第27卷第2期 数 学 研 究 与 评 论 Vol.27, No.2
2007年5月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION May, 2007

Article ID: 1000-341X(2007)02-0253-11 **Document code**: A

Counting Dyck Paths with Strictly Increasing Peak Sequences

SUN Yi-dong, JIA Cang-zhi
(Department of Applied Mathematics, Dalian University of Technology, Liaoning 116024, China)
(E-mail: sydmath@yahoo.com.cn)

Abstract: In this paper we consider the enumeration of subsets of the set, say $\mathcal{D}_m$, of those Dyck paths of arbitrary length with maximum peak height equal to m and having a strictly increasing sequence of peak height (as one goes along the path). Bijections and the methods of generating trees together with those of Riordan arrays are used to enumerate these subsets, resulting in many combinatorial structures counted by such well-known sequences as the Catalan nos., Narayana nos., Motzkin nos., Fibonacci nos., Schröder nos., and the unsigned Stirling numbers of the first kind. In particular, we give two configurations which do not appear in Stanley's well-known list of Catalan structures.

Key words: Generating tree; Riordan array; Catalan numbers; Schröder numbers.
MSC(2000): 05A05
CLC number: O157

1. Introduction

Dyck paths are very well-known combinatorial objects that have been widely studied in the literature. Stanley's book Enumerative Combinatorics and its addendum (http://www-math.mit.edu/ rstan/ec/catadd.pdf) list over 95 collections of objects equivalent to Dyck paths of a fixed length, all of which are counted by the Catalan sequence $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$. Many various points of view[14,16,17,21,22] have thrown light on Dyck paths, such as area[6,7,13,32], bounce score[15,18], bounce count[18], pyramid weight and number of exterior pairs[9], number of udu's[29]. There are many others[10,23] who have made important and earlier studies regarding statistics on Dyck paths.

Definition 1.1 *A Dyck path is a lattice path in the first quadrant, which begins at the origin* $(0,0)$, *ends at* $(2n,0)$ *and consists of steps* $(1,1)$ *and* $(1,-1)$. *We can encode each* $(1,1)$ *step by the letter* u *(for up), each* $(1,-1)$ *step by the letter* d *(for down), obtaining the encoding of Dyck path by a so-called Dyck word. A peak in a Dyck path is an occurrence of* ud, *a valley is an occurrence of* du. *By the height of a peak or of a valley we mean the height of the intersection point of its two steps. By a return step we mean a* d *step ending at height* 0. *A Dyck path with one return step is called primitive.*

In the sequel, we will at times use interchangeably a Dyck path W and its Dyck word W. We will refer to $l(W) = n$ as the semilength of the path.

Received date: 2005-02-26; **Accepted date**: 2005-07-19

Given a Dyck path(word) $W = u^{\alpha_1}d^{\beta_1}\cdots u^{\alpha_i}d^{\beta_i}\cdots u^{\alpha_t}d^{\beta_t}$ with $t(\geq 1)$ peaks, where α_i, β_i are positive integers for $i \in [t]$, it is clear that the following conditions must be satisfied:

(1) $\sum_{j=1}^{i}(\alpha_j - \beta_j) \geq 0$ for $i \in [t]$.

(2) $\alpha_1 + \alpha_2 + \cdots + \alpha_t = \beta_1 + \beta_2 + \cdots + \beta_t = l(W)$.

Define the blocks of up steps $(\alpha_1, \alpha_2, \ldots, \alpha_t)$ to be the *U-sequence*, the blocks of down steps $(\beta_1, \beta_2, \ldots, \beta_t)$ to be the *D-sequence*, the height of peaks $(p_1, p_2, \ldots, p_t)$ to be the *P-sequence* and the height of valleys $(v_1, v_2, \ldots, v_t)$ to be the *V-sequence*, where $p_i = \sum_{j=1}^{i}(\alpha_j - \beta_{j-1})$ for $i \in [t]$ is the height of the i-th peak of W and $v_i = \sum_{j=1}^{i}(\alpha_j - \beta_j)$ for $i \in [t-1]$ is the height of the i-th valley of W (left to right). For convenience we set $v_0 = \beta_0 = 0$ and sometimes we set $v_0 = -1$ if necessary.

Conversely, any two of these four sequences can uniquely determine α_i and β_i for $i \in [t]$, which in turn can uniquely determine a Dyck path, if they satisfy the conditions (1) and (2).

In this paper we consider the enumeration of subsets of the set, say $\mathcal{D}_m$, of those Dyck paths of any length with maximum peak height equal to m and having a strictly increasing sequence of peak height (as one goes along the path). Bijections and the methods of generating trees together with those of Riordan arrays are used to enumerate these subsets, resulting in such well-known sequences as the Catalan nos., Narayana nos., Motzkin nos., Fibonacci nos., Schröder nos., and the unsigned Stirling numbers of the first kind. In particular, we give two configurations which do not appear in Stanley's well-known list of Catalan structures. The first of these configurations is the set of those paths in $\mathcal{D}_m$ with strictly increasing blocks of down steps (as one goes along the path). The other configuration corresponds to the set of those paths in $\mathcal{D}_m$ with increasing valley heights (as one goes along the path).

The paper will precisely be organized as follows. The first section will dedicate to the necessary machinery such as Riordan arrays and generating trees, which play an important role in the paper. In Section 3 we consider the set $\mathcal{D}_m$ which turns out to be equivalent to $\mathcal{S}_m$, the set of permutations of $[m] = \{1, 2, \ldots, m\}$; and in the consequent subsections the following statistics related to $\mathcal{D}_m$ are of interest: *D-sequence*, *P-sequence* and *V-sequence*. Subject to some restricted conditions in these sequences, we investigate many new settings counted by the above-mentioned classical numbers.

2. Riordan arrays and generating trees

Definition 2.1 *A proper Riordan array is an infinite lower triangular array $\{R_{m,k}\}_{m,k\in\mathbb{N}}$, defined by a pair of formal power series $R = (R_{m,k}) = (d(x), h(x))$, with the generic element $R_{m,k}$ satisfying*

$$R_{m,k} = [x^m]d(x)(h(x))^k,$$

where $d(x) = d_0 + d_1x + d_2x^2 + \cdots$ and $h(x) = h_1x + h_2x^2 + \cdots$ with $d_0, h_1 \neq 0$.

Proper Riordan array can also be defined in terms of two sequences $A = \{a_0, a_1, a_2, \ldots\}$ with $a_0 \neq 0$ and $Z = \{z_0, z_1, z_2, \ldots\}$[20,26,28] such that every entry $R_{m+1,k+1}$ or $R_{m+1,0}$ can be expressed as a linear combination as follows:

$$R_{m+1,k+1} = a_0 R_{m,k} + a_1 R_{m,k+1} + a_2 R_{m,k+2} + \cdots,$$
$$R_{m+1,0} = z_0 R_{m,0} + z_1 R_{m,1} + z_2 R_{m,2} + \cdots.$$

Let $A(x)$ and $Z(x)$ be the generating functions of these two sequences A and Z, respectively. Then the pair $(d(x), h(x))$ can be determined uniquely by the following formulas:

$$h(x) = xA(h(x)), \quad d(x) = \frac{d_0}{1 - xZ(h(x))}.$$

Definition 2.2 *A generating tree is a rooted labelled tree with the property that the labels of the set of children of each node v can be determined from the label of v itself. To specify a generating tree, it suffices to specify*

(i) the label of the root, and

(ii) a set of succession rules explaining how to derive, given the label of a father, the quantity of children and their labels.

Chung et al[7] introduced the technique of generating tree, which was later used and named by West[30,31]. This approach has been exploited further in related problems[4,5,11,12], and developed further by the authors[1−3,24]. Using the notation of West[30], we are interested in the number of nodes on level m, denoted by Σ_m with the root on level 0. Sometimes it is more convenient to start with the root on level 1.

Example 2.3 The Catalan generating tree is generated by the rule:

$$\begin{array}{ll} \text{Root}: & (2). \\ \text{Rule}: & (k) \rightarrow (2)(3)\cdots(k+1). \end{array}$$

Example 2.4 The Schröder generating tree is generated by the rule:

$$\begin{array}{ll} \text{Root}: & (2). \\ \text{Rule}: & (k) \rightarrow (3)(4)\cdots(k+1)(k+1). \end{array}$$

In order to obtain Σ_m, the following lemma[19] will be very useful.

Lemma 2.5 *Let $c \in \mathbb{N}$, $A_j,\ Z_j \in \mathbb{N}$ for $j \geq 0$, $A_0 \neq 0$ and $k \geq c$ and let*

$$\begin{array}{ll} \textit{Root}: & (c) \\ \textit{Rule}: & (k) \rightarrow (c)^{Z_{k-c}}(c+1)^{A_{k-c}}(c+2)^{A_{k-c-1}}\cdots(k+1)^{A_0} \end{array}$$

be a generating tree specification. Define $A(x) = \sum_{i\geq 0} A_i x^i$ and $Z(x) = \sum_{i\geq 0} Z_i x^i$. Then a proper Riordan array $R = (R_{m,k}) = (d(x), h(x))$ is obtained. Namely,

$$R_{m,k} = [x^m]d(x)(h(x))^k,$$

where $h(x)$ and $d(x)$ are defined by

$$h(x) = xA(h(x)), \quad d(x) = \frac{1}{1 - xZ(h(x))}.$$

Specially, one can obtain the level number

$$\Sigma_m = \sum_{k=0}^{m} R_{m,k} = \sum_{k=0}^{m} [x^m] d(x)(h(x))^k = [x^m] \frac{d(x)}{1-h(x)}.$$

Example 2.6 Considering the Catalan generating tree, by Lemma 3.1, we have

$$A(x) = \sum_{i\geq 0} x^i = \frac{1}{1-x}, \quad Z(x) = \sum_{i\geq 0} x^i = \frac{1}{1-x}; \quad h(x) = xC(x), \quad d(x) = C(x).$$

Then for $m \geq 1$, taking the root on level 1, one obtains

$$\Sigma_m = [x^{m-1}] \frac{d(x)}{1-h(x)} = [x^{m-1}] \frac{C(x)}{1-xC(x)} = [x^{m-1}] \frac{C(x)-1}{x} = C_m,$$

where $C(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{m\geq 0} C_m x^m$ is the generating function for the Catalan numbers.

Example 2.7 Considering the Schröder generating tree, by Lemma 3.1, we have

$$A(x) = 1 + \sum_{i\geq 0} x^i = \frac{2-x}{1-x}, \quad Z(x) = 0; \quad h(x) = \frac{1+x-\sqrt{1-6x+x^2}}{2}, \quad d(x) = 1.$$

Then for $m \geq 0$, taking the root on level 0, one obtains

$$\Sigma_m = [x^m] \frac{d(x)}{1-h(x)} = [x^m] \frac{1}{1 - \frac{1+x-\sqrt{1-6x+x^2}}{2}} = [x^m] \frac{1-x-\sqrt{1-6x+x^2}}{2x} = r_m,$$

where r_m is the mth large Schröder number, defined by $\sum_{m\geq 0} r_m x^m = \frac{1-x-\sqrt{1-6x+x^2}}{2x}$.

3. Dyck paths with the height of peaks strictly increasing

In this section we study Dyck paths with the height of peaks strictly increasing from left to right. Let $\mathcal{D}_m$ denote the set of such paths with the last peak at height m and let D_m be its cardinality. It turns out that $\mathcal{D}_m$ is equivalent to the set $\mathcal{S}_m$ of permutations of $[m]$. Let $\mathrm{Inv}(\pi)$ denote the number of inversions of $\pi \in \mathcal{S}_m$, where an *inversion* of π is a pair (i, j) with $1 \leq i < j \leq m$ such that $\pi(i) > \pi(j)$.

Then the relation between $\mathcal{D}_m$ and $\mathcal{S}_m$ is described as follows

Theorem 3.1 *There exists a simple bijection φ between $\bigcup_{m\geq 1} \mathcal{D}_m$ and $\bigcup_{m\geq 1} \mathcal{S}_m$ such that $\varphi(\mathcal{D}_m) = \mathcal{S}_m$ and $\mathrm{Inv}(\varphi(W)) = l(W) - m$ for $W \in \mathcal{D}_m$.*

Proof We now construct recursively such a bijection as follows. For $W' \in \mathcal{D}_{m+1}$, i.e., $W' = u^{\alpha_1} d^{\beta_1} \cdots u^{\alpha_t} d^{\beta_t} u^{\alpha_{t+1}} d^{m+1}$ for some $0 \leq t \leq m$ such that $0 = p_0 < p_1 < \cdots < p_t < p_{t+1} = m+1$. Let

$$W = \begin{cases} u^{\alpha_1} d^{\beta_1} \cdots u^{\alpha_t} d^{\beta_t} u^{\alpha_{t+1}-1} d^m, & \text{if } \ p_t < m, \\ u^{\alpha_1} d^{\beta_1} \cdots u^{\alpha_t} d^m, & \text{if } \ p_t = m, \end{cases}$$

which is a member of $\mathcal{D}_m$.

Suppose that $\varphi(W) = \pi \in \mathcal{S}_m$. If $p_t < m$, then $\varphi(W') = \pi\overline{m+1} \in \mathcal{S}_{m+1}$; If $p_t = m$, then $\varphi(W') = \pi' \in \mathcal{S}_{m+1}$ by inserting $m+1$ into π in the position between $m+1-\alpha_{t+1}$ and $m+2-\alpha_{t+1}$.

On the other hand, for $\pi' \in \mathcal{S}_{m+1}$, let $\pi'(m+1-\gamma) = m+1$ for some $0 \le \gamma \le m$ and let π denote the permutations obtained by deleting $m+1$ in π'. Suppose $\varphi^{-1}(\pi) = W = u^{\alpha_1}d^{\beta_1}\cdots u^{\alpha_{t-1}}d^{\beta_{t-1}}u^{\alpha_t}d^m$, then $\varphi^{-1}(\pi') = W'$ is defined by

$$W' = \begin{cases} u^{\alpha_1}d^{\beta_1}\cdots u^{\alpha_t}d^{\gamma}u^{\gamma+1}d^{m+1}, & \text{if } \gamma \in [m], \\ u^{\alpha_1}d^{\beta_1}\cdots u^{\alpha_t+1}d^{m+1}, & \text{if } \gamma = 0, \end{cases}$$

which belongs to $\mathcal{D}_{m+1}$.

Hence φ is indeed a bijection. Note that from $\varphi(W')$ to $\varphi(W)$, the number of inversions and semilength decrease, respectively, by γ and $\gamma+1$. So it is easy to prove, by induction on m, that $\mathrm{Inv}(\varphi(W)) = l(W) - m$ for $W \in \mathcal{D}_m$. □

For example $W = u^2d^1u^3d^4u^5d^3u^4d^3u^4d^7 \in \mathcal{D}_7$, then $\varphi(W) = 5167324$ and $\mathrm{Inv}(\varphi(W)) = 11$. The detailed construction is described as follows

$$\begin{array}{lllllll} W & \leftarrow u^2d^1u^3d^4u^5d^3u^4d^6 & \leftarrow u^2d^1u^3d^4u^5d^5 & \leftarrow u^2d^1u^3d^4 & \leftarrow u^2d^1u^2d^3 & \leftarrow u^2d^2 & \leftarrow ud \\ 5167324 & \leftarrow 516324 & \leftarrow 51324 & \leftarrow 1324 & \leftarrow 132 & \leftarrow 12 & \leftarrow 1. \end{array}$$

For some restricted conditions, say $*$, let $\mathcal{D}_m(*)$ denote the set of Dyck paths in $\mathcal{D}_m$ satisfying $*$ and let $D_m(*)$ denote its cardinality. For examples, $\mathcal{D}_m(D\uparrow)$ denotes the set of Dyck paths in $\mathcal{D}_m$ such that their D-sequence strictly increases; $\mathcal{D}_m(D\uparrow, V\nearrow)$ denotes the set of Dyck paths in $\mathcal{D}_m$ such that their D-sequence strictly increases and their V-sequence is not decreasing. In the following subsections a lot of counting problems are considered, and classical numbers such as the unsigned Stirling numbers of the first kind, Catalan numbers, ballot numbers, Fibonacci numbers, Narayana numbers, Motzkin numbers, and Schröder numbers are obtained.

3.1 Stirling distribution of the first kind

In this subsection we consider three statistics counted by the unsigned Stirling number of the first kind, i.e., number of udu's, number of peaks, number of return steps of Dyck paths in $\mathcal{D}_m$. Note that it seems to be the first time that the unsigned Stirling number of the first kind counts these configurations related to Dyck paths in the literature.

The unsigned Stirling numbers of the first kind $c_{m,k}$ are defined by

$$\sum_{k=0}^{m} c_{m,k}x^k = \prod_{i=0}^{m-1}(x+i),$$

which leads to the recurrence relation

$$c_{m+1,k} = mc_{m,k} + c_{m,k-1}$$

with the initial values $c_{m,0} = 0$, $c_{m,m} = 1$ for $m \ge 1$ with $c_{0,0} = 1$.

Let $\lambda_{m,k}$, $\eta_{m,k}$ and $\zeta_{m,k}$ denote the number of Dyck paths in $\mathcal{D}_m$ with $k-1$ udu's, $m-k+1$ peaks, and respectively k return steps for $k \in [m]$. Then we have

Theorem 3.2 *For integers $m,k\ge 1$,*

$$\lambda_{m,k}=\eta_{m,k}=\zeta_{m,k}=c_{m,k}.$$

Proof Following the proof of Theorem 3.1, from $\varphi^{-1}(\pi)$ to $\varphi^{-1}(\pi')$, we can see that the number of peaks does not change when $\gamma=0$ and increases by one when $\gamma\in[m]$; the number of udu's does not change when $\gamma\ne 1$ and increases by one when $\gamma=1$; the number of return steps does not change when $\gamma\ne m$ and increases by one when $0\le\gamma<m$. It is easily verifiable that $\lambda_{m,k}$, $\eta_{m,k}$ and $\zeta_{m,k}$ satisfy the same recurrence relation as $c_{m,k}$ with the same initial conditions. Hence they must be equal. □

3.2 Counting $\mathcal{D}_m(D\uparrow)$

Theorem 3.3 *For any integer $m\ge 1$, the cardinality of $\mathcal{D}_m(D\uparrow)$ is a Catalan number, i.e.,*

$$D_m(D\uparrow)=C_m.$$

Proof Here we give a bijection between $\mathcal{D}_m(D\uparrow)$ and $\tilde{\mathcal{W}}_m$, the set of Dyck paths of semilength m. Let $W\in\mathcal{D}_m(D\uparrow)$ with P-sequence $(p_0=0,p_1,p_2,\ldots,p_t=m)$ and D-sequence $(d_0=0,d_1,d_2,\ldots,d_t=m)$ for some $t\in[m]$. Clearly, $p_i\ge d_i$ for $i\in[t]$. Define $\tilde{W}=u^{\alpha_1}d^{\beta_1}\cdots u^{\alpha_i}d^{\beta_i}\cdots u^{\alpha_t}d^{\beta_t}$ with $\alpha_i=p_i-p_{i-1}$ and $\beta_i=d_i-d_{i-1}$ for $i\in[t]$. A routine verification shows that α_i's and β_i's satisfy the conditions (1) and (2). So we have $\tilde{W}\in\tilde{\mathcal{W}}_m$.

Conversely, for $\tilde{W}=u^{\alpha_1}d^{\beta_1}u^{\alpha_2}d^{\beta_2}\cdots u^{\alpha_t}d^{\beta_t}\in\tilde{\mathcal{W}}_m$ with t peaks, define a Dyck path W with its P-sequence $(p_1,p_2,\ldots,p_j,\ldots,p_t)$ and D-sequence $(d_1,d_2,\ldots,d_j,\ldots,d_t)$ such that $p_j=\sum_{i=1}^{j}\alpha_i$ and $d_j=\sum_{i=1}^{j}\beta_i$ for $j\in[t]$. Clearly, $1\le p_1<p_2<\cdots<p_t=m$ and $1\le d_1<d_2<\cdots<d_t=m$. So we have $W\in\mathcal{D}_m(D\uparrow)$. □

Theorem 3.4 *The number of Dyck paths in $\mathcal{D}_m(D\uparrow)$ with k peaks is the Narayana number $N_{m,k}=\frac{1}{m}\binom{m}{k}\binom{m}{k-1}$ for $m\ge k\ge 1$.*

Proof Note that in the bijection from W to $\tilde{W}$ in the proof of Theorem 3.3, the number of peaks does not change. It is well known that the number of Dyck paths of semilength m with k peaks is the Narayana number $N_{m,k}=\frac{1}{m}\binom{m}{k}\binom{m}{k-1}$. □

Theorem 3.5 *The number of Dyck paths in $\mathcal{D}_{m+1}(D\uparrow)$ with D-sequence $(d_0=0,d_1,d_2,\ldots,d_t=m+1)$ for some $t\in[m+1]$ such that $d_j-d_{j-1}\ge 2$ for $j\in[t-1]$ is the mth Motzkin numbers M_m defined by $M(x)=\sum_{m\ge 0}M_mx^m=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$.*

Proof Note that in the bijection from W to $\tilde{W}$ in the proof of Theorem 3.3, if $d_j-d_{j-1}\ge 2$ for $j\in[t-1]$, then all $\beta_j\ge 2$ except for β_t, in other words, $\tilde{W}$ has no udu's. It has been proved in [25,29] that the number of Dyck paths of semilength $m+1$ without udu's is the mth Motzkin number. □

Theorem 3.6 *The number of Dyck paths in $\mathcal{D}_{\Updownarrow+\infty}(\mathcal{D}\uparrow)$ with D-sequence $(d_0=0,d_1,d_2,\ldots,d_t=m+1)$ for some $t\in[m+1]$ such that $d_{t-1}=k$ is counted by the ballot number*

$b_{m,k} = \frac{m-k+1}{m+1}\binom{m+k}{m}$.

Proof Note that in the bijection from W to $\tilde{W}$ in the proof of Theorem 3.3, if $d_{t-1} = k$, then $\tilde{W}$ ends with $m+1-k$ consecutive d steps. It is known that the number of Dyck paths of semilength $m+1$ ending with $m+1-k$ consecutive d steps is determined by $[x^k]C(x)^{m+1-k} = \frac{m-k+1}{m+1}\binom{m+k}{m}$[10]. □

3.3 Counting $\mathcal{D}_m(V\uparrow)$

Theorem 3.7 *For any integer $m \geq 1$, the cardinality of $\mathcal{D}_m(V\uparrow)$ is a Catalan number, i.e.,*

$$D_m(V\uparrow) = C_m.$$

First proof We present a generating tree proof. Let $W = u^{\alpha_1}d^{\beta_1}u^{\alpha_2}d^{\beta_2}\cdots u^{\alpha_t}d^m \in \mathcal{D}_m(V\uparrow)$ for some $t \in [m]$. Let $\alpha_t = k$, clearly, $2 \leq k \leq m$ for $m \geq 2$ and the height of the last valley of W is $v_{t-1} = m-k$. Note that W can produce k Dyck paths $W_i = u^{\alpha_1}d^{\beta_1}u^{\alpha_2}d^{\beta_2}\cdots u^k d^i u^{i+1}d^{m+1} \in \mathcal{D}_{m+1}(V\uparrow)$ for $0 \leq i \leq k-1$. If $i \in [k-1]$, then each W_i can generate $i+1$ Dyck paths in $\mathcal{D}_{m+2}(V\uparrow)$; and if $i=0$, then $W_0 = u^{\alpha_1}d^{\beta_1}u^{\alpha_2}d^{\beta_2}\cdots u^{k+1}d^{m+1}$ can generate $k+1$ Dyck paths in $\mathcal{D}_{m+2}(V\uparrow)$. So the rule $(k) \to (2)(3)\cdots(k+1)$ with the root (2) corresponding to the Dyck path ud is obtained. Hence, by Examples 2.3 and 2.6, the result holds. □

Second proof Here we give a simple bijection ρ between $\mathcal{D}_m(V\uparrow)$ and $\mathcal{D}_m(D\uparrow)$. For $W \in \mathcal{D}_m(V\uparrow)$ with V-sequence $(-1 = v_0, v_1, v_2, \ldots, v_{t-1})$ for some $t \in [m]$, define $W' = u^{\alpha_1}d^{\beta_1}u^{\alpha_2}d^{\beta_2}\cdots u^{\alpha_t}d^m$ with D-sequence $(\beta_1, \beta_2, \ldots, \beta_{t-1}, \beta_t = m)$ such that $\beta_i = v_i + 1$ for $0 \leq i < t$ and that W' has the same P-sequence as W. A routine verification shows that $W' \in \mathcal{D}_m(D\uparrow)$. Conversely, it is also easy to determine ρ^{-1}. Hence the result holds. □

Corresponding to Theorems 3.4, 3.5 and 3.6, we have

Theorem 3.8 *(i). The number of Dyck paths in $\mathcal{D}_m(V\uparrow)$ with k peaks is the Narayana number $N_{m,k} = \frac{1}{m}\binom{m}{k}\binom{m}{k-1}$ for $m \geq k \geq 1$.*

(ii). The number of Dyck paths in $\mathcal{D}_{m+1}(V\uparrow)$ with V-sequence $(v_0 = -1, v_1, v_2, \ldots, v_{t-1})$ for some $t \in [m+1]$ such that $v_j - v_{j-1} \geq 2$ for $j \in [t-1]$ is the mth Motzkin numbers M_m.

(iii). The number of Dyck paths in $\mathcal{D}_{m+1}(V\uparrow)$ with V-sequence $(v_0 = -1, v_1, v_2, \ldots, v_{t-1})$ for some $t \in [m+1]$ such that $v_{t-1} = k-1$ is counted by the ballot number $b_{m,k} = \frac{m-k+1}{m+1}\binom{m+k}{m}$.

3.4 Counting $\mathcal{D}_m(V\nearrow)$ and $\mathcal{D}_m(D\nearrow)$

Theorem 3.9 *For any integer $m \geq 1$, the cardinality of $\mathcal{D}_m(V\nearrow)$ is a Schröder number, i.e.,*

$$D_m(V\nearrow) = r_{m-1}.$$

Proof We present a generating tree proof. Let $W = u^{\alpha_1}d^{\beta_1}u^{\alpha_2}d^{\beta_2}\cdots u^{\alpha_t}d^m \in \mathcal{D}_m(V\nearrow)$ for some $t \in [m]$. Let $\alpha_t = k-1$, then the height of the last valley of W is $v_{t-1} = m -$

$k+1$. Clearly, $3 \le k \le m+1$ for $m \ge 2$. Note that W can produce k Dyck paths $W_i = u^{\alpha_1}d^{\beta_1}u^{\alpha_2}d^{\beta_2}\cdots u^{k-1}d^iu^{i+1}d^{m+1} \in \mathcal{D}_{m+1}(V\nearrow)$ for $0 \le i \le k-1$. If $1 \le i \le k-1$, then each W_i can yield $i+2$ Dyck paths in $\mathcal{D}_{m+2}(V\nearrow)$; and if $i=0$, then $W_0 = u^{\alpha_1}d^{\beta_1}u^{\alpha_2}d^{\beta_2}\cdots u^kd^{m+1}$ can generate $k+1$ Dyck paths in $\mathcal{D}_{m+2}(V\nearrow)$. So the rule $(k) \to (3)(4)\cdots(k+1)(k+1)$ with the root (2) corresponding to the Dyck path ud is obtained. Hence, by Examples 2.4 and 2.7, the result follows. $\square$

Theorem 3.10 *For any integer $m \ge 1$, the cardinality of $\mathcal{D}_m(D\nearrow)$ is a Schröder number, i.e.,*

$$D_m(D\nearrow) = r_{m-1}.$$

Proof Similar to the second proof of Theorem 3.8, a simple bijection between $\mathcal{D}_m(V\nearrow)$ and $\mathcal{D}_m(D\nearrow)$ can be obtained. $\square$

3.5 Counting $\mathcal{D}_m(D\nearrow, V\nearrow)$

Theorem 3.11 *For any integer $m \ge 1$, we have*

$$D_m(D\nearrow, V\nearrow) = \sum_{j=0}^{[(m-1)/2]} \binom{m-1}{2j} 2^{m-j-1}.$$

Proof We give a generating tree proof. Let $W = u^{\alpha_1}d^{\beta_1}u^{\alpha_2}d^{\beta_2}\cdots u^{\alpha_t}d^m \in \mathcal{D}_m(D\nearrow, V\nearrow)$ for some $t \in [m]$. Let

$$k = \begin{cases} \alpha_t - \beta_{t-1} + 2, & \text{if } \beta_{t-1} \ge 1, \\ \alpha_t + 1, & \text{if } \beta_{t-1} = 0, \end{cases}$$

Clearly, $3 \le k \le m+1$ when $m \ge 2$ and $k = 2$ when $m = 1$. Note that when $\beta_{t-1} \ge 1$, W can produce k Dyck paths in $\mathcal{D}_{m+1}(D\nearrow, V\nearrow)$. Namely,

$$W_i = \begin{cases} u^{\alpha_1}d^{\beta_1}\cdots u^{\alpha_t}d^{\beta_{t-1}+i}u^{\beta_{t-1}+i+1}d^{m+1}, & \text{if } 0 \le i \le k-2, \\ u^{\alpha_1}d^{\beta_1}\cdots u^{\alpha_t+1}d^{m+1}, & \text{if } i = k-1; \end{cases}$$

and when $\beta_{t-1} = 0$, i.e., $W = u^md^m = u^{k-1}d^{k-1}$ which can yield k Dyck paths $W_i^{'} = u^md^{k-i-1}u^{k-i}d^{m+1} \in \mathcal{D}_{m+1}(D\nearrow, V\nearrow)$ for $0 \le i \le k-1$.

It is easy to see that W_i and $W_i^{'}$ can exactly generate 3 Dyck paths for $0 \le i \le k-2$, while W_{k-1} and $W_{k-1}^{'}$ can generate $k+1$ Dyck paths, all of which belong to $\mathcal{D}_{m+1}(D\nearrow, V\nearrow)$. So we obtain the rule $(k) \to (3)^{k-1}(k+1)$ with the root (2) corresponding to the Dyck path ud, namely,

$$\begin{array}{ll} \text{Root}: & (2). \\ \text{Rule}: & (k) \to (3)^{k-1}(k+1). \end{array}$$

Regarding the root on level one, taking (3) on level 2 to be the new root, one obtains two subtrees. Making use of Lemma 2.5, we have

$$A(x) = 1, \quad Z(x) = \sum_{i \ge 0}(i+2)x^i = \frac{2-x}{(1-x)^2}; \quad h(x) = x, \quad d(x) = \frac{(1-x)^2}{1-4x+2x^2}.$$

For $m \geq 2$, one obtains

$$\begin{aligned}
D_m(D \nearrow, V \nearrow) &= 2\Sigma_{m-2} = 2[x^{m-2}]\frac{d(x)}{1-h(x)} = 2[x^{m-2}]\frac{1-x}{1-4x+2x^2} \\
&= [x^{m-1}](1+2x\frac{1-x}{1-4x+2x^2}) = [x^{m-1}](\frac{1-2x}{1-4x+2x^2}) \\
&= [x^{m-1}]\frac{1}{1-2x}\frac{1}{1-\frac{2x^2}{(1-2x)^2}} = [x^{m-1}]\sum_{i\geq 0}\frac{(2x^2)^i}{(1-2x)^{2i+1}} \\
&= [x^{m-1}]\sum_{i\geq 0}\sum_{j\geq 0}\binom{2i+j}{j}2^{i+j}x^{2i+j} \\
&= \sum_{i\geq 0}\binom{m-1}{2i}2^{m-i-1}.
\end{aligned}$$

This is the desired result. □

Remark 1 It is easy to show that $D_m(D \nearrow, V \nearrow) = L_{m-1}$, where L_m satisfies the recurrence relation $L_{m+1} = 4L_m - 2L_{m-1}$ with $L_0 = 1, L_2 = 2$. See sequence A006012[27], where Callan pointed out that L_m counts the set $\mathcal{L}_{m+1}$ of permutations π on $[m]$ for which the pairs $(i, \pi(i))$ with $i < \pi(i)$, considered as half-open intervals $(i, \pi(i)]$, do not overlap; equivalently, for each $i \in [m]$ there is at most one $j \leq i$ with $\pi(j) > i$; and Kociemba showed that L_m also counts the set $\mathcal{H}_m$ of sequences $(s(0), s(1), \ldots, s(2m))$ such that $0 < s(i) < 8$ and $|s(i) - s(i-1)| = 1$ for $i \in [2m]$, provided that $s(0) = s(2m) = 4$. It may be interesting to find bijections between $\mathcal{D}_m(D \nearrow, V \nearrow)$, $\mathcal{L}_m$ and $\mathcal{H}_{m-1}$ for $m \geq 1$.

3.6 Counting $\mathcal{D}_m(D \uparrow, V \uparrow)$

Theorem 3.12 *For any integer $m \geq 1$, we have*

$$D_m(D \uparrow, V \uparrow) = 2^{m-1}.$$

First proof Similar to the proof of Theorem 3.11 the generating tree of $\mathcal{D}_m(D \uparrow, V \uparrow)$ is given by the rule:

$$\begin{aligned}
&\text{Root}: \quad (1) \\
&\text{Rule}: \quad (k) \to (1)^{k-1}(k+1),
\end{aligned}$$

where the root (1) corresponds to the empty Dyck path.

Making use of Lemma 2.5, we have

$$A(x) = 1, \quad Z(x) = \sum_{i\geq 0} ix^i = \frac{x}{(1-x)^2}; \quad h(x) = x, \quad d(x) = \frac{(1-x)^2}{1-2x}.$$

For $m \geq 1$, one obtains

$$D_m(D \uparrow, V \uparrow) = \Sigma_m = [x^m]\frac{d(x)}{1-h(x)} = [x^m]\frac{1-x}{1-2x} = 2^{m-1},$$

which completes the proof. □

Second proof Here we give a bijection λ between $\mathcal{D}_m(D\uparrow, V\uparrow)$ and the set $\mathcal{J}_m$ of compositions of m. For $W = u^{\alpha_1}d^{\beta_1}\dots u^{\alpha_t}d^m \in \mathcal{D}_m(D\uparrow, V\uparrow)$ for some $t \in [m]$ with V-sequence $(v_1, v_2, \dots, v_{t-1})$, define

$$\lambda(W) = \begin{cases} (\beta_1, \beta_2-\beta_1, \dots, \beta_{t-1}-\beta_{t-2}, m-\beta_{t-1}-v_{t-1}, v_{t-1}-v_{t-2}, \dots, v_2-v_1, v_1), & \text{if } v_1 \neq 0, \\ (\beta_2-\beta_1, \dots, \beta_{t-1}-\beta_{t-2}, m-\beta_{t-1}-v_{t-1}, v_{t-1}-v_{t-2}, \dots, v_3-v_2, v_2, \beta_1), & \text{if } v_1 = 0, \end{cases}$$

then $\lambda(W) \in \mathcal{J}_m$.

Conversely, for $J = (J_1, J_2, \dots, J_k) \in \mathcal{J}_m$, define $\lambda^{-1}(J) = u^{\alpha_1}d^{\beta_1}\cdots u^{\alpha_t}d^m$ with V-sequence $(v_1, v_2, \dots, v_{t-1})$ for some $t \in [m]$ such that

$$(\beta_j, v_j) = \begin{cases} (\sum_{i=1}^{j} J_i, \sum_{i=0}^{j-1} J_{k-i}), & \text{if } k = 2t-1, \\ (J_k + \sum_{i=1}^{j-1} J_i, \sum_{i=1}^{j-1} J_{k-i}), & \text{if } k = 2t. \end{cases}$$

A routine verification shows that $\lambda^{-1}(J) \in \mathcal{D}_m(D\uparrow, V\uparrow)$. $\square$

3.7 Counting $\mathcal{D}_m(D\uparrow, V\nearrow)$ and $\mathcal{D}_m(D\nearrow, V\uparrow)$

Theorem 3.13 *For any integer $m \geq 1$, the Fibonacci number with even subscript is realized as*

$$D_m(D\uparrow, V\nearrow) = D_m(D\nearrow, V\uparrow) = F_{2m-2}.$$

Proof Similar to the proof of Theorem 3.11, the generating trees of $\mathcal{D}_m(D\uparrow, V\nearrow)$ and $\mathcal{D}_m(D\nearrow, V\uparrow)$ are given by the rule:

$$\begin{array}{ll} \text{Root}: & (2) \\ \text{Rule}: & (k) \to (2)^{k-1}(k+1), \end{array}$$

where the root (2) corresponds to the Dyck path ud.

Making use of Lemma 2.5, we have

$$A(x) = 1, \quad Z(x) = \sum_{i\geq 0}(i+1)x^i = \frac{1}{(1-x)^2}; \quad h(x) = x, \quad d(x) = \frac{(1-x)^2}{1-3x+x^2}.$$

For $m \geq 1$, one obtains

$$D_m(D\uparrow, V\nearrow) = D_m(D\nearrow, V\uparrow) = \Sigma_{m-1} = [x^{m-1}]\frac{d(x)}{1-h(x)} = [x^{m-1}]\frac{1-x}{1-3x+x^2} = F_{2m-2}.$$

This is the desired result. $\square$

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峰严格递增的 Dyck 路的计数

孙怡东， 贾藏芝

(大连理工大学应用数学系，辽宁 大连 116024)

摘要: 本文考虑了由最高峰的高度为 m, 并且峰的高度沿着 Dyck 路严格递增的所有 Dyck 路组成的集合，即集合 $\mathcal{D}_m$ 的子集的计数问题. 利用双射、生成树以及 Riordan 阵的方法来对集合 $\mathcal{D}_m$ 的一些子集进行计数，得到了一些以经典的序列如 Catalan 数、 Narayana 数、 Motzkin 数、 Fibonacci 数、 Schröder 数以及第一类无符号 Stirling 数来计数的组合结构. 特别地，我们给出了两个新的 Catalan 结构，它们并没有明显地出现在 Stanley 关于 Catalan 结构的列表中.

关键词: 生成树； Riordan 阵； Catalan 数； Schröder 数.