Let A be a simple 4-dimensional 3-Lie algebra, (V_1,ρ_1) and (V_2,ρ_2) be irreducible A-modules, then $(V_1,\rho_1)\cong(V_2,\rho_2)$ if and only if* $\mathrm{dim}V_1=\mathrm{dim}V_2$.

We are now studying the $(n+1)$-dimensional simple n-Lie algebra with the case $n=2m+1>3$. We take the multiplication table as (2.26). From paper [6], $L(A)\cong D_{m+1}$. From Lemma 2.1 and Theorem 2.4, let (V,ρ) be an irreducible A-module, then V is an irreducible $L_\rho(A)$-module under the action $(\star)$. And $H=\sum_{i=1}^{m+1}Fh_{\alpha_i}$ is a Cartan subalgebra of $L_\rho(A)$, where $\{\alpha_1,\cdots,\alpha_{m+1}\}\subset H^*$ is a simple root system of H, $h_{\alpha_i},\quad e_{\alpha_i},\quad f_{\alpha_i},\quad i=1,\cdots,m+1$, is a Chevalley basis of $L_\rho(A)$:

$$h_{\alpha_i}=\rho(\hat e_{2i-1},\hat e_{2i})-\rho(\hat e_{2i+1},\hat e_{2i+2}),e_{\alpha_i}=\rho(\hat e_{2i},\hat e_{2i+1}),1\le i\le m;$$

$$h_{\alpha_{m+1}}=\rho(\hat e_{2m-1},\hat e_{2m})+\rho(\hat e_{2m+1},\hat e_{2m+2}),e_{\alpha_{m+1}}=\rho(\hat e_{2m},\hat e_{2m+2}),\tag{$*$}$$

$$f_{\alpha_i}=\rho(\hat e_{2i-1},\hat e_{2i+2}),1\le i\le m;\quad f_{\alpha_{m+1}}=\rho(\hat e_{2m-1},\hat e_{2m+1}),$$

where $\rho(\hat e_i,\hat e_j)=\rho(e_1,\cdots,e_{i-1},e_{i+1},\cdots,e_{j-1},e_{j+1},\cdots,e_{n+1})$.

$$[e_{\alpha_i},f_{\alpha_i}]=h_{\alpha_i},[e_{\alpha_i},h_{\alpha_i}]=2e_{\alpha_i},[f_{\alpha_i},h_{\alpha_i}]=-2f_{\alpha_i},1\le i\le m+1;[e_{\alpha_i},f_{\alpha_j}]=0,i\ne j.$$

$$[f_{\alpha_i},h_{\alpha_j}]=-<\alpha_i,\alpha_j>f_{\alpha_i},\quad [e_{\alpha_i},h_{\alpha_j}]=<\alpha_i,\alpha_j>e_{\alpha_i},\quad 1\le i,j\le m+1,$$

where $<\alpha_i,\alpha_j>$ are Cartan integers of D_{m+1}.

Let $\lambda\in H^*$ be a highest weight of H, v_0 a maximal vector of H in V respect to λ, and $\lambda_i=\lambda(h_{\alpha_i}),\quad i=1,\cdots,m+1$, then λ_i are nonnegative integers. By the above notations, we have

Lemma 3.3 *Let A be an $(n+1)$-dimensional simple n-Lie algebra, $n=2m+1>3$, (V,ρ) be an irreducible A-module, and λ be a highest weight of H, then*

$$\lambda_m=\lambda_{m+1}.\tag{3.12}$$

Proof Let v_0 be the maximal vector of H in V respect with λ, then $(v_0)\rho(\hat e_{2i},\hat e_{2i+1})=(v_0)\rho(\hat e_{2m},\hat e_{2m+2})=0,i=1,\cdots,m$; $(v_0)(\rho(\hat e_{2i-1},\hat e_{2i})-\rho(\hat e_{2i+1},\hat e_{2i+2}))=\lambda_i v_0,\quad i=1,\cdots,m$; $(v_0)(\rho(\hat e_{2m-1},\hat e_{2m})+\rho(\hat e_{2m+1},\hat e_{2m+2}))=\lambda_{m+1}v_0$,

$$(v_0)\rho(\hat e_{2m-1},\hat e_{2m})=\frac{\lambda_m+\lambda_{m+1}}{2},(v_0)\rho(\hat e_{2m+1},\hat e_{2m+2})=\frac{\lambda_{m+1}-\lambda_m}{2}v_0.\tag{3.13}$$

From Equations (2.6), (2.7), (2.26) and (3.13)

$$(\lambda_m+\lambda_{m+1})(\lambda_m-\lambda_{m+1})=-2(\lambda_m-\lambda_{m+1}).\tag{3.14}$$

Because λ_m,λ_{m+1} are nonnegative integers, we get $\lambda_m=\lambda_{m+1}$. □

From identities (3.12) and (3.13), we get the useful identities:

$$(v_0)\rho(\hat e_{2m+1},\hat e_{2m+2})=0,\quad (v_0)\rho(\hat e_{2m-1},\hat e_{2m})=\lambda_m v_0.\tag{3.15}$$

Lemma 3.4 *If A is an $(n+1)$-dimensional simple n-Lie algebra with the multiplication table (2.26), $n=2m+1>3$, (V,ρ) is an irreducible A-module with the highest weight λ, and v_0 is a maximal vector of λ in V, then we have for any $1\le i\le m-2$*

$$\rho(\hat e_{2i},\hat e_{2i+2})\in Fe_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_{m-1}+\alpha_m+\alpha_{m+1}},\tag{3.16}$$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2i+1})\in Ff_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_{m-1}+\alpha_m+\alpha_{m+1}}, \tag{3.17}$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2i+2}),\rho(\hat{e}_{2i-1},\hat{e}_{2i+1})]=h_{\alpha_i}+2h_{\alpha_{i+1}}+\cdots+2h_{\alpha_{m-1}}+h_{\alpha_m}+h_{\alpha_{m+1}}. \tag{3.18}$$

$$\rho(\hat{e}_{2m-2},\hat{e}_{2m})\in Fe_{\alpha_{m-1}+\alpha_m+\alpha_{m+1}},\quad \rho(\hat{e}_{2m-3},\hat{e}_{2m-1})\in Ff_{\alpha_{m-1}+\alpha_m+\alpha_{m+1}}, \tag{3.19}$$

$$[\rho(\hat{e}_{2m-2},\hat{e}_{2m}),\rho(\hat{e}_{2m-3},\hat{e}_{2m-1})]=h_{\alpha_{m-1}}+h_{\alpha_m}+h_{\alpha_{m+1}}. \tag{3.20}$$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2i})=h_{\alpha_i}+\cdots+h_{\alpha_m}+\rho(\hat{e}_{2m+1},\hat{e}_{2m+2}). \tag{3.21}$$

For $1\le i<k\le m-1$ we have

$$\rho(\hat{e}_{2i},\hat{e}_{2k})=r_{ik}e_{\alpha_i+\cdots+\alpha_{k-1}+2\alpha_k+\cdots+2\alpha_{m-1}+\alpha_m+\alpha_{m+1}}, \tag{3.22}$$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2k-1})=s_{ik}f_{\alpha_i+\cdots+\alpha_{k-1}+2\alpha_k+\cdots+2\alpha_{m-1}+\alpha_m+\alpha_{m+1}}, \tag{3.23}$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2k}),\rho(\hat{e}_{2i-1},\hat{e}_{2k-1})]=h_{\alpha_i}+\cdots+h_{\alpha_{k-1}}+2h_{\alpha_k}+\cdots+2h_{\alpha_{m-1}}+h_{\alpha_m}+h_{\alpha_{m+1}}, \tag{3.24}$$

where $r_{ik},s_{ik}\in F, 1\le i<k\le m-1$.

$$\rho(\hat{e}_{2i},\hat{e}_{2m})=[e_{\alpha_i},[\cdots[e_{\alpha_{m-2}},\rho(\hat{e}_{2m-2},\hat{e}_{2m}]\cdots]]=r_{im}e_{\alpha_i+\cdots+\alpha_{m-1}+\alpha_m+\alpha_{m+1}}, \tag{3.25}$$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2m-1})=[f_{\alpha_i},[\cdots[f_{\alpha_{m-2}},\rho(\hat{e}_{2m-3},\hat{e}_{2m-1}]\cdots]]=s_{im}f_{\alpha_i+\cdots+\alpha_{m-1}+\alpha_m+\alpha_{m+1}}, \tag{3.26}$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2m}),\rho(\hat{e}_{2i-1},\hat{e}_{2m-1})]=h_{\alpha_i}+\cdots+h_{\alpha_{m-1}}+h_{\alpha_m}+h_{\alpha_{m+1}},\quad 1\le i\le m-2; \tag{3.37}$$

$$\rho(\hat{e}_{2i},\hat{e}_{2(m+1)})=[e_{\alpha_i},[\cdots[e_{\alpha_{m-1}},e_{\alpha_{m+1}}]\cdots]]=t_ie_{\alpha_i+\cdots+\alpha_{m-1}+\alpha_{m+1}}, \tag{3.28}$$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2m+1})=[f_{\alpha_i},[\cdots[f_{\alpha_{m-1}},f_{\alpha_{m+1}}]\cdots]]=l_if_{\alpha_i+\cdots+\alpha_{m-1}+\alpha_{m+1}}, \tag{3.29}$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2(m+1)}),\rho(\hat{e}_{2i-1},\hat{e}_{2m+1})]=h_{\alpha_i}+\cdots+h_{\alpha_{m-1}}+h_{\alpha_{m+1}}, \tag{3.30}$$

$t_i,l_i\in F,\quad 1\le i\le m-1;$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2k})=[f_{\alpha_i},[f_{\alpha_{i+1}},\cdots,[f_{\alpha_{k-3}},[f_{\alpha_{k-2}},f_{\alpha_{k-1}}],\cdots]]=l_{ik}f_{\alpha_i+\cdots+\alpha_{k-1}}, \tag{3.31}$$

$$\rho(\hat{e}_{2i},\hat{e}_{2k-1})=[e_{\alpha_i},[e_{\alpha_{i+1}},\cdots,[e_{\alpha_{k-3}},[e_{\alpha_{k-2}},e_{\alpha k-1}],\cdots]]=t_{ik}e_{\alpha_i+\cdots+\alpha_{k-1}}, \tag{3.32}$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2k-1}),\rho(\hat{e}_{2i-1},\hat{e}_{2k})]=\rho(\hat{e}_{2i-1},\hat{e}_{2i})-\rho(\hat{e}_{2k-1},\hat{e}_{2k})=h_{\alpha_i}+\cdots+h_{\alpha_{k-1}}, \tag{3.33}$$

$l_{ik},t_{ik}\in F,\quad 1\le i<k\le m+1.$

$$(v_0)\rho(\hat{e}_{2i},\hat{e}_{2k-1})=(v_0)\rho(\hat{e}_{2i},\hat{e}_{2k})=0,\quad 1\le i<k\le m+1. \tag{3.34}$$

Proof For any $i\le m-1$, from the multiplication table (2.26), identities (2.6) and $(*)$,

$$[\rho(\hat{e}_{2i},\hat{e}_{2i+2}),h_{\alpha_{i-1}}]=[\rho(\hat{e}_{2i},\hat{e}_{2i+2}),\rho(\hat{e}_{2i-3},\hat{e}_{2i-2})-\rho(\hat{e}_{2i-1},\hat{e}_{2i})]=-\rho(\hat{e}_{2i},\hat{e}_{2i+2}), \tag{3.35}$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2i+2}),h_{\alpha_i}]=[\rho(\hat{e}_{2i},\hat{e}_{2i+2}),\rho(\hat{e}_{2i-1},\hat{e}_{2i})-\rho(\hat{e}_{2i+1},\hat{e}_{2i+2})]=0, \tag{3.36}$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2i+2}),h_{\alpha_{i+1}}]=[\rho(\hat{e}_{2i},\hat{e}_{2i+2}),\rho(\hat{e}_{2i+1},\hat{e}_{2i+2})-\rho(\hat{e}_{2i+3},\hat{e}_{2i+4})]=\rho(\hat{e}_{2i},\hat{e}_{2i+2}), \tag{3.37}$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2i+2}),h_{\alpha_j}]=0,\ j\ne i-1,i,i+1. \tag{3.38}$$

Then we get $\rho(\hat{e}_{2i},\hat{e}_{2i+2})\in Fe_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_{m-1}+\alpha_m+\alpha_{m+1}}$. Similarly by direct computation, we get $\rho(\hat{e}_{2i-1},\hat{e}_{2i+1})\in Ff_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_{m-1}+\alpha_m+\alpha_{m+1}}$, and

$$[\rho(\hat{e}_{2i},\hat{e}_{2i+2}),\rho(\hat{e}_{2i-1},\hat{e}_{2i+1})]=h_{\alpha_i}+2h_{\alpha_{i+1}}+\cdots+2h_{\alpha_{m-1}}+h_{\alpha_m}+h_{\alpha_{m+1}}.$$

$$[\rho(\hat{e}_{2m-2},\hat{e}_{2m}),h_{m-2}]=-\rho(\hat{e}_{2m-2},\hat{e}_{2m}),\quad [\rho(\hat{e}_{2m-2},\hat{e}_{2m}),h_{m-1}]=0,$$

$$[\rho(\hat{e}_{2m-2},\hat{e}_{2m}),h_{\alpha_m}]=\rho(\hat{e}_{2m-2},\hat{e}_{2m}),\quad [\rho(\hat{e}_{2m-2},\hat{e}_{2m}),h_{\alpha_{m+1}}]=\rho(\hat{e}_{2m-2},\hat{e}_{2m}),$$

$[\rho(\hat{e}_{2m-2},\hat{e}_{2m}),h_{\alpha_j}]=0,\quad j\neq m-2,m-1,m,m+1$. From Equation (3.16) and by induction on i,k, we get

$$\rho(\hat{e}_{2i},\hat{e}_{2k})=r_{ik}e_{\alpha_i+\cdots+\alpha_{k-1}+2\alpha_k+\cdots+2\alpha_{m-1}+\alpha_m+\alpha_{m+1}},$$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2k-1})=s_{ik}f_{\alpha_i+\cdots+\alpha_{k-1}+2\alpha_k+\cdots+2\alpha_{m-1}+\alpha_m+\alpha_{m+1}},r_{ik},s_{ik}\in F,1\leq i<k\leq m-1;$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2k}),\rho(\hat{e}_{2i-1},\hat{e}_{2k-1})]=h_{\alpha_i}+\cdots+h_{\alpha_{k-1}}+2h_{\alpha_k}+\cdots+2h_{\alpha_{m-1}}+h_{\alpha_m}+h_{\alpha_{m+1}},$$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2k})=l_{ik}f_{\alpha_i+\cdots+\alpha_{k-1}},$$

$$\rho(\hat{e}_{2i},\hat{e}_{2k-1})=t_{ik}e_{\alpha_i+\cdots+\alpha_{k-1}},$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2k-1}),\rho(\hat{e}_{2i-1},\hat{e}_{2k})]=h_{\alpha_i}+\cdots+h_{\alpha_{k-1}},l_{ik},t_{ik}\in F,\quad 1\leq i<k\leq m+1.$$

From the above discussion and by induction on i, we get

$$\rho(\hat{e}_{2i},\hat{e}_{2m})=r_{im}e_{\alpha_i+\cdots+\alpha_{m-1}+\alpha_m+\alpha_{m+1}},$$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2m-1})=s_{im}f_{\alpha_i+\cdots+\alpha_{m-1}+\alpha_m+\alpha_{m+1}},$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2m}),\rho(\hat{e}_{2i-1},\hat{e}_{2m-1})]=h_{\alpha_i}+\cdots+h_{\alpha_{m-1}}+h_{\alpha_m}+h_{\alpha_{m+1}},1\leq i\leq m-2;$$

$$\rho(\hat{e}_{2i},\hat{e}_{2(m+1)})=t_ie_{\alpha_i+\cdots+\alpha_{m-1}+\alpha_{m+1}},$$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2m+1})=l_if_{\alpha_i+\cdots+\alpha_{m-1}+\alpha_{m+1}},$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2(m+1)}),\rho(\hat{e}_{2i-1},\hat{e}_{2m+1})]=h_{\alpha_i}+\cdots+h_{\alpha_{m-1}}+h_{\alpha_{m+1}},t_i,l_i\in F,\quad 1\leq i\leq m-1.$$

From identities (3.16),(3.22), (3.25), (3.28), (3.32), and noting that v_0 is a maximal vector of λ, we get identity (3.34). Identity (3.21) is obvious by the definition of h_{α_i}. □

Theorem 3.5 *Let A be an $(n+1)$-dimensional simple n-Lie algebra with the multiplication table (2.26), $n=2m+1>3$, and (V,ρ) be an irreducible A-module with the highest weight λ. Then there exist only the following cases up to isomorphisms:*

$$\lambda_1>0,\quad \lambda_i=0,\quad i=2,\cdots,m+1, \tag{3.39}$$

where λ_1 is any positive integer.

Proof Step 1. If (V,ρ) is an irreducible representation of A and λ is the highest weight of $H=\sum_{i=1}^{m+1}Fh_{\alpha_i}$, then from Lemma 3.4,

$$\rho(\hat{e}_1,\hat{e}_2)=h_{\alpha_1}+\cdots+h_{\alpha_m}+\rho(\hat{e}_{2m+1},\hat{e}_{2m+2}).$$

$$(v_0)\rho(\hat{e}_1,\hat{e}_2)=(\lambda_1+\cdots+\lambda_m)(v_0).$$

$$[\rho(\hat{e}_2,\hat{e}_4),\rho(\hat{e}_1,\hat{e}_3)]=h_{\alpha_1}+2h_{\alpha_2}+\cdots+2h_{\alpha_{m-1}}+h_{\alpha_m}+h_{\alpha_{m+1}}. \tag{3.40}$$

By identities (2.7) and (2.26)

$$\begin{aligned}(v_0)\rho(\hat{e}_1,\hat{e}_2)&=(v_0)\rho([e_1,e_2,e_3,\hat{e}_4,e_5,\cdots,e_{n+1}],\hat{e}_1,\hat{e}_2,\hat{e}_3,e_4,\cdots,e_{n+1})\\&=(v_0)(h_{\alpha_1}+2h_{\alpha_2}+\cdots+2h_{\alpha_{m-1}}+h_{\alpha_m}+h_{\alpha_{m+1}})+\\&\quad(\lambda_1+\cdots+\lambda_m)(\lambda_2+\cdots+\lambda_m)(v_0)\\&=(\lambda_1+2\lambda_2+\cdots+2\lambda_{m-1}+\lambda_m+\lambda_{m+1}+(\lambda_1+\cdots+\lambda_m)(\lambda_2+\cdots+\lambda_m))(v_0).\end{aligned} \tag{3.41}$$

From equations (3.40) and (3.41), we get

$$(1+\lambda_1+\cdots+\lambda_m)(\lambda_2+\cdots+\lambda_m)=0. \tag{3.42}$$

Because $\lambda\neq 0$, λ_i are nonnegative integers, so $\lambda_1>0,\quad \lambda_i=0,\quad i=2,\cdots,m+1.$

Step 2. We now prove that for any λ satisfying (3.39), there exists an irreducible representation (V,ρ) of A with the highest weight λ.

Suppose V is a $L(A)$-module with the highest weight λ satisfying (3.39) respect to $H(A)$, where $H(A)=\sum_{i=1}^{m+1}F\bar{h}_{\alpha_i}$ is a Cartan subalgebra of $L(A)$, $\{\alpha_1,\cdots,\alpha_{m+1}\}\subset H(A)^*$ is a simple root system, $\bar{h}_{\alpha_i},\bar{e}_{\alpha_i},\bar{f}_{\alpha_i}, i=1,\cdots,m+1$, is a Chevalley basis of $L(A)$, $\bar{h}_{\alpha_i}=R(\hat{e}_{2i-1},\hat{e}_{2i})-R(\hat{e}_{2i+1},\hat{e}_{2i+2}), i=1,\cdots,m$; $\bar{h}_{\alpha_{m+1}}=R(\hat{e}_{2m-1},\hat{e}_{2m})+R(\hat{e}_{2m+1},\hat{e}_{2m+2})$. $\bar{e}_{\alpha_i}=R(\hat{e}_{2i},\hat{e}_{2i+1}), i=1,\cdots,m$; $\quad\bar{e}_{\alpha_{m+1}}=R(\hat{e}_{2m},\hat{e}_{2m+2})$, $\bar{f}_{\alpha_i}=R(\hat{e}_{2i-1},\hat{e}_{2i+2}), i=1,\cdots,m$; $\bar{f}_{\alpha_{m+1}}=R(\hat{e}_{2m-1},\hat{e}_{2m+1})$. $R(\hat{e}_i,\hat{e}_j)=R(e_1,\cdots,e_{i-1},e_{i+1},\cdots,e_{j-1},e_{j+1},\cdots,e_{n+1})$. We define

$$\rho: A^{n-1}\longrightarrow \mathrm{End}(V),\quad (v)\rho(a_1,\cdots,a_{n-1})=(v)R(a_1,\cdots,a_{n-1}). \tag{3.43}$$

We now prove (V,ρ) is an irreducible A-module with the highest weight λ.

From (3.43) $L_\rho(A)$ generated by $\rho(a_1,\cdots,a_{n-1}), a_i\in A$ is a Lie algebra satisfying identity (2.6), and it is clear that

$$\pi: L(A)\longrightarrow L_\rho(A),\quad R(\hat{e}_i,\hat{e}_j)\longmapsto \rho(\hat{e}_i,\hat{e}_j), \tag{3.44}$$

is a Lie isomorphism. Write $h_{\alpha_i}=(\bar{h}_{\alpha_i})\pi, e_{\alpha_i}=(\bar{e}_{\alpha_i})\pi, f_{\alpha_i}=(\bar{f}_{\alpha_i})\pi, i=1,\cdots,m+1$, then $H=\sum_{i=1}^{m+1}Fh_{\alpha_i}$ is a Cartan subalgebra of $L_\rho(A)$, and $h_{\alpha_i},e_{\alpha_i},f_{\alpha_i}, i=1,\cdots,m+1$ is a Chevalley basis. From the property of the regular representation of simple n-Lie algebra and the above discussion, $L(A)$ satisfies identities (3.16–34). Hence Lie algebra $L_\rho(A)$ satisfies identities (3.16-34).

If v_0 is a maximal vector of H in V with respect to the highest weight λ, then V is spanned by the vectors $(v_0)f_{\alpha_{k_1}}\cdots f_{\alpha_{k_s}},\quad k_s\geq 0$. By the properties of weights on Lie algebras, Lemma 3.4 and the character of λ, we have $(v_0)f_{\alpha_1}\neq 0, (v_0)f_{\alpha_i}=0, i>1$; $(v_0)\rho(\hat{e}_{2i-1},\hat{e}_k)=0, 1<i, 2i-1<k\leq 2(m+1)$; $(v_0)\rho(\hat{e}_{2i},\hat{e}_k)=0, 1\leq i, 2i<k\leq 2(m+1)$. We now prove identity (2.7) by induction on k_s. When $k_s=0$, suppose $1\leq j<i\leq k\leq m+1$, $(a_1,\cdots,a_n)=(e_1,\cdots,\hat{e}_{2j},\cdots,e_{n+1})$, $(b_2,\cdots,b_{n-1})=(e_1,\cdots,\hat{e}_{2j-1},\cdots,\hat{e}_{2i-1},\cdots,\hat{e}_{2k},\cdots,e_{n+1})$. Then

$$(v_0)\rho([a_1,\cdots,a_n],b_2,\cdots,b_{n-1})=l_{ik}(v_0)f_{\alpha_i+\cdots+\alpha_{k-1}}=0. \tag{3.45}$$

$$(v_0)\sum_{i=1}^{n}(-1)^{i+1}\rho(a_i,b_2,\cdots,b_{n-1})\rho(a_1,\cdots,\hat{a}_i,\cdots,a_n)$$
$$=(v_0)(\rho(\hat{e}_{2i-1},\hat{e}_{2k})-\rho(\hat{e}_{2i-1},\hat{e}_{2k}))=0. \tag{3.46}$$

Therefore, (2.7) holds for this case. If $i=1,\quad 1<j<k\le m+1$,

$$(a_1,\cdots,a_n)=(e_1,\cdots,\hat{e}_{2j},\cdots,e_{n+1}),$$

$$(b_2,\cdots,b_{n-1})=(\hat{e}_1,e_2,\cdots,\hat{e}_{2j-1},\cdots,\hat{e}_{2k},\cdots,e_{n+1}),$$
$$(v_0)\rho([a_1,\cdots,a_n],b_2,\cdots,b_{n-1})=-(v_0)\rho(\hat{e}_1,\hat{e}_{2k}). \tag{3.47}$$

$$(v_0)\sum_{i=1}^{n}(-1)^{i+1}\rho(a_i,b_2,\cdots,b_{n-1})\rho(a_1,\cdots,\hat{a}_i,\cdots,a_n)$$
$$=-(v_0)(\rho(\hat{e}_{2j-1},\hat{e}_{2j})\rho(\hat{e}_1,\hat{e}_{2k})+[\rho(\hat{e}_1,\hat{e}_{2k}),\rho(\hat{e}_{2j-1},\hat{e}_{2j})]+\rho(\hat{e}_1,\hat{e}_{2k}))$$
$$=-(v_0)\rho(\hat{e}_1,\hat{e}_{2k}), \tag{3.48}$$

from identities (3.47), (3.48), the identity (2.7) holds. By similar discussion, when $(a_1,\cdots,a_n)=(e_1,\cdots,\hat{e}_l,\cdots,e_{n+1})$, $(b_2,\cdots,b_{n-1})=(e_1,\cdots,\hat{e}_r,\cdots,\hat{e}_t,\cdots,\hat{e}_s,\cdots,e_{n+1})$, the identity (2.7) holds, where $l=2j,r=2j-1$, or $l=2j-1,r=2j$; $t=2i$, or $2i-1$; $s=2k-1$, or $2k$. This proves that for any $a_i,b_j\in A$, (2.7) hold when $k_s=0$.

Suppose (2.7) holds for k_s-1, we now prove the case k_s. Set $v_1=(v_0)f_{\alpha_1}\cdots f_{\alpha_{k_s-1}}$. By inductive supposition and the discussion above, we get

$$((v_0)f_{\alpha_{k_1}}\cdots f_{\alpha_{k_s}})\sum_{i=1}^{n}(-1)^{i+1}\rho(a_i,b_2,\cdots,b_{n-1})\rho(a_1,\cdots,\hat{a}_i,\cdots,a_n)$$
$$=\sum_{i=1}^{n}(-1)^{i+1}(v_1)f_{\alpha_{k_s}}\rho(a_i,b_2,\cdots,b_{n-1})\rho(a_1,\cdots,\hat{a}_i,\cdots,a_n)$$
$$=\sum_{i=1}^{n}(-1)^{i+1}(v_1)\rho(a_i,b_2,\cdots,b_{n-1})f_{\alpha_{k_s}}\rho(a_1,\cdots,\hat{a}_i,\cdots,a_n)+$$
$$\sum_{i=1}^{n}(-1)^{i+1}(v_1)[f_{\alpha_{k_s}},\rho(a_i,b_2,\cdots,b_{n-1})]\rho(a_1,\cdots,\hat{a}_i,\cdots,a_n)$$
$$=(v_1)f_{\alpha_{k_s}}\rho([a_1,\cdots,a_n],b_2,\cdots,b_{n-1})$$
$$=(v_0)f_{\alpha_{k_1}}\cdots f_{\alpha_{k_s}}\rho([a_1,\cdots,a_n],b_2,\cdots,b_{n-1}).$$

The proof is completed.

The discussion of the case $n=2m>3$ is simlar to the case $n=2m+1$.

Lemma 3.6 *If A is an $(n+1)$-dimensional simple n-Lie algebra with the multiplication table (2.30), $n=2m>3$, and (V,ρ) is an irreducible A-module with the highest weight λ, v_0 is a maximal vector of λ in V, then we have*

$$\rho(\hat{e}_{2i},\hat{e}_{2i+2})\in Fe_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_{m-1}+2\alpha_m}, \tag{3.49}$$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2i+1})\in Ff_{\alpha_i+2\alpha_{i+1}+\cdots+2\alpha_{m-1}+2\alpha_m}, \tag{3.50}$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2i+2}),\rho(\hat{e}_{2i-1},\hat{e}_{2i+1})]=h_{\alpha_i}+2h_{\alpha_{i+1}}+\cdots+2h_{\alpha_{m-1}}+h_{\alpha_m}, \tag{3.51}$$

where $1\le i\le m-2$.

$$\rho(\hat{e}_{2m-2},\hat{e}_{2m})\in Fe_{\alpha_{m-1}+2\alpha_m},\quad \rho(\hat{e}_{2m-3},\hat{e}_{2m-1})\in Ff_{\alpha_{m-1}+2\alpha_m}, \tag{3.52}$$

$$[\rho(\hat{e}_{2m-2},\hat{e}_{2m}),\rho(\hat{e}_{2m-3},\hat{e}_{2m-1})]=h_{\alpha_{m-1}}+h_{\alpha_m}. \tag{3.53}$$

For $1\le i<k\le m-1$ *we have*

$$\begin{aligned}\rho(\hat{e}_{2i},\hat{e}_{2k})&=[e_{\alpha_i},[e_{\alpha_{i+1}},\cdots,[e_{\alpha_{k-2}},\rho(\hat{e}_{2k-2},\hat{e}_{2k})]\cdots]]\\&=r_{ik}e_{\alpha_i+\cdots+\alpha_{k-1}+2\alpha_k+\cdots+2\alpha_{m-1}+\alpha_m},\end{aligned} \tag{3.54}$$

$$\begin{aligned}\rho(\hat{e}_{2i-1},\hat{e}_{2k-1})&=[f_{\alpha_i},[f_{\alpha_{i+1}},\cdots,[f_{\alpha_{k-2}},\rho(\hat{e}_{2(k-1)-1},\hat{e}_{2k-1})]\cdots]]\\&=s_{ik}f_{\alpha_i+\cdots+\alpha_{k-1}+2\alpha_k+\cdots+2\alpha_{m-1}+\alpha_m},\end{aligned} \tag{3.55}$$

$$\begin{aligned}[\rho(\hat{e}_{2i},\hat{e}_{2k}),\rho(\hat{e}_{2i-1},\hat{e}_{2k-1})]&=\rho(\hat{e}_{2i-1},\hat{e}_{2i})+\rho(\hat{e}_{2k-1},\hat{e}_{2k})\\&=h_{\alpha_i}+\cdots+h_{\alpha_{k-1}}+2h_{\alpha_k}+\cdots+2h_{\alpha_{m-1}}+h_{\alpha_m},\end{aligned} \tag{3.56}$$

where $r_{ik},s_{ik}\in F,1\le i<k\le m-1$.

$$\rho(\hat{e}_{2i},\hat{e}_{2m})=[e_{\alpha_i},[\cdots[e_{\alpha_{m-2}},\rho(\hat{e}_{2m-2},\hat{e}_{2m}]\cdots]]=r_{im}e_{\alpha_i+\cdots+\alpha_{m-1}+2\alpha_m}, \tag{3.57}$$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2m-1})=[f_{\alpha_i},[\cdots[f_{\alpha_{m-2}},\rho(\hat{e}_{2m-3},\hat{e}_{2m-1}]\cdots]]=s_{im}f_{\alpha_i+\cdots+\alpha_{m-1}+2\alpha_m}, \tag{3.58}$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2m}),\rho(\hat{e}_{2i-1},\hat{e}_{2m-1})]=h_{\alpha_i}+\cdots+h_{\alpha_{m-1}}+h_{\alpha_m},\quad 1\le i\le m-2; \tag{3.59}$$

$$\rho(\hat{e}_{2i},\hat{e}_{2m+1})=[e_{\alpha_i},[\cdots[e_{\alpha_{m-1}},e_{\alpha_m}]\cdots]]=t_ie_{\alpha_i+\cdots+\alpha_{m-1}+\alpha_m}, \tag{3.60}$$

$$\rho(\hat{e}_{2i-1},\hat{e}_{2m+1})=(-\frac{1}{2})[f_{\alpha_i},[\cdots[f_{\alpha_{m-1}},f_{\alpha_m}]\cdots]]=l_if_{\alpha_i+\cdots+\alpha_{m-1}+\alpha_m}, \tag{3.61}$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2m+1}),\rho(\hat{e}_{2i-1},\hat{e}_{2m+1})]=-(h_{\alpha_i}+\cdots+h_{\alpha_{m-1}}+\frac{1}{2}h_{\alpha_m}), \tag{3.62}$$

$t_i,l_i\in F$, $1\le i\le m$;

$$\rho(\hat{e}_{2i-1},\hat{e}_{2k})=[f_{\alpha_i},[f_{\alpha_{i+1}},\cdots,f_{\alpha_{k-3}},[f_{\alpha_{k-2}},f_{\alpha_{k-1}}]\cdots]]=l_{ik}f_{\alpha_i+\cdots+\alpha_{k-1}}, \tag{3.63}$$

$$\rho(\hat{e}_{2i},\hat{e}_{2k-1})=[e_{\alpha_i},[e_{\alpha_{i+1}},\cdots,[e_{\alpha_{k-3}},[e_{\alpha_{k-2}},e_{\alpha_{k-1}}]\cdots]]=t_{ik}e_{\alpha_i+\cdots+\alpha_{k-1}}, \tag{3.64}$$

$$[\rho(\hat{e}_{2i},\hat{e}_{2k-1}),\rho(\hat{e}_{2i-1},\hat{e}_{2k})]=\rho(\hat{e}_{2i-1},\hat{e}_{2i})-\rho(\hat{e}_{2k-1},\hat{e}_{2k})=h_{\alpha_i}+\cdots+h_{\alpha_{k-1}}, \tag{3.65}$$

$l_{ik},t_{ik}\in F,\quad 1\le i<k\le m.$

$$(v_0)\rho(\hat{e}_{2i},\hat{e}_j)=0,\quad 1<2i<j\le m+1. \tag{3.66}$$

$$(v_0)\rho(\hat{e}_{2i-1},\hat{e}_{2i})=(\lambda_i+\cdots+\lambda_{m-1}+\frac{1}{2}\lambda_m)(v_0). \tag{3.67}$$

Proof The proof is similar to the proof of Lemma 3.4.

Theorem 3.7 *Let A be an $(n+1)$-dimensional n-Lie algebra with the multiplication table (2.30), $n = 2m > 3$, and (V, ρ) be an irreducible A-module with the highest weight λ. Then there exist only following cases up to an isomorphism:*

$$\lambda_1 > 0, \quad \lambda_i = 0, \quad i = 2, \cdots, m, \tag{3.72}$$

where λ_1 is any positive integer.

Proof The proof is similar to the proof of Theorem 3.5.

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单的 $n+1$ 维 n- 李代数的表示

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摘要: 本文证明了不可约的 $L(A)$- 模是 A- 模的充要条件，给出了单的 $n+1$- 维 n- 李代数的有限维不可约表示的分类.

关键词: n- 李代数；表示；半单；完全可约.