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The Real Solutions of Functional Equation $f^{[m]} = 1/f$

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Abstract The authors study the functional equation $f^{[m]} = 1/f$ and analyze the features of its piecewise continuous solutions. All the piecewise continuous real solutions are obtained constructively. The results generalize the ones in [2]. Moreover, the conclusion is drawn that there is no circuit iterative roots for those functions not satisfying Babbage equation.

Keywords iterate; iterative equation; k-circuit; Babbage equation; piecewise continuous.

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1. Introduction

When the inverse of a function and its reciprocal do mean the same thing is an interesting problem, which proposes a functional equation $f^{-1} = 1/f$. This equation was investigated by many authors[1–3],[10]. Euler and Foran[3] demonstrated these functions f do exist and showed that such a solution f on $(0, \infty)$ may have an infinite number of discontinuities. In 1998, the equation in more general subsets of the real line $\mathbf{R}$ or of the complex plane $\mathbf{C}$ was discussed in [2]. Naturally, people would like to ask when an iterate $f^{[m]}$ of a function f and the power f^n actually agree, i.e.,

$$f^{[m]} = f^n, \tag{1}$$

where $m, n \in \mathbf{Z}$. Here the iterate $f^{[m]}$ is defined by $f^{[m]}(x) = f(f^{[m-1]}(x))$ and $f^{[0]}(x) = x$ for all real or complex x and we note that $f^{[-1]} = f^{-1}$ when f is invertible. For $m \geq 2$ and $n \geq 1$, Ng and Zhang[4] considered equation (1) and described all continuous real solutions.

In this paper we continue their work and consider equation (1) for $m \in \mathbf{Z}$ and $n = -1$, i.e., the functional equation

$$f^{[m]} = \frac{1}{f}. \tag{2}$$

We obtain all solutions in the class of piecewise continuous functions which are one to one from some subset of the real line onto itself. We investigate the circuit number $k(m)$ and k-circuit solutions for different k. We give a construction of the piecewise continuous solutions on the k-circuit domain intervals on the real line. This construction is universal which means that

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every piecewise continuous solution can be obtained in this way. Thus the main results in [2] are generalized. Moreover, we get that there is no circuit iterative roots for those functions not satisfying Babbage equation.

2. Preliminaries

If the equation (2) has a bijective solution on a subset D of $\mathbf{R}$ or $\mathbf{C}$, then f has its inverse $f^{[-1]}: D \to D$, and $0 \notin D$. As in [2], set $1/D := \{1/x : x \in D\}$. Then $D = 1/D$ if $f : D \to D$ is a solution of equation (2). In fact, since $f(D) = D$, it is shown by induction that $f^{[m]}(D) = D$. On the other hand, $1/f(D) = 1/D$. Thus $D = 1/D$ by (2).

Lemma 1 *If $f : D \to D$ is bijective, then equation (2) has the following equivalent propositions for given integers m and for all $x \in D$:*

$$f^{[m-1]}(x) = \frac{1}{x}, \tag{3}$$

$$f^{[1-m]}(x) = \frac{1}{x}. \tag{4}$$

Proof Suppose that (2) holds. For all $y \in D$, there exists $x \in D$ such that $y = f(x)$. By (2), $f^{[m-1]}(y) = 1/y$, which implies (3) holds.

Let (3) hold. For all $x \in D$, $1/x$ is also in D. Then $f^{[1-m]}(x) = f^{[1-m]}(f^{[m-1]}(1/x)) = 1/x$, which shows that (4) holds.

Let (4) hold. For all $x \in D$, $f(x) = f^{[1-m]}(f^{[m]}(x)) = 1/f^{[m]}(x)$. Thus (2) holds. □

In virtue of Lemma 1, the equation (2) is simplified to the iterative roots of the special function $1/x$ limited on some functional class. On the other hand, by Lemma 1, if $f : D \to D$ satisfies equation (2), then for every integer s:

$$f^{[2s(m-1)]}(x) = x, \quad \text{and} \quad f^{[(2s+1)(m-1)]}(x) = \frac{1}{x}, \quad \forall x \in D, \tag{5}$$

which implies that $2m-2$ is an iterative period of f. Without loss of generality, we only consider (2) with the case $m > 0$. In fact, for $m < 0$, by Lemma 1, we need to consider its equivalent equation (4), where the index $1-m$ is positive.

If $f : D \to D$ is a solution of equation (2), by (5), the following relation holds for all $x \in D$:

$$\begin{tikzcd}
x \arrow[r, "f"] & f(x) \arrow[r, "f"] & \cdots \arrow[r, "f"] & f^{[m-3]}(x) \arrow[r, "f"] & f^{[m-2]}(x) \arrow[d, "f"] \\
f^{[2m-3]}(x) \arrow[u, "f"] & f^{[2m-4]}(x) \arrow[l, "f"'] & \cdots \arrow[l, "f"'] & f^{[m]}(x) \arrow[l, "f"'] & f^{[m-1]}(x) = \frac{1}{x} \arrow[l, "f"']
\end{tikzcd} \tag{6}$$

Thus, we need consider the Babbage equation which satisfies (6):

$$f^{[2(m-1)]}(x) = x. \tag{7}$$

With respect to (3) and (7), a natural problem is: Are there smaller integers k such that $f^{[2(k-1)]}(x) = x$? If such integers k do exist, what are they?

Let F be a self-mapping on interval I. In the monographs[6,7], more general results on real continuous iterative root $f : I \to I$ of F are given, and the solutions can be constructed by piecewise defining. Assume that the domain of the function contains two disjoint connected intervals. While two ranges of the function on two disjoint connected intervals do not contain each other, the discussion becomes more difficult. As pointed out in [2], the solutions become interesting only if they interact with different connected intervals, for example, they map one subinterval to another subinterval and return to its beginning interval after finite times of iteration. We will consider such solutions of (2) and give their complete descriptions.

Define piecewise continuous functional class $\mathcal{E}(D)$. Function $f \in \mathcal{E}(D)$ is called piecewise continuous if the domain of f is the union of disjoint intervals $\{J_1, J_2, \ldots\}$ and the restriction of f to each other J_i is continuous. We insist that each interval J_i be maximal in the sense that if $J_i \subseteq J$ for some interval J, and f is continuous on J, then $J_i = J$. For simplicity, assume that each J_i is also nonsingular. Set $\mathcal{E}_*(D)$ is a subset containing all bijective functions in $\mathcal{E}(D)$. Note that for each $f \in \mathcal{E}_*(D)$, the restriction of f to every subinterval $J \subseteq D$ is continuous and strictly monotonic.

Definition 1 *A finite sequence $\{I_1, I_2, \ldots, I_k\}$ of pairwise disjoint nondegenerate intervals is called a k-circuit of $f : D \to D$, if $\cup_{j=1}^{k} I_j \subseteq D$, $f|_{I_j}$ is continuous such that*

$$f(I_j) = I_{j+1}, \quad \forall j = 1, 2, \ldots, k-1, \quad \text{and} \quad f(I_k) = I_1.$$

k is called circuit number.

3. Piecewise continuous solutions

If a bijective function f is a solution of equation (2), by the graph of (6), f must have k-circuit for some positive integer k. We shall look for the set of these circuit numbers k for integer m in (2).

Lemma 2 *For $m = 1\mathrm{mod}(4)$, equation (2) has no 2-circuit solution.*

Proof Suppose that $f : D \to D$ satisfies equation (2) on the 2-circuit $\{I_1, I_2\}$ and $m = 4n + 1$ for some integer n. By (3), f satisfies

$$f^{[m-1]}(x) = f^{[4n]}(x) = \frac{1}{x}. \tag{8}$$

Set $f_1 = f|_{I_1}$, $f_2 = f|_{I_2}$ and $\phi := f_2 \circ f_1 : I_1 \to I_1$, where $\circ$ denotes the composition of functions. Then ϕ is continuous and monotonic on I_1. From (8) we have that

$$f^{[4n]}(x) = \phi^{[2n]}(x) = \frac{1}{x}. \tag{9}$$

By the results in [6, Chp. XV, §4], equation (9) has no continuous and monotonic solutions because $1/x$ is a strictly decreasing function. This contradicts the hypotheses. $\square$

For a given integer m, consider the irreducible decomposition of $m - 1$:

$$m - 1 = 2^{\ell_1} p_2^{\ell_2} \cdots p_s^{\ell_s}, \tag{10}$$

where p_i $(s \geq i \geq 2)$ are primes such that $2 < p_2 < p_3 < \cdots < p_s$, all ℓ_i $(2 \leq i \leq s)$ are integers, and $\ell_1 \geq 0$, $\ell_i > 0$, $2 \leq i \leq s$.

Set

$$\mathcal{P}(m) = \begin{cases} P(m) := \{2^{\ell_1+1}p_2^{t_2}\cdots p_s^{t_s} \in \mathbf{N}\backslash\{2\}: \ 0 \le t_i \le \ell_i, \ \forall i \ge 2\}, & m = 1 \pmod 4, \\ P(m) \cup \{2\}, & m \ne 1 \pmod 4. \end{cases} \tag{11}$$

By the definition of $\mathcal{P}(m)$, we have

$$\begin{array}{lllll} \mathcal{P}(2) = \{2\}, & \mathcal{P}(3) = \{2,4\}, & \mathcal{P}(4) = \{2,6\}, & \mathcal{P}(5) = \{8\}, & \mathcal{P}(6) = \{2,10\}, \\ \mathcal{P}(7) = \{2,4,12\}, & \mathcal{P}(8) = \{2,14\}, & \mathcal{P}(9) = \{16\}, & \mathcal{P}(10) = \{2,6,18\}, & \end{array}$$

Similarly, for $m \ge 11$, it is easy to calculate $\mathcal{P}(m)$.

The following lemma is important for constructing the domain of solutions of (2).

Lemma 3 *Suppose $f \in \mathcal{E}_*(D)$ satisfies equation (2). k is a positive integer. If f has k-circuit, then $k \in \mathcal{P}(m)$. Furthermore, for $k \in \mathcal{P}(m)$, $k > 2$, k-circuit of f has the following forms:*

$$\{I_1, \, I_2, \, \ldots, I_{\frac{k}{2}}, \, 1/I_1, \, 1/I_2, \, \ldots, 1/I_{\frac{k}{2}}\}; \tag{12}$$

For $k = 2$, k-circuit of f has two forms:

$$\{I_1, 1/I_1\}, \quad \textit{if} \ \ m \equiv 0 \pmod 2, \tag{13}$$

$$\{I_1, I_2\}, \quad \textit{where} \quad I_1 = 1/I_1, \quad I_2 = 1/I_2, \textit{if} \quad m \equiv 3 \pmod 4. \tag{14}$$

Proof Suppose that $\{I_1, I_2, \ldots, I_k\}$ is k-circuit of f. Note that $I_s = I_t$ if $s \equiv t \pmod k$. Then

$$I_k = f(I_{k-1}) = \cdots = f^{[m]}(I_{k-m}) = f^{[m]}(I_j) = 1/f(I_j) = 1/I_{j+1},$$

where

$$j \equiv k - m \pmod k \equiv -m \pmod k, \quad 0 \le j \le k-1, \tag{15}$$

the symbol I_0 denotes $j \equiv 0 \pmod k$, i.e., $I_0 = I_k$.

On the other hand,

$$I_{j+1} = f^{[m]}(I_{j+1-m}) = f^{[m]}(I_{2j+1}) = 1/f(I_{2j+1}) = 1/I_{2j+2}.$$

This implies that $I_k = 1/I_{j+1} = I_{2j+2}$. Therefore

$$2j + 2 = k \pmod k \equiv 0 \pmod k. \tag{16}$$

By (15) and (16), we have that $k = j + 1$ or

$$k = 2j + 2 = \frac{2(m-1)}{2t-1}, \tag{17}$$

where t is some integer.

If $k = j + 1$, then for all $1 \le i \le k$,

$$I_i = f^{[m]}(I_{i-m}) = f^{[m]}(I_{i+j}) = 1/f(I_{i+j}) = 1/I_{i+j+1} = 1/I_{i+k} = 1/I_i.$$

Thus I_i must contain -1 or 1 for every $i \in \{1, 2, \ldots, k\}$. Since those $I_i, i \in \{1, 2, \ldots, k\}$ are disjoint to each other, by the Principle of Pigeon Nest, k must equal 2 and $j = 1$. Then from (15), n is odd. In view of Lemma 2, we get that $m = 3 \mathrm{mod}(4)$, which implies (12) holds.

If $k = 2j + 2 = (2(m-1))/(2t-1)$, for all $1 \le i \le \frac{k}{2}$, we have

$$I_i = f(I_{i-1}) = f^{[m]}(I_{i-m}) = f^{[m]}(I_{i+j}) = 1/f(I_{i+j}) = 1/I_{i+j+1} = 1/I_{i+\frac{k}{2}}$$

which implies form (13) occurs when $j = 0$ and form (14) occurs when $j > 0$. By (10) and (17), we get that $k \in \mathcal{P}(m)$. □

By Lemma 1, $f^{[3]} = 1/f$ if and only if $f^{[-1]} = 1/f$. The result in [2, Lemma 4.1] is the special case of $m = 3$ and $\mathcal{P}(3) = \{2, 4\}$ in Lemma 3. In view of Lemma 3, we have found the smaller integer k than $2(m-1)$ in (7), such that $f^{[\frac{k}{2}]}(x) = \frac{1}{x}$. Therefore, equation (2) is simplified.

Further, we can find that if k equals 2 in (12), the form (13) can be merged into form (12). Next, we only consider forms (12) and (14).

Lemma 4 *Suppose $f \in \mathcal{E}_*(\cup_{i=1}^{i=k} I_i)$, $k \in \mathcal{P}(m)$ and $(I_1, I_2, \ldots, I_k)$ be a k-circuit of function f.*

(i) If k-circuit is of form (12). then f satisfies (2) if and only if

$$f^{[\frac{k}{2}]}(x) = \frac{1}{x}, \quad \forall x \in \bigcup_{i=1}^{i=k} I_i. \tag{18}$$

(ii) If k-circuit is of form (14), then f satisfies (2) if and only if

$$f^{[2]}(x) = \frac{1}{x}, \quad \forall x \in I_1 \cup I_2. \tag{19}$$

Proof For (i), if equation (18) holds, equation (2) also holds in virtue of (5).

Conversely, if equation (2) holds, by Lemma 3 and (17), that is,

$$f^{[m-1]}(x) = f^{[(2t-1)\frac{k}{2}]}(x) = \frac{1}{x},$$

then

$$f^{[k(2t-1)]}(x) = x. \tag{20}$$

Define $\varphi := f^{[k]} : I_i \to I_i$. Then φ is continuous and monotonic. By (20), we have

$$\varphi^{[2t-1]}(x) = x.$$

Following the result of Babbage equation in [6, Theorem 15.2], we have that $f^{[k]}(x) = x$ for every $x \in I_i$. Then

$$f^{[\frac{k}{2}]}(x) = f^{[(t-1)k]} \circ f^{[\frac{k}{2}]}(x) = f^{[(t-1)k+\frac{k}{2}]}(x) = f^{[(2t-1)\frac{k}{2}]}(x) = \frac{1}{x}, \quad x \in \cup_{i=1}^{i=k} I_i,$$

which completes the proof of (i). (ii) can be similarly proved. □

For any $k \in \mathcal{P}(m)$, we consider the solutions of equation (2) which contain one or more disjoint piecewise continuous circuits. First of all, we shall describe these continuous solutions of equation (2) defined over a single k-circuit. Note that if $k = 2$ in Lemma 4, then it follows from (18) that $f = 1/x$ is a unique piecewise continuous solution of equation (2) on a 2-circuit with form (12). Therefore, we shall consider the case with form (12) only for $k \ge 4$.

Theorem 1 *Let $k \in \mathcal{P}(m)$, $k \ge 4$ and $g(x) = 1/x$. Suppose $I_1, I_2, \ldots, I_{\frac{k}{2}}$ are any mutually disjoint intervals which do not contain ± 1 or 0. $f_1, f_2, \ldots, f_{\frac{k}{2}-1}$ are arbitrary, continuous and* *strictly monotonic on intervals* $I_1, I_2, \ldots, I_{\frac{k}{2}-1}$ *respectively, and fulfilling the condition*

$$f_i(I_i) = I_{i+1}, \quad i = 1, 2, \ldots, \frac{k}{2} - 1. \tag{21}$$

Then function

$$f(x) = f_i(x), \qquad x \in I_i, i = 1, 2, \ldots, k, \tag{22}$$

where $I_{\nu+\frac{k}{2}} := g(I_\nu),\ \nu = 1, 2, \ldots, \frac{k}{2}$ *and function* $f_{\nu-1+\frac{k}{2}}$ *are defined on interval* $I_{\nu-1+\frac{k}{2}}$ *as:*

$$f_{\nu-1+\frac{k}{2}}(x) = g \circ f_\nu^{[-1]} \circ f_{\nu+1}^{[-1]} \circ \cdots \circ f_{\nu-2+\frac{k}{2}}^{[-1]}(x),\ \nu = 1, 2, \ldots, k, \tag{23}$$

is the k*-circuit solution of equation (2). Further, every* k*-circuit solution in functional class* $\mathcal{E}_*(\cup_{i=1}^k I_i)$ *arises in this way.*

Proof First the definition of (23) is well-defined. In fact, note that $g \circ g = \mathrm{id}$, for all $i \in \mathbf{N}$, we have

$$\begin{aligned} f_{k+i} &= g \circ f_{\frac{k}{2}+i+1}^{[-1]} \circ f_{\frac{k}{2}+i+2}^{[-1]} \circ \cdots \circ f_{k+i-2}^{[-1]} \circ f_{k+i-1}^{[-1]} \\ &= g \circ (f_{\frac{k}{2}+i+1}^{[-1]} \circ f_{\frac{k}{2}+i+2}^{[-1]} \circ \cdots \circ f_{k+i-2}^{[-1]}) \circ (f_{k+i-2} \circ f_{k+i-3} \circ \cdots \circ f_{\frac{k}{2}+i} \circ g^{[-1]}) \\ &= g \circ f_{\frac{k}{2}+i} \circ g^{[-1]}. \end{aligned}$$

With the similar computation, we have $f_{\frac{k}{2}+i} = g \circ f_i \circ g^{[-1]}$. Thus,

$$f_{k+i} = g \circ f_{\frac{k}{2}+i} \circ g^{[-1]} = g \circ g \circ f_i \circ (g \circ g)^{[-1]} = f_i,$$

therefore when $s = t \pmod k$, $f_s = f_t$.

Next, we will prove the function f in (22) is a solution of (2). In fact, $f|_{I_i} = f_i$. By (21) and (23), for every x_i in I_i, $i = 1, 2, \ldots, k$,

$$\begin{aligned} f^{[\frac{k}{2}]}(x_i) &= f_{i+\frac{k}{2}-1} \circ f_{i+\frac{k}{2}-2} \circ \cdots \circ f_{i+1} \circ f_i(x_i) \\ &= \big(g \circ f_i^{[-1]} \circ f_{i+1}^{[-1]} \circ \cdots \circ f_{i+\frac{k}{2}-2}^{[-1]}\big) \circ f_{i+\frac{k}{2}-2} \circ \cdots \circ f_{i+1} \circ f_i(x_i) \\ &= \frac{1}{x_i}. \end{aligned}$$

Consequently, by Lemma 4, f satisfies equation (2) on the k-circuit.

Finally, we shall show that every k-circuit solution of equation (2) can be obtained in this manner. Because intervals $I_1, I_2, \ldots, I_k$ are pairwise disjoint, $k \geq 4$, By the Principle of Pigeon Nest, for $i = 1, 2, \ldots, k/2$, $I_i \cap \{-1, 1\} = \emptyset$. By (22), functions $f_1, f_2, \ldots, f_{\frac{k}{2}-1}$ are continuous, strictly monotonic on intervals $I_1, I_2, \ldots, I_{\frac{k}{2}-1}$, and satisfy condition (21). In view of (18) and (22), the relation (23) holds. □

With the similar argument for 2-circuit of (12), we have the following results.

Theorem 2 *Let* $g(x) = 1/x$*, and* I_1 *and* I_2 *be two arbitrary disjoint intervals satisfying* $I_i = 1/I_i, i = 1, 2$*. Suppose* f_1 *is an arbitrary, continuous and strictly monotonic function on* I_1*, fulfilling the condition* $f_1(I_1) = I_2$*. Define the function* $f_2(x)$ *for* $x \in I_2$*, by*

$$f_2(x) = g \circ f_1^{[-1]}(x). \tag{24}$$

Then the formula

$$f(x) = f_i(x), \quad x \in I_i,\ i = 1, 2, \tag{25}$$

defines 2*-circuit solution of equation (2). Further, every* 2*-circuit solution in* $\mathcal{E}_*(\cup_{i=1}^{2} I_i)$ *arises in this way.*

Piecewise continuous solutions of equation (2) enjoy the following characterization.

Theorem 3 *Let f be a piecewise continuous function and, $\mathcal{J} = \{J_1, J_2, \dots\}$ be the associated sequence of maximal intervals of continuity of f. Then f satisfies equation (2) if and only if $\mathcal{J}$ can be partitioned into $\{\mathcal{J}_i | i = 1, 2, \dots\}$ such that for every i, $\mathcal{J}_i$ is a k-circuit for f, where $k \in \mathcal{P}(m)$ is defined in Lemma 3, $f|_{\mathcal{J}_i}$ is continuous and satisfies equation (2).*

Proof It is easy to see that if f has the structure stated above, then f satisfies (2).

On the other hand, suppose that f satisfies equation (2). Then for every i, the connected set J_i is mapped to an interval. Since f is piecewise continuous, f^{-1} is also piecewise continuous with the same sequence of maximal intervals. It follows that $f(J_i)$ must be a member of $\mathcal{J}$. Then there exits $k \in \mathcal{P}(m)$ such that $\{J_i, f(J_i), \dots, f^{k-1}(J_i)\}$ is a k-circuit of f. $\square$

Example 1 There is a piecewise continuous solution to equation (2) for $m = 4k+3$, $k = 1, 2, \dots$, on $\mathbf{R}\backslash\{0\}$:

$$f(x) = \begin{cases} -x^3, & x \in (0, +\infty), \\ -\frac{1}{\sqrt[3]{x}}, & x \in (-\infty, 0). \end{cases}$$

Example 2 Let $m = 5$ in equation (2). There exists a piecewise continuous solutions f:

$$f(x) = \begin{cases} x + \frac{1}{2}, & x \in (0, \frac{1}{4}), \\ \frac{12x-3}{16x+4}, & x \in (\frac{1}{4}, \frac{1}{2}), \\ \frac{4}{4x+1}, & x \in (\frac{1}{2}, \frac{3}{4}), \\ \frac{x}{6-4x}, & x \in (\frac{3}{4}, 1), \\ 6x - 4, & x \in (1, \frac{4}{3}), \\ \frac{x+4}{4x}, & x \in (\frac{4}{3}, 2), \\ \frac{4x+16}{12-3x}, & x \in (2, 4), \\ \frac{2x}{x+2}, & x \in (4, +\infty). \end{cases}$$

Example 3 Let $m = 10$ in equation (2). There are three solutions f_1, f_2, f_3:

$$f_1(x) = \begin{cases} \frac{1}{4x}, & x \in [\frac{1}{12}, \frac{1}{8}], \\ \frac{1}{2}x, & x \in [\frac{1}{6}, \frac{1}{4}], \\ \frac{1}{2}x, & x \in [\frac{1}{3}, \frac{1}{2}], \\ 2x, & x \in [2, 3], \\ 2x, & x \in [4, 6], \\ \frac{4}{x}, & x \in [8, 12], \end{cases} \qquad f_2(x) = \begin{cases} \frac{3}{3x+1}, & x \in (0, \frac{1}{3}], \\ \frac{9x}{3x+2}, & x \in (\frac{1}{3}, \frac{2}{3}], \\ \frac{3x-1}{1-x}, & x \in (\frac{2}{3}, 1), \\ 1, & x = 1, \\ \frac{x-1}{3-x}, & x \in (1, \frac{3}{2}), \\ \frac{2x+3}{9}, & x \in [\frac{3}{2}, 3), \\ \frac{x+3}{3x}, & x \in [3, +\infty), \end{cases}$$

$$f_3(x)=\begin{cases}\frac{1}{x+1}, & x\in(3n-2,3n-1],\\ \frac{1}{x+1}, & x\in(3n-1,3n],\\ \frac{1}{x-2}, & x\in(3n,3n+1],\\ 1, & x=1,\\ \frac{x+1}{x}, & x\in[\frac{1}{3n-1},\frac{1}{3n-2}),\\ \frac{x+1}{x}, & x\in[\frac{1}{3n},\frac{1}{3n-1}),\\ \frac{1-2x}{x}, & x\in[\frac{1}{3n+1},\frac{1}{3n}),\end{cases}$$

where $n\in\mathbf{N}$. The domains of f_2 and f_3 are connected. f_2 has five discontinuities: $1/3, 2/3, 1, 3/2, 3$ on $(0,\infty)$. While f_3 has infinitely many discontinuities. What's more, the domain of f_3 is the union of $\{1\}$ and a sequence of 6-circuits.

Remark When integer $n>1$, there is no k-circuit solutions for equation (1). In fact, set $y=f(x)$. Then equation (1) can be deduced to $f^{[m-1]}(y)=y^n$. that $f^{[j(m-1)]}(y)=y$ does not hold for every positive integer j. In general, if function $F(x)$ does not satisfy Babbage equation, then the iterative equation $f^{[m]}(x)=F(x)$ has no circuit solution.

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