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Asymptotic Behavior of Second Order Neutral Difference Equations with Maxima

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Abstract: The authors consider the following second order neutral difference equation with maxima

$$\Delta(a_n\Delta(y_n+p_ny_{n-k}))-q_n\max_{[n-\ell,n]}y_s=0,\quad n=0,1,2,\cdots, \tag{$*$}$$

where $\{a_n\},\{p_n\}$ and $\{q_n\}$ are sequences of real numbers, and k and ℓ are integers with $k\geq 1$ and $\ell\geq 0$. And the asymptotic behavior of nonoscillatory solutions of $(*)$. An example is given to show the difference between the equations with and without "maxima" is studied.

Key words: asymptotic behavior; nonoscillation; neutral difference equation; maxima.
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1. Introduction

Consider the difference equation

$$\Delta(a_n\Delta(y_n+p_ny_{n-k}))-q_n\max_{[n-\ell,n]}y_s=0,n=0,1,2,\cdots, \tag{1.1}$$

where k and ℓ are integers with $k\geq 1$ and $\ell\geq 0$; $[n-\ell,n]=\{n-\ell,n-\ell+1,n-\ell+2,\cdots,n\}$; $\{a_n\},\{p_n\}$ and $\{q_n\}$ are real sequences; and Δ denotes the forward difference operator defined by $\Delta y_n=y_{n+1}-y_n$.

Let $\theta=\max\{k,\ell\}$. Then by a solution of Equation (1.1), we mean a real sequence $\{y_n\}$ defined for $n\geq-\theta$ that satisfies Equation (1.1) for $n=0,1,2,\cdots$. Clearly, in this case if we are given real numbers

$$y_n=b_n,\quad n=-m_0,-m_0+1,\cdots,0 \tag{1.2}$$

as a set of initial conditions, then Equation (1.1) has a unique solution satisfying (1.2).

We often say that a function eventually satisfies a certain property if there exists an integer n_0 such that for $n\geq n_0$, the function f satisfies the stated property. A solution $\{y_n\}$ of Equation (1.1) is said to be nonoscillatory if the terms y_n of the sequence $\{y_n\}$ are eventually positive or eventually negative, and to be oscillatory otherwise.

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In this paper, we investigate asymptotic behavior of the nonoscillatory solutions of Equation (1.1). We also establish sufficient conditions to ensure that every bounded/unbounded solutions of Equation (1.1) to be oscillatory. There has been a substantial amount of theory developed for neutral differential equation with maxima, see for example[1−4,6] and the references cited therein. However it seems that very few results are available for corresponding neutral difference equations with maxima, eventhough such equations are often met in applications, for instance, in the theory of automatic control[5,7].

Note that since Equation (1.1) is nonlinear, assuming a solution is of one sign requires that the cases $y_n > 0$ and $y_n < 0$ must both be considered. We shall say that conditions (H) are met if the following conditions hold:

[(H_1)] $\{a_n\}$ is a positive sequence of real numbers such that $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$;

[(H_2)] $\{q_n\}$ is a sequence of nonnegative real numbers such that $\sum_{n=n_0}^{\infty} q_n = \infty$.

We often use the sequence $\{z_n\}$ which is defined as follows:

$$z_n = y_n + p_n y_{n-k}. \tag{1.3}$$

Then Equation (1.1) implies that

$$\Delta(a_n \Delta z_n) = q_n \max_{[n-\ell,n]} y_s, \tag{1.4}$$

$$a_n \Delta z_n = a_{n_0} \Delta z_{n_0} + \sum_{s=n_0}^{n-1} q_s \max_{[s-\ell,s]} y_t. \tag{1.5}$$

2. Basic lemmas

In this section we state and prove some lemmas which are needed in the sequel to prove our main results.

Lemma 2.1 *Suppose that conditions (H) hold and that there exists a constant p such that $p \le p_n \le 0$.*

(a) If $\{y_n\}$ is an eventually positive solution of Equation (1.1), then the sequences $\{z_n\}$ and $\{a_n \Delta z_n\}$ are eventually monotonic and either

$$z_n > 0, \Delta z_n > 0, \Delta(a_n \Delta z_n) \ge 0 \ \text{ and } \ \lim_{n\to\infty} z_n = \lim_{n\to\infty} a_n \Delta z_n = \infty \tag{2.1}$$

or

$$z_n > 0, \Delta z_n < 0, \Delta(a_n \Delta z_n) \ge 0 \quad \text{and} \quad \lim_{n\to\infty} z_n = \lim_{n\to\infty} a_n \Delta z_n = 0. \tag{2.2}$$

(b) If $\{y_n\}$ is an eventually negative solution of equation (1.1), then the sequences $\{z_n\}$ and $\{a_n \Delta z_n\}$ are eventually monotonic and either

$$z_n < 0, \Delta z_n < 0, \Delta(a_n \Delta z_n) \le 0 \quad \text{and} \quad \lim_{n\to\infty} z_n = \lim_{n\to\infty} a_n \Delta z_n = -\infty \tag{2.3}$$

or

$$z_n < 0, \Delta z_n > 0, \Delta(a_n \Delta z_n) \leq 0 \quad \textit{and} \quad \lim_{n\to\infty} z_n = \lim_{n\to\infty} a_n \Delta z_n = 0. \tag{2.4}$$

Proof (a) Let $\{y_n\}$ be an eventually positive solution of Equation (1.1). From (1.4), it follows that $\Delta(a_n\Delta z_n) = q_n \max_{[n-\ell,n]} y_s \geq 0$ eventually and $a_n\Delta z_n$ is a nondecreasing sequence. On the other hand, (H$_2$) implies that $q_n \not\equiv 0$ and therefore $\{a_n\Delta z_n\}$ is eventually of one sign and in consequence $\{z_n\}$ is eventually monotonic.

First suppose that there exists an integer $n_1 \geq n_0$ such that $a_n\Delta z_n > 0$ for $n \geq n_1$. Then there exists an integer $n_2 > n_1$ such that $a_n\Delta z_n \geq a_{n_2}\Delta z_{n_2} = c > 0$ for $n \geq n_2$. Summing the last inequality, by (H$_1$) we have

$$z_n \geq z_{n_2} + c\sum_{s=n_2}^{n-1} \frac{1}{a_s} \to \infty, \quad n \to \infty,$$

so $z_n \to \infty$ as $n \to \infty$.

Since $y_n \geqslant z_n$, we have $y_n \to \infty$ as $n \to \infty$. From (1.5) and (H$_2$), we see that $a_n\Delta z_n \to \infty$ as $n \to \infty$, and thus (2.1) holds.

Now if $a_n\Delta z_n < 0$ for $n \geq n_0$, then $a_n\Delta z_n \to L \leq 0$ as $n \to \infty$. Suppose that $L < 0$. Then $a_n\Delta z_n < L$ and by (H$_1$), $\lim_{n\to\infty} z_n = -\infty$. From (1.3) it follows that the inequality

$$z_n > p_n y_{n-k} > p y_{n-k}$$

is valid and therefore $\lim_{n\to\infty} y_n = \infty$. From (1.5) we obtain that $\lim_{n\to\infty} a_n\Delta z_n = \infty$. The contradiction obtained shows that $\lim_{n\to\infty} a_n\Delta z_n = 0$ and since $\{a_n\Delta z_n\}$ is a nondecreasing sequence, we have $a_n\Delta z_n < 0$ and $\{z_n\}$ is a decreasing sequence. Suppose $\lim_{n\to\infty} z_n = L'$, where L' is finite. Let $L' > 0$. The inequality $y_n > z_n$ implies that $y_n > L'$. From (1.5) and (H$_2$) it follows that the relation $\lim_{n\to\infty} a_n\Delta z_n = \infty$ is valid and we obtain a contradiction. Then $L' \leq 0$. Let $L' < 0$. The estimate

$$\frac{L'}{2} > z_n = y_n + p_n y_{n-k} > p_n y_{n-k} > p y_{n-k}$$

is valid. From the inequality $y_{n-k} > \frac{L'}{2p} > 0$ as above, we obtain that $\lim_{n\to\infty} a_n\Delta z_n = \infty$, a contradiction. Thus $L' = 0$ and since $\{z_n\}$ is a decreasing sequence, then $z_n > 0$. Suppose that $\lim_{n\to\infty} z_n = -\infty$. As above the inequality $y_{n-k} > \frac{z_n}{p}$ holds and $\lim_{n\to\infty} y_n = \infty$. From (1.5), it follows that $\lim_{n\to\infty} a_n\Delta z_n = \infty$ and we again obtain a contradiction. Then (2.2) is valid if $\Delta z_n < 0$.

The proof of (b) is similar to that of (a) and hence the details are omitted.

Lemma 2.2 *The sequence $\{y_n\}$ is a negative solution of Equation (1.1) if and only if $\{-y_n\}$ is a positive solution of the equation*

$$\Delta(a_n\Delta(y_n + p_n y_{n-k})) - q_n \min_{[n-\ell,n]} y_s = 0, \tag{1$'$}$$

Proof The proof is straightforward and hence the details are omitted.

3. Asymptotic behavior of nonoscillatory solutions

Here we give some oscillatory and asymptotic properties of solutions of Equation (1.1).

Theorem 3.1 *Let conditions (H) hold. If there exists a constant p such that $p \le p_n \le -1$, then every nonoscillatory solution $\{y_n\}$ of Equation (1.1) satisfies $|y_n| \to \infty$ as $n \to \infty$.*

Proof Let $\{y_n\}$ be an eventually negative solution of (1.1). Then Lemma 2.1 implies that (2.3) or (2.4) is valid. Suppose that (2.3) holds. Then from the inequality $y_n < z_n$ it follows that $\lim_{n\to\infty} y_n = -\infty$ and the assertion of the theorem is proved. Suppose that (2.4) is valid and $c = \limsup_{n\to\infty} y_n$. If $c < 0$, then $y_n < \frac{c}{2}$ and from (1.5) we obtain $\lim_{n\to\infty} a_n \Delta z_n = -\infty$ which contradicts the relation $\lim_{n\to\infty} a_n \Delta z_n = 0$ proved in Lemma 2.1. Hence $c = 0$, that is, $\limsup_{n\to\infty} y_n = 0$. Then there is an increasing sequence of positive integers $\{n_j\}$ such that $y_{n_j} \to 0$ as $j \to \infty$ and

$$\max_{[n_1,n_j]} y_s = y_{n_j}. \tag{3.1}$$

On the other hand, since $z_n < 0$, $y_n < -p_n y_{n-k} \le y_{n-k}$. But the inequality $y_{n_j} < y_{n_j-k}$ contradicts (3.1). Thus only relation (2.3) holds and $\lim_{n\to\infty} y_n = -\infty$. The case when $\{y_n\}$ is eventually positive can be considered analogously. This completes the proof.

From Theorem 3.1, we immediately obtain

Corollary 3.1 *Under the assumptions of Theorem 3.1 all bounded solutions of Equation (1.1) are oscillatory.*

Theorem 3.2 *Let conditions (H) hold and let $\{p_n\}$ satisfy one of the following conditions*

$$-1 < p \le p_n \le 0 \tag{3.2}$$

or

$$0 \le p_n \le p < 1 \quad \text{and} \quad k \le \ell. \tag{3.3}$$

Then for each nonoscillatory solution $\{y_n\}$ of Equation (1.1) either $\lim_{n\to\infty} y_n = 0$ or $\lim_{n\to\infty} |y_n| = \infty$.

Proof We shall first consider the case when (3.2) is satisfied. Let $\{y_n\}$ be an eventually bounded positive solution of (1.1). Clearly in this case, in the relations (2.1) and (2.2), only (2.2) is valid and thus $\lim_{n\to\infty} z_n = 0$. Suppose that $c = \limsup_{n\to\infty} y_n > 0$. Then there is an increasing sequence of integers $\{n_j\}$ such that $\lim_{j\to\infty} y_{n_j} = c$. Choose a constant α such that $1 < \alpha < -\frac{1}{p}$ (if $p_n \equiv 0$ then p could be any constant in $(-1, 0)$). Then $y_n < \alpha c$ for sufficiently large n and we have

$$z_n = y_n + p_n y_{n-k} \geqslant y_n + p\alpha c.$$

Hence

$$z_{n_j} \geqslant y_{n_j} + p\alpha c$$

as $j \to \infty$ and we obtain $0 \geqslant c + p\alpha c = c(1 + p\alpha) > 0$. This contradiction shows that $\limsup_{n\to\infty} y_n = 0$ and $\lim_{n\to\infty} y_n = 0$. Next let us assume that $\{y_n\}$ is an unbounded solution of (1.1). We shall show that in this case relation (2.1) is valid. Assume this is not true. Since $\{y_n\}$ is unbounded there is an increasing sequence of positive integers $\{n_i\}$ such that $y_{n_i} \to \infty$ as $i \to \infty$ and $y_{n_i} = \max_{[n_1,n_i]} y_n$. Then we have

$$z_{n_i} = y_{n_i} + p_{n_i} y_{n_i-k} \geqslant y_{n_i} + p_{n_i} y_{n_i} \geqslant y_{n_i}(1 + p). \tag{3.4}$$

From (3.2), (3.4) implies that $\lim_{i\to\infty} z_{n_i} = \infty$ which contradicts the relation $\lim_{n\to\infty} z_n = 0$. Hence (2.1) is valid and $\lim_{n\to\infty} z_n = \infty$. From the inequality $y_n > z_n$, it follows that $\lim_{n\to\infty} y_n = \infty$. The proof is similar when $\{y_n\}$ is an eventually negative solution of Equation (1.1).

Now assume that (3.3) holds. Let $\{y_n\}$ be an eventually positive solution of (1.1). From (1.4), it follows that $\Delta(a_n\Delta z_n) \geqslant 0$ and $a_n\Delta z_n$ is nondecreasing. Condition (H_2) then implies that either $a_n\Delta z_n > 0$ or $a_n\Delta z_n < 0$. Let $a_n\Delta z_n > 0$. Clearly, $\lim_{n\to\infty} z_n = \infty$ and $\{z_n\}$ is an increasing sequence. From (1.3), we have

$$y_n = z_n - p_n y_{n-k} \geqslant z_n - p_n z_{n-k} \geqslant (1 - p) z_n, \tag{3.5}$$

where we have used the increasing nature of $\{z_n\}$ and $z_n \geq y_n$. Since $\lim_{n\to\infty} z_n = \infty$, from (3.5) we have $\lim_{n\to\infty} y_n = \infty$.

Next, assume that $\{a_n\Delta z_n\}$ is eventually negative. In this case, we obtain that $\{z_n\}$ is a positive decreasing sequence. If $\lim_{n\to\infty} a_n\Delta z_n = c < 0$, then by (H_1), we have $\lim_{n\to\infty} z_n = -\infty$. Therefore $\lim_{n\to\infty} a_n\Delta z_n = 0$. Secondly, we prove that $\lim_{n\to\infty} z_n = 0$. Since $\{z_n\}$ is a positive decreasing sequence, the $\lim_{n\to\infty} z_n = d$ exists with $d \geq 0$. Assume $d > 0$. Then $z_n > d$ eventually and

$$d < y_n + p y_{n-k} < (1 + p) \max\{y_n, y_{n-k}\}.$$

Thus $\max\{y_n, y_{n-k}\} > \frac{d}{1+p}$. Since $k \leq \ell$, from the previous inequality it follows that

$$\max\{y_{n-\ell}, y_{n-\ell+1}, y_{n-\ell+2}, \cdots, y_n\} > \frac{d}{1+p}.$$

From (1.5) and (H_2) we obtain $\lim_{n\to\infty} a_n\Delta z_n = \infty$. This contradiction shows that $\lim_{n\to\infty} z_n = 0$. Then (1.3) implies that $\lim_{n\to\infty} y_n = 0$. A similar argument treats the case of negative solution of Equation (1.1). This completes the proof of the theorem.

Theorem 3.3 *Let $p_n \equiv 1$ and conditions (H_1) and (H_2) hold with the condition $\sum_{n=n_0}^{\infty} q_n = \infty$ replaced by*

$$\sum_{n=n_0}^{\infty} \bar{q}_n = \infty \quad when \quad \bar{q}_n = \min\{q_n, q_{n+k}\}. \tag{3.6}$$

Then for each bounded positive solution $\{y_n\}$ of Equation (1.1), $\lim_{n\to\infty} y_n = 0$.

Proof Since $\{y_n\}$ is an eventually positive solution of (1.1), we have $\Delta(a_n\Delta z_n) \geqslant 0$ and

$\{a_n\Delta z_n\}$ is a nondecreasing sequence. Condition (3.6) implies that $q_n \not\equiv 0$ eventually. Then either $a_n\Delta z_n > 0$ or $a_n\Delta z_n < 0$ eventually. If $a_n\Delta z_n > 0$ then $\lim_{n\to\infty} z_n = \infty$ which contradicts the boundedness of $\{y_n\}$. Hence $a_n\Delta z_n < 0$ and $\{z_n\}$ is a positive decreasing sequence. Let $\lim_{n\to\infty} z_n = c > 0$. From (1.4) it follows that

$$\Delta(a_n\Delta z_n) + q_{n-k} \max_{[n-\ell-k,n-k]} y_s = q_n \max_{[n-\ell,n]} y_s + q_{n-k} \max_{[n-\ell-k,n-k]} y_s.$$

Then using the definition of $\bar{q}_n$ and of (1.3), we obtain that

$$\begin{aligned}\Delta(a_n\Delta z_n) + q_{n-k} \max_{[n-\ell-k,n-k]} y_s &\geqslant \bar{q}_{n-k}[\max_{[n-\ell,n]} y_s + \max_{[n-\ell-k,n-k]} y_s]\\ &= \bar{q}_{n-k}[\max_{[n-\ell,n]} y_s + \max_{[n-\ell,n]} y_{s-k}]\\ &= \bar{q}_{n-k} \max_{[n-\ell,n]}(y_s + y_{s-k})\\ &= \bar{q}_{n-k} \max_{[n-\ell,n]} z_s = \bar{q}_{n-k} z_{n-\ell}.\end{aligned}$$

Since $\{z_n\}$ is a decreasing sequence and $\lim_{n\to\infty} z_n = c$, then $z_n > c$ and the last inequality takes the form

$$\Delta(a_n\Delta z_n) + q_{n-k} \max_{[n-\ell-k,n-k]} y_s \geqslant c\bar{q}_{n-k}.$$

Summing the last inequality from n_1 to $n-1$, we obtain

$$a_n\Delta z_n - a_{n_1}\Delta z_{n_1} + \sum_{s=n_1}^{n-1} q_{s-k} \max_{[s-\ell-k,s-k]} y_t > c\sum_{s=n_1}^{n-1} \bar{q}_{s-k}$$

or

$$a_n\Delta z_n - a_{n_1}\Delta z_{n_1} + \sum_{s=n_1-k}^{n-1-k} q_s \max_{[s-\ell,s]} y_t \geqslant c\sum_{s=n_1-k}^{n-1-k} \bar{q}_s. \tag{3.7}$$

Since $\{a_n\Delta z_n\}$ is a negative nondecreasing sequence, then $\{a_n\Delta z_n\}$ is a bounded sequence. On the other hand (3.6) implies that the right hand side of (3.7) tends to infinity as $n \to \infty$. Thus from (3.7) we obtain

$$\sum_{n=n_1}^{\infty} q_n \max_{[n-\ell,n]} y_s = \infty.$$

Summing from (1.4) from n_1 to $n-1$, we obtain

$$a_n\Delta z_n - a_{n_1}\Delta z_{n_1} = \sum_{s=n_1}^{n-1} q_s \max_{[s-\ell,s]} y_t.$$

Then (3.8) implies that $\lim_{n\to\infty} a_n\Delta z_n = \infty$. The contradiction obtained shows that $c = 0$, that is, $\lim_{n\to\infty} z_n = 0$. But from the inequality $y_n < z_n$ it follows that $\lim_{n\to\infty} y_n = 0$ and the proof is complete.

Remark 3.1 In contrast to neutral equations without "maxima", the assertion of Theorem 3.3 is not valid for bounded negative solutions of Equation (1.1) even under stronger conditions

$q_n \geq q > 0$ for all $n = 0, 1, 2, \cdots$. We shall illustrate this fact with the following example.

Example 3.1 Consider the difference equation

$$\Delta^2(y_n + y_{n-1}) - q_n \min_{[n-1,n]} y_s = 0 \tag{3.9}$$

where $q_n = e^{-n-2}(1-e)^2(e+1)(\min_{[n-1,n]}(\varphi_s + e^{-n}))^{-1}$ and $\{\varphi_n\}_{n\geqslant 1}$ is the sequence defined by

$$\varphi_{2k+1} = 0, \quad \varphi_{2k+2} = \frac{1}{2}, \quad k = 0, 1, 2, \cdots.$$

It is easy to verify that $\{y_n\} = \{\varphi_n + e^{-n}\}$ is a positive solution of equation (3.9). Further more, obviously $\lim_{n\to\infty} \inf y_n = 0$ and $\lim_{n\to\infty} \sup y_n = \frac{1}{2}$. On the other hand, the inequality

$$e^{-n} \leqslant \min_{[n-1,n]} \{\varphi_s + e^{-s}\} \leqslant e^{-n+1}$$

implies that $\frac{(e+1)(1-e)^2}{e^3} \leqslant q_n \leqslant \frac{(e+1)(1-e)^2}{e^2}$. Thus condition (3.6) is valid (in fact, even the stronger conditions $q_n \geqslant q > 0$ holds). Clearly, the function $\{z_n\} = \{-\varphi_n - e^{-n}\}$ is a negative solution of the equation

$$\Delta^2(x_n + x_{n-1}) - q_n \max_{[n-1,n]} x_s = 0.$$

Thus, although the conditions of Theorem 3.3 are met, Equation (1.1) could have negative bounded solution which does not tend to zero.

Theorem 3.4 *Suppose that conditions (H) hold and that there exists constants p_1 and p_2 such that*

$$1 < p_1 \leqslant p_n \leqslant p_2. \tag{3.10}$$

Then, if $\{y_n\}$ is a bounded nonoscillatory solution of (1.1), we have $y_n \to 0$ as $n \to \infty$.

Proof Let $\{y_n\}$ be an eventually positive solution of Equation (1.1). As in Theorem 3.3 it can be proved that if $\{y_n\}$ is bounded positive solution of (1.1) then $\Delta(a_n\Delta z_n) \geqslant 0, a_n\Delta z_n < 0$ and $z_n > 0$ eventually. Suppose $d = \lim_{n\to\infty} \inf y_n > 0$. Then $y_n > \frac{d}{2}$. From this inequality and (1.5) it follows that $\lim_{n\to\infty} a_n\Delta z_n = \infty$ which is a contradiction. Hence $\lim_{n\to\infty} \inf y_n = 0$. Then there is an increasing sequence of integers $\{n_i\}$ such that $\lim_{i\to\infty} y_(n_i - k) = 0$. Suppose that $c = \lim_{n\to\infty} z_n > 0$. Passing to the limit in the equality $z_{n_i} = y_{n_i} + p_{n_i} y_{n_i-k}$, we obtain that $\lim_{i\to\infty} y_{n_i} = c$. On the other hand

$$z_{n_i+k} = y_{n_i+k} + p_{n_i+k} y_{n_i} > p_1 y_{n_i}.$$

Taking the limit in the last inequality we obtain $c > p_1 c > c$. Hence $\lim_{n\to\infty} z_n = 0$ and since $z_n > y_n$, we have $\lim_{n\to\infty} y_n = 0$. The case that $\{y_n\}$ is eventually negative can be proved analogously and the proof is complete.

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含最大值项二阶中立型差分方程的渐近性

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摘要: 考虑含最大值项二阶中立型差分方程

$$\Delta\left(a_n\Delta\left(y_n+p_ny_{n-k}\right)\right)-q_n\max_{[n-\ell,n]}y_s=0,\quad n=0,1,2,\cdots, \tag{$*$}$$

其中 $\{a_n\},\{p_n\}$ 和 $\{q_n\}$ 为实数列， k 和 ℓ 为整数且 $k\geq 1$, $\ell\geq 0$, 我们研究了方程 $(*)$ 非振动解的渐近性. 通过例子说明了含最大值项的方程和相应的不含最大值项方程之间的区别.

关键词: 渐近性; 非振动; 中立型差分方程; 最大值.