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Schatten Class Weighted Composition Operators on the Bergman Space of the Unit Ball

LIU Yong-min[1], *YU Yan-yan*[2]

(1. Department of Mathematics, Xuzhou Normal University, Jiangsu 221116, China;
2. Department of Computation Science, Xuzhou Institute of Technology, Jiangsu 221008, China)
(E-mail: minliu@xznu.edu.cn)

Abstract: We consider the Schatten class weighted composition operators on the Bergman space of the unit ball. The main result is several necessary and sufficient conditions for such kind of weighted composition operators belong to the Schatten-Von Neumann ideal S_p. As a corollary, we now have that $W_{\varphi,\psi}$ is a Hilbert-Schmidt operator if and only if

$$\int_{B_n} \frac{|\psi(w)|^2}{(1-|\varphi(w)|^2)^{n+1}} \mathrm{d}V(w) < \infty.$$

Key words: Bergman space; weighted composition operator; Schatten class; Hilbert-Schmidt operator.
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1. Introduction

Throughout this paper, we fix $n \in N$, and let B_n be the unit ball of C^n and $\mathrm{d}V$ the normal Lebesgue measures on B_n. Let $H(B_n)$ be the set of holomorphic functions defined on B_n, and $H(B_n, B_n)$ the set of holomorphic maps from B_n to B_n. We will always assume that $\psi \in H(B_n)$ and $\varphi \in H(B_n, B_n)$. The weighted composition operator is defined as $W_{\varphi,\psi}$: $H(B_n) \to H(B_n)$, where $W_{\varphi,\psi}(f)(z) = \psi(z)f(\varphi(z))$, $\forall f \in H(B_n)$. These operators can be considered as a combination of a multiplication operator and a composition operator[1−3]. When $\psi \equiv 1$, $W_{\varphi,\psi} = C_\varphi$ is the usual composition operator. When $\varphi(z) \equiv z$, $W_{\varphi,\psi} = M_\psi$ is the usual multiplication operator induced by ψ. Let $0 < p < \infty$. We consider the Lebesgue spaces $L^p(B_n)$ defined as the Banach space of Lebesgue measurable functions f on B_n with

$$\|f\|_{n,p} = \{\int_{B_n} |f(z)|^p \mathrm{d}V(z)\}^{1/p} < \infty.$$

The weighted Bergman spaces $A^p(B_n)$ is the subspace of $L^p(B_n)$ consisting of holomorphic functions on B_n. It is easy to prove that $A^p(B_n)$ is the closed subspace of $L^p(B_n)$ generated by

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polynomials. Denote by P an orthogonal projection from $L^2(B_n)$ onto $A^2(B_n)$, defined as the following integral operator

$$Pf(z)=\int_{B_n} f(w)K(z,w)\mathrm{d}V(w),\quad f\in L^2(B_n),$$

where $K_w(z)=K(z,w)=1/(1-z\overline{w})^{n+1}$ is the reproducing kernel of $A^2(B_n)$ and $K(z,w)=\overline{K(w,z)}$. Let $k_z(w)=K(z,w)/K(z,z)^{1/2}$, $z,w\in B_n$. This is the normalized reproducing kernel for $A^2(B_n)$. Given a positive measure μ on B_n , we define an operator T_μ as follows:

$$T_\mu f(z)=\int_{B_n} f(w)K_w(z)\mathrm{d}\mu(w),\quad \forall f\in L^2(B_n).$$

T_μ is called the Toeplitz operator with symbol μ. The Berezin transform of μ is defined as

$$\mu^{\sim}(z)=\int_{B_n}|k_z(w)|^2\mathrm{d}\mu(w),\quad \forall z\in B_n.$$

To describe the background about the Schatten class weighted composition operators and to state our results we need some notations. We begin with a trace formula for operators on $A^2(B_n)$ which are either positive or in the trace class. Let T be a bounded operator on a Hilbert space $A^2(B_n)$. Then the singular numbers of the operator T on $A^2(B_n)$ are defined by $t_n(T)=\inf\{\|T-B\|:\mathrm{rank}B\le n\}$. For any $1\le p<\infty$, the Schatten ideal $S_p(A^2(B_n))$ is defined to be the set of all compact operators T on $A^2(B_n)$ such that $\sum_{n=1}^\infty(t_n(T))^p<\infty$. $S_p(A^2(B_n))$ is a Banach space with the norm $\|T\|_{S_p}=\{\sum_{n=1}^\infty|t_n(T)|^p\}^{1/p}$. Hereafter, we will simply write $S_p=S_p(A^2(B_n))$. If $T\in S_1$ and $\{e_n\}$ is an orthonormal basis for $A^2(B_n)$, then $\mathrm{tr}(T)=\sum_{n=1}^\infty\langle Te_n,e_n\rangle$ is convergent and independent of $\{e_n\}$. If $T\in S_1$ and $T\ge 0$, then $\|T\|_{S_1}=\mathrm{tr}(T)$. In general we have $\|T\|_{S_p}=\{\mathrm{tr}(T^*T)^{p/2}\}^{1/p}$. Another elementary fact about the Schatten class is that if $T\in S_p$, then $T^*\in S_p$. Moreover, $\|T^*\|_{S_p}=\|T\|_{S_p}$, see [9]. This theme has attracted the attention of many researchers, but there are still many problems left for investigation. For more information on the Schatten ideals, see [5–8, 10–12].

It is of great interest to find some simple conditions on the functions φ and ψ such that $W_{\varphi,\psi}$ belongs to the Schatten ideal S_p. In [4], H. M. Xu and T. S. Liu considered the weighted composition operators between Bergman space on the unit ball, and characterized the boundedness and compactness of the weighted composition operator in term of Carleson measures. The main purpose of this paper is to give a characterization of those mappings φ and ψ so that $W_{\varphi,\psi}$ belongs to the Schatten ideal S_p on the Bergman spaces $A^2(B_n)$. Finally, we would like to acknowledge the fact that we are borrowing heavily the techniques of the proofs of [11]. For any map φ and ψ , we define a function on B_n as follows:

$$Q_{\varphi,\psi}(z)=\left\{\int_{B_n}K(z,z)^{-1}|K(z,\varphi(w))|^2|\psi(w))|^2\mathrm{d}V(w)\right\}^{1/2}.$$

2. Some basic lemmas

We collect some facts mostly well known, that will be used in the sequel.

Lemma 1[9] *If T is a compact operator on the Hilbert space H and $p>1$, then $T\in S_p$ if and only if $|T|^p=(T^*T)^{p/2}\in S_1$, and if and only if $T^*T\in S_{p/2}$.*

Lemma 2 *For any $\psi\in H(B_n)$, $\varphi\in H(B_n,B_n)$, let the weighted composition operator $W_{\varphi,\psi}$ be bounded operator on $A^2(B_n)$. Then*

$$Q^2_{\varphi,\psi}(z)=\langle (W_{\varphi,\psi})^*W_{\varphi,\psi}k_z,k_z\rangle_{A^2(B_n)}.$$

Proof For any $\psi\in H(B_n)$, $\varphi\in H(B_n,B_n)$, we have

$$\begin{aligned}&\langle (W_{\varphi,\psi})^*W_{\varphi,\psi}k_z,k_z\rangle_{A^2(B_n)}=\langle W_{\varphi,\psi}K_z,W_{\varphi,\psi}K_z\rangle_{A^2(B_n)}K(z,z)^{-1}\\&=\int_{B_n}K(z,z)^{-1}K_z(\varphi(w))\overline{K_z(\varphi(w))}\mathrm{d}V_\psi(w)\\&=\int_{B_n}K(z,z)^{-1}|K(z,\varphi(w))|^2\mathrm{d}V_\psi(w)=Q^2_{\varphi,\psi}(z),\end{aligned}$$

where $\mathrm{d}V_\psi(z)=|\psi(z)|^2\mathrm{d}V(z)$.

Lemma 3[9,13] *Suppose T is a positive operator on a Hilbert space H and x is a unit vector in H. Then (1) $\langle T^px,x\rangle\ge\langle Tx,x\rangle^p$, for all $p\ge1$; (2) $\langle T^px,x\rangle\le\langle Tx,x\rangle^p$, for all $0<p\le1$.*

This Lemma follows directly from the spectral decomposition of the positive operator T.

The following Lemma is due to K. H. Zhu[6, Lemma 13] for $\Omega=B_n$.

Lemma 4 *Suppose T is an operator in the trace class of $A^2(B_n,\mathrm{d}V_\alpha)$. Then*

$$\begin{aligned}\mathrm{tr}(T)&=\int_{B_n}\langle TK_\alpha(\cdot,z),K_\alpha(\cdot,z)\rangle_{L^2(B_n,\mathrm{d}V_\alpha)}\mathrm{d}V_\alpha(z)\\&=\int_{B_n}\langle Tk^\alpha_z,k^\alpha_z\rangle_{L^2(B_n,\mathrm{d}V_\alpha)}K_\alpha(z,z)\mathrm{d}V_\alpha(z).\end{aligned}$$

The following Lemma is due to K. H. Zhu[6, Theorem 12] for $\Omega=B_n$.

Lemma 5 *Suppose $\mu\ge0$ is a finite Borel measure on B_n. If $p\ge1$ and $r>0$, then the following conditions are equivalent:*

(1) $T_\mu\in S_p$; (2) $\mu^\sim_\alpha\in L^p(B_n,\mathrm{d}\lambda(z))$; (3) $\mu^\wedge_r\in L^p(B_n,\mathrm{d}\lambda(z))$;

(4) $\sum_{n=1}^\infty\left(\frac{\mu(D(a_n,r))}{|D(a_n,r)|^\alpha}\right)^p=\sum_{n=1}^\infty(\mu^\wedge_r(a_n))^p<\infty$,

where $\{a_n\}$ is the sequence given by Lemma 2.6 of [6], $\mu^\sim_\alpha(z)=\int_{B_n}|k_z(w)|^{2\alpha}\mathrm{d}\mu(w)$, $\mu^\wedge_r(z)=\frac{\mu(D(z,r))}{|D(z,r)|^\alpha}$.

Lemma 6[4] *For any $\psi\in A^2(B_n)$, $\varphi\in H(B_n,B_n)$, g is a positive Lebesgue measurable function on B_n, then*

$$\int_{B_n}gd\mu_{\varphi,\psi}=\int_{B_n}(g\circ\varphi)|\psi|^2\mathrm{d}V,$$

where $\mu_{\varphi,\psi}(A)=\int_{\varphi^{-1}(A)}|\psi|^2\mathrm{d}V$ for given Borel set $A\subset B_n$.

This is just Lemma 4.2 of H. M. Xu and T. S. Liu[4] for the case $p=2$.

3. Schatten class weighted composition operators

Our problem in this section is to find necessary and sufficient conditions on ψ and φ that will ensure that the weighted composition operator $W_{\varphi,\psi}$ belongs to S_p. We can now show the main results of the paper.

Theorem 1 *For any $\psi \in H(B_n)$, $\varphi \in H(B_n, B_n)$, let the weighted composition operator $W_{\varphi,\psi}$ be compact operator on $A^2(B_n)$, then*

(a) If $0 < p \le 2$ and $Q_{\varphi,\psi} \in L^p(B_n, \mathrm{d}\lambda)$, then $W_{\varphi,\psi} \in S_p$;

(b) If $2 \le p < \infty$ and $W_{\varphi,\psi} \in S_p$, then $Q_{\varphi,\psi} \in L^p(B_n, \mathrm{d}\lambda)$,

where $\mathrm{d}\lambda(z) = K(z,z)\mathrm{d}V(z)$.

Proof If $0 < p \le 2$, and $Q_{\psi,\varphi} \in L^p(B_n, \mathrm{d}\lambda)$, using Lemmas 2, 3 and 4, we have that

$$\begin{aligned}
\|W_{\varphi,\psi}\|_{S_p}^p &= \operatorname{tr}\left(((W_{\varphi,\psi})^* W_{\varphi,\psi})^{p/2}\right)\\
&\le \int_{B_n} \langle (W_{\varphi,\psi})^* W_{\varphi,\psi} k_z, k_z\rangle_{A^2(B_n)}^{p/2} K(z,z)\mathrm{d}V(z)\\
&= \int_{B_n} |Q_{\varphi,\psi}(z)|^p K(z,z)\mathrm{d}V(z) = \int_{B_n} |Q_{\varphi,\psi}(z)|^p \mathrm{d}\lambda(z),
\end{aligned}$$

so (a) follows. If $2 \le p < \infty$ and $W_{\varphi,\psi} \in S_p$, by using Lemmas 2, 3 and 4, it follows that

$$\begin{aligned}
\|W_{\varphi,\psi}\|_{S_p}^p &= \operatorname{tr}\left(((W_{\varphi,\psi})^* W_{\varphi,\psi})^{p/2}\right)\\
&\ge \int_{B_n} \langle (W_{\varphi,\psi})^* W_{\varphi,\psi} k_z, k_z\rangle_{A^2(B_n)}^{p/2} K(z,z)\mathrm{d}V(z)\\
&= \int_{B_n} |Q_{\varphi,\psi}(z)|^p K(z,z)\mathrm{d}V(z) = \int_{B_n} |Q_{\varphi,\psi}(z)|^p \mathrm{d}\lambda(z).
\end{aligned}$$

So $Q_{\varphi,\psi} \in L^p(B_n, \mathrm{d}\lambda)$, and (b) immediately follows.

Remark In Theorem 2 we will prove that $W_{\varphi,\psi} \in S_p$ is also a necessary condition for $Q_{\varphi,\psi}$ to be in $L^p(B_n, \mathrm{d}\lambda)$, if $2 \le p < \infty$.

Theorem 2 *Suppose $p \ge 2$. For any $\psi \in H(B_n)$, $\varphi \in H(B_n, B_n)$, let the weighted composition operator $W_{\varphi,\psi}$ be a compact operator on $A^2(B_n)$. Then $W_{\varphi,\psi} \in S_p$ if and only if $Q_{\varphi,\psi} \in L^p(B_n, \mathrm{d}\lambda)$, where $\mathrm{d}\lambda(z) = K(z,z)\mathrm{d}V(z)$.*

Proof Given a Borel set $A \subset B_n$, recall that

$$\mu_{\varphi,\psi}(A) = \int_{\varphi^{-1}(A)} |\psi|^2 \mathrm{d}V.$$

Then for any $f, g \in A^2(B_n)$, we can write by Lemma 6

$$\begin{aligned}
&\langle (W_{\varphi,\psi})^* W_{\varphi,\psi} f, g\rangle_{A^2(B_n)} = \langle W_{\varphi,\psi} f, W_{\varphi,\psi} g\rangle_{A^2(B_n)}\\
&= \int_{B_n} W_{\varphi,\psi} f(z)\overline{W_{\varphi,\psi} g(z)}\mathrm{d}V(z) = \int_{B_n} f(\varphi(z))\overline{g(\varphi(z))}|\psi(z)|^2 \mathrm{d}V(z)\\
&= \int_{B_n} f(w)\overline{g(w)}\mathrm{d}\mu_{\varphi,\psi}(w).
\end{aligned}$$

Considering the Toeplitz operator

$$T_{\mu_{\varphi,\psi}}(z)=\int_{B_n} f(w)K_w(z)\mathrm{d}\mu_{\varphi,\psi}(w),$$

we have that by Fubini's theorem

$$\begin{aligned}\langle T_{\mu_{\varphi,\psi}}f,g\rangle_{A^2(B_n)}&=\int_{B_n} T_{\mu_{\varphi,\psi}}f(z)\overline{g(z)}\mathrm{d}V(z)\\&=\int_{B_n}\left(\int_{B_n} f(w)K_w(z)\mathrm{d}\mu_{\varphi,\psi}(w)\right)\overline{g(z)}\mathrm{d}V(z)\\&=\int_{B_n} f(w)\overline{\left(\int_{B_n} g(z)K_z(w)\mathrm{d}V(z)\right)}\mathrm{d}\mu_{\varphi,\psi}(w)\\&=\int_{B_n} f(w)\overline{g(w)}\mathrm{d}\mu_{\varphi,\psi}(w).\end{aligned}$$

Thus $(W_{\varphi,\psi})^*W_{\varphi,\psi}=T_{\mu_{\varphi,\psi}}$. We obtain that by Lemmas 1 and 5, $W_{\varphi,\psi}\in S_p$ if and only if $T_{\mu_{\varphi,\psi}}\in S_{p/2}$, and if and only if $\mu^{\sim}_{\varphi,\psi}\in L^{p/2}(B_n,\mathrm{d}\lambda)$. We rewrite the Berezin transform of $\mu^{\sim}_{\varphi,\psi}$ in the form

$$\begin{aligned}\mu^{\sim}_{\varphi,\psi}(z)&=\int_{B_n}|k_z(w)|^2\mathrm{d}\mu_{\varphi,\psi}(w)=\int_{B_n}|k_z(\varphi(w)|^2|\psi(w)|^2\mathrm{d}V(w)\\&=\int_{B_n}\frac{(1-|z|^2)^{n+1}}{|1-z\overline{\varphi(w)}|^{2n+2}}|\psi(w)|^2\mathrm{d}V(w)=Q^2_{\varphi,\psi}(z).\end{aligned}$$

Thus, $W_{\varphi,\psi}\in S_p$ if and only if $Q_{\varphi,\psi}\in L^p(B_n,\mathrm{d}\lambda)$. The proof is completed. □

The corollary below is a direct consequence of Theorem 2, Fubini's theorem and the normalized reproducing kernel.

Corollary *For any $\psi\in H(B_n)$, $\varphi\in H(B_n,B_n)$, let the weighted composition operator $W_{\varphi,\psi}$ be a compact operator on $A^2(B_n)$. Then $W_{\varphi,\psi}$ is a Hilbert-Schmidt operator if and only if*

$$\int_{B_n}\frac{|\psi(w)|^2}{(1-|\varphi(w)|^2)^{n+1}}\mathrm{d}V(w)<\infty.$$

Proof By Theorem 2, $W_{\varphi,\psi}\in S_2$ if and only if $Q_{\varphi,\psi}\in L^2(B_n,\mathrm{d}\lambda)$ which means that

$$\begin{aligned}I&=\int_{B_n}Q^2_{\varphi,\psi}(z)\mathrm{d}\lambda(z)\\&=\int_{B_n}\left(\int_{B_n}\frac{(1-|z|^2)^{n+1}}{|1-z\overline{\varphi(w)}|^{2n+2}}|\psi(w)|^2\mathrm{d}V(w)\right)\frac{1}{(1-|z|^2)^{n+1}}\mathrm{d}V(z)\\&<\infty.\end{aligned}$$

By Fubini's theorem,

$$\begin{aligned}I&=\int_{B_n}\left(\int_{B_n}\frac{1}{|1-z\overline{\varphi(w)}|^{2n+2}}\mathrm{d}V(z)\right)|\psi(w)|^2\mathrm{d}V(w)\\&=\int_{B_n}\frac{|\psi(w)|^2}{(1-|\varphi(w)|^2)^{n+1}}\mathrm{d}V(w).\end{aligned}$$

Thus, $W_{\varphi,\psi} \in S_2$ if and only if

$$\int_{B_n} \frac{|\psi(w)|^2}{(1-|\varphi(w)|^2)^{n+1}} \mathrm{d}V(w) < \infty,$$

as we claimed.

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References:

[1] CONTRERAS M D, HERNÁNDEZ-DIAZ A G. *Weighted composition operators on Hardy spaces* [J]. J. Math. Anal. Appl., 2001, **263**(1): 224–233.

[2] ZHOU Ze-hua. *Weighted Composition Operators on the Lipschitz Space in Polydiscs* [M]. Geometric Function Theory in Several Complex Variables, 333–342, World Sci. Publ., River Edge, NJ, 2004.

[3] LI Song-xiao, ZHU Xiang-ling. *Essential norm of weighted composition operator between α-Bloch space and β-Bloch space in polydiscs* [J]. Int. J. Math. Math. Sci., 2004, **69-72**: 3941–3950.

[4] XU Hui-ming, LIU Tai-shun. *Weighted composition operators between Hardy spaces on the unit ball* [J]. Chinese Quart. J. Math., 2004, **19**(2): 111–119.

[5] GOHBERG I C, KREIN M G. *Introduction to the Theory of Linear Non-Selfadjoint Operators* [M]. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18 American Mathematical Society, Providence, R.I. 1969.

[6] ZHU Ke-he. *Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains* [J]. J. Operator Theory, 1988, **20**(2): 329–357.

[7] Xia Jing-bo. *On the Schatten class membership of Hankel operators on the unit ball* [J]. Illinois J. Math., 2002, **46**(3): 913–928.

[8] XU Xian-min. *Schatten classes of composition operators on weighted Bergman spaces* [J]. Chinese Ann. Math. Ser. A, 1999, **20**(3): 283–286. (in Chinese)

[9] ZHU Ke-he. *Operator Theory in Function Spaces* [M]. Marcel Dekker, New York, 1990.

[10] LIU Yong-min. *Schatten class of Toeplitz operators on weighted Bergman spaces* [J]. Chinese Ann. Math. Ser. A, 1999, **20**(1): 53–56. (in Chinese)

[11] Liu Yong-min. *Schatten class composition operators on $A^2(\Omega, \mathrm{d}V_\alpha)$* [J]. Chinese Ann. Math. Ser. A, 2001, **22**(6): 735–742. (in Chinese)

[12] LUECKING D H, ZHU Ke-he. *Composition operators belonging to the Schatten ideals* [J]. Amer. J. Math., 1992, **114**(5): 1127–1145.

[13] ARAZY J, FISHER S D, PEETRE J. *Hankel operators on weighted Bergman spaces* [J]. Amer. J. Math., 1988, **110**(6): 989–1053.

单位球的 Bergman 空间上 Schatten 类加权复合算子

刘永民 [1]， 于燕燕 [2]

(1. 徐州师范大学数学系，江苏 徐州 221116; 2. 徐州工程学院计算科学系，江苏 徐州 221008)

摘要: 本文研究了单位球的 Bergman 空间上 Schatten 类加权复合算子, 得到了这种加权复合算子属于 Schatten-Von Neumann 理想 S_p 的几个充要条件. 作为推论给出了 $W_{\varphi,\psi}$ 是一个 Hilbert-Schmidt 算子的充要条件是

$$\int_{B_n} \frac{|\psi(w)|^2}{(1-|\varphi(w)|^2)^{n+1}} \mathrm{d}V(w) < \infty.$$

关键词: Bergman 空间；加权复合算子； Schatten- 类； Hilbert-Schmidt 算子.