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A Note on Necessary and Sufficient Conditions for the Jacobi Matrix Inverse Eigenvalue Problem

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Abstract: This paper deals with an inverse eigenvalue problem of a specially structured Jacobi matrix, which arises from the discretization of the differential equation governing the axial of a rod with varying cross section. This problem was also studied by Lu L. Z. and Sun W.W.. We give some necessary conditions for such inverse eigenvalue problem to have solutions, and present some numerical examples to show that the sufficient conditions and algorithm presented by Lu is incorrect when the order of matrix is greater than 3.

Key words: inverse problem; inverse eigenvalue problem; Jacobi matrix.
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1. Introduction

A Jacobi matrix inverse eigenvalue problem is to determine the elements of a Jacobi matrix from prescribed spectrum data. Some good conclusions on this problem have been made [1–5]. In this paper, we study an inverse eigenvalue problem of a specially structured Jacobi matrix as follows:

Problem 1 *Find $n-1$ positive numbers $\beta_1, \beta_2, \cdots, \beta_{n-1}$ so that the Jacobi matrix*

$$J_n(\beta_1, \beta_2, \cdots, \beta_{n-1}) = \begin{bmatrix} 1 & \beta_1 & & & \\ \beta_1 & 1+\beta_1^2 & \beta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \beta_{n-2} & 1+\beta_{n-2}^2 & \beta_{n-1} \\ & & & \beta_{n-1} & 1+\beta_{n-1}^2 \end{bmatrix} \tag{1}$$

has the prescribed positive eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$.

This problem was first discussed by Ram and Elhay[6], arising from a discretization of the eigenvalue problem associated with the axial oscillations of a nonuniform rod. They did not study the solvability of the problem but gave an iterative algorithm to solve it. Lu and Sun[7] gave some necessary conditions for Problem 1 having solutions. Lu and Michael[8] gave some

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new necessary and sufficient conditions for such inverse eigenvalue problem to have solutions. They also present an algorithm based on their theoretical results. In this paper, we give some more necessary conditions when $n > 3$ and simpler proofs to the results obtained in [7] and [8]. We present some examples to show the incorrectness of some theorems and algorithm in [8].

2. Main results

Here we use the terminology in [8]. The special Jacobi matrix $J_n(\beta_1, \beta_2, \cdots, \beta_{n-1})$ define in (1) is said to realize a set of n positive numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$, if there exist $n-1$ positive numbers $\beta_1, \beta_2, \cdots, \beta_{n-1}$ such that $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of $J_n(\beta_1, \beta_2, \cdots, \beta_{n-1})$. Denote by S_n the collection of all n-tuples of positive numbers that are realized by some $J_n(\beta_1, \beta_2, \cdots, \beta_{n-1})$.

Let Δ_k denote the sum of all $k \times k$ principal minors of $J_n(\beta_1, \beta_2, \cdots, \beta_{n-1})$, and δ_k denote the elementary symmetric function of degree k of the n positive numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$, i.e.

$$\delta_k(\lambda_1, \lambda_2, \cdots, \lambda_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}, k = 1, \cdots, n. \tag{2}$$

By Vieta's Theorem, we have

Theorem 1 *A necessary and sufficient condition for $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in S_n$ is that the equation*

$$\Delta_k(\beta_1, \beta_2, \cdots, \beta_{n-1}) = \delta_k(\lambda_1, \lambda_2, \cdots, \lambda_n), \quad k = 1, 2, \cdots, n \tag{3}$$

has positive solution.

The following two lemmas can be verified by direct calculation:

Lemma 1 *For the matrix $J_n(\beta_1, \beta_2, \cdots, \beta_{n-1})$ in (1), we have*

$$\Delta_1 = \mathrm{trace}(J_n) = n + \sum_{i=1}^{n-1} \beta_i^2, \tag{4}$$

$$\Delta_2 = \frac{n(n-1)}{2} + (n-2) \sum_{i=1}^{n-1} \beta_i^2 + \sum_{1 \le i < j \le n-1} \beta_i^2 \beta_j^2, \tag{5}$$

$$\Delta_{n-1} = n + \sum_{i=1}^{n-1} \beta_i^2 + \sum_{i=1}^{n-2} \beta_i^2 \beta_{i+1}^2 + \sum_{i=1}^{n-3} \beta_i^2 \beta_{i+1}^2 \beta_{i+2}^2 + \cdots + \beta_1^2 \cdots \beta_{n-1}^2, \tag{6}$$

$$\Delta_n = \det(J_n) = 1. \tag{7}$$

Lemma 2[8] *If $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in S_n$, then*

$$\sum_{i=1}^{n} \lambda_i = \mathrm{trace}(J_n(\beta_1, \beta_2, \cdots, \beta_{n-1})) = n + \sum_{i=1}^{n-1} \beta_i^2, \tag{8}$$

$$\sum_{i=1}^{n} \lambda_i^2 = \mathrm{trace}(J_n^2(\beta_1, \beta_2, \cdots, \beta_{n-1})) = n + 4 \sum_{i=1}^{n-1} \beta_i^2 + \sum_{i=1}^{n-1} \beta_i^4. \tag{9}$$

Combining (4) and (7) with taking $k = 1, n$ in (3), we have

Theorem 2[8] *A necessary condition for* $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in S_n$ *is*

$$\prod_{i=1}^{n} \lambda_i = 1, \ \sum_{i=1}^{n} \lambda_i - n > 0. \tag{10}$$

Theorem 3 *For* $n \geq 3$*, a necessary condition for* $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in S_n$ *is*

$$b - 4a < a^2 \leq (n-1)(b-4a), \tag{11}$$

where $a = \sum_{i=1}^{n} \lambda_i - n, b = \sum_{i=1}^{n} \lambda_i^2 - n$. *The equality holds if and only if* $\beta_1 = \beta_2 = \cdots = \beta_{n-1}$.

Proof When $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in S_n$, from (8) and (9) we have

$$a = \sum_{i=1}^{n} \lambda_i - n = \sum_{i=1}^{n-1} \beta_i^2, \tag{12}$$

$$b - 4a = \sum_{i=1}^{n-1} \beta_i^4 > 0. \tag{13}$$

So

$$a^2 - (b-4a) = 2 \sum_{1 \leq i < j \leq n-1} \beta_i^2 \beta_j^2 > 0, \tag{14}$$

$$\begin{aligned} (n-1)(b-4a) - a^2 &= (n-1)\sum_{i=1}^{n-1} \beta_i^4 - \left(\sum_{i=1}^{n-1} \beta_i^2\right)^2 = (n-2)\sum_{i=1}^{n-1} \beta_i^4 - 2\sum_{1 \leq i < j \leq n-1} \beta_i^2 \beta_j^2 \\ &= \sum_{1 \leq i < j \leq n-1} (\beta_i^2 - \beta_j^2)^2 \geq 0. \end{aligned} \tag{15}$$

The equality holds if and only if $\beta_1 = \beta_2 = \cdots = \beta_{n-1}$.

Theorem 4 (10) *is a necessary and sufficient condition for* $(\lambda_1, \lambda_2) \in S_2$. (10) *and* (11) *are necessary and sufficient conditions for* $(\lambda_1, \lambda_2, \lambda_3) \in S_3$.

Proof We only have to verify the sufficiency. When $n = 2$, (10) becomes $\lambda_1 \lambda_2 = 1, \lambda_1 + \lambda_2 - 2 > 0$. Take $\beta_1 = \sqrt{\lambda_1 + \lambda_2 - 2}$, then λ_1, λ_2 are eigenvalues of $\begin{bmatrix} 1 & \beta_1 \\ \beta_1 & 1+\beta_1^2 \end{bmatrix}$. When $n = 3$, (10),(11) become

$$\lambda_1 \lambda_2 \lambda_3 = 1, \ a > 0, \ b - 4a < a^2 \leq 2(b-4a). \tag{16}$$

To show $(\lambda_1, \lambda_2, \lambda_3) \in S_3$, we only have to show that the equation

$$\begin{cases} 3 + \beta_1^2 + \beta_2^2 = \lambda_1 + \lambda_2 + \lambda_3 \\ 3 + \beta_1^2 + \beta_2^2 + \beta_1^2 \beta_2^2 = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \end{cases} \tag{17}$$

has positive solution. From the definitions of a, b we have

$$\lambda_1 + \lambda_2 + \lambda_3 = a + 3, \tag{18}$$

$$\lambda_1^2+\lambda_2^2+\lambda_3^2=b+3. \tag{19}$$

so

$$\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3=\frac{1}{2}[(\lambda_1+\lambda_2+\lambda_3)^2-(\lambda_1^2+\lambda_2^2+\lambda_3^2)=\frac{1}{2}(a^2+6a-b+6). \tag{20}$$

(17) is equivalent to

$$\begin{cases}\beta_1^2+\beta_2^2=a\\ \beta_1^2\beta_2^2=\frac{1}{2}(a^2+4a-b).\end{cases} \tag{21}$$

Under the condition (16), one positive solution of the equation (21) is

$$\beta_{1,2}=\sqrt{\frac{a\pm\sqrt{2(b-4a)-a^2}}{2}}. \tag{22}$$

Remark 1 $a^2=(n-1)(b-4a)$ is a necessary but not sufficient condition that $(\lambda_1,\cdots,\lambda_n)$ is the spectrum of $J_n(\beta,\cdots,\beta)$ when $n>3$. (Compare with Corollary 5 of [8].)

Example 1 Let $n=5$, $(\lambda_1,\cdots,\lambda_5)=(0.1047,0.4640,2.0000,3.0000,3.4313)$, where we take λ_1,λ_2 and λ_5 as the computed eigenvalues of matrix

$$\begin{bmatrix}0&0&\frac{1}{6}\\1&0&-2\\0&1&4\end{bmatrix} \tag{23}$$

so that

$$\lambda_1+\lambda_2+\lambda_5=4, \tag{24}$$

$$\lambda_1\lambda_2+\lambda_1\lambda_5+\lambda_2\lambda_5=2, \tag{25}$$

$$\lambda_1\lambda_2\lambda_5=\frac{1}{6}. \tag{26}$$

Therefore, $a=4,b=20,(n-1)(b-4a)=16=a^2$. If $(\lambda_1,\cdots,\lambda_5)$ is the spectrum of $J_5(\beta,\cdots,\beta)$ for some $\beta>0$, then $\beta=\sqrt{\frac{a}{4}}=1$. But the matrix

$$\begin{bmatrix}1&1&&&\\1&2&1&&\\&1&2&1&\\&&1&2&1\\&&&1&2\end{bmatrix} \tag{27}$$

has eigenvalues $(0.0810,0.6903,1.7154,2.8308,3.6825)$.

Now denote by $J_{n,m}(\beta,\gamma)$ the Jacobi matrix $J_n(\beta_1,\beta_2,\cdots,\beta_{n-1})$ with $\beta_1=\cdots=\beta_m=\beta,\beta_{m+1}=\cdots=\beta_{n-1}=\gamma$. About this special Jacobi matrix, we have the following theorem:

Theorem 5 *A necessary condition for $(\lambda_1,\cdots,\lambda_n)$ being the spectrum of $J_{n,m_1}(\beta,\gamma)$ with $m_1\le(n-1)/2$ is*

$$a>0,m_1(b-4a)<a^2\le(n-1)(b-4a). \tag{28}$$

Proof We only have to verify the left of the second inequality. Let $m_2 = n - m_1 - 1$, then $m_1 \leq m_2$. By (12) and (13) we have

$$\left\{\begin{array}{l} m_1\beta^2 + m_2\gamma^2 = a, \\ m_1\beta^4 + m_2\gamma^4 = b - 4a. \end{array}\right. \tag{29}$$

Hence

$$\begin{aligned} a^2 - m_1(b-4a) &= (m_2^2 - m_1m_2)\gamma^4 + 2m_1m_2\beta^2\gamma^2 \\ &= m_2\gamma^2[(m_2 - m_1)\gamma^2 + 2m_1\beta^2] > 0. \end{aligned}$$

Remark 2 (28) is a necessary but not sufficient condition for $(\lambda_1, \cdots, \lambda_n)$ being the spectrum of $J_{n,m_1}(\beta, \gamma)$.

Example 2 Let $n = 8$ and the prescribed eight eigenvalues be

$$1.6914 * 10^{-8}, 1.6474 * 10^0, 4.5406 * 10^0, 9.5176 * 10^0;$$

$$1.6511 * 10^1, 2.5538 * 10^1, 3.6936 * 10^1, 5.3282 * 10^1$$

which are also given in Example 4 in [6] and Example 3 in [8]. Here,

$$a = 139.9996, b - 4a = 4675.9546, a^2 = 19599.888.$$

So, $a > 0, 4(b-4a) < a^2 < 7(b-4a)$. Lu in [8], taking $m_1 = 1, 3$, calculated two solutions of Problem 1 as

$$J_8 = (7.7524, 3.6492, \cdots, 3.6492) \quad \text{and} \quad J_8(6.2373, 6.2373, 6.2373, 2.4130, \cdots, 2.4130).$$

But by our calculation, these two matrices have eigenvalues of

$$2.7109 * 10^{-9}, 7.7290 * 10^0, 9.7250 * 10^0, 1.26245 * 10^1,$$

$$1.58683 * 10^1, 1.88182 * 10^1, 2.08767 * 10^1, 6.23580 * 10^1$$

and

$$1.1925 * 10^{-8}, 2.8950 * 10^0, 5.2674 * 10^0, 8.2458 * 10^0,$$

$$1.06993 * 10^1, 3.14401 * 10^1, 4.04823 * 10^1, 4.89721 * 10^1$$

respectively, which are different from the prescribed data. This example shows that neither (28) in this paper nor Theorem 7 and Corollary 8 in [8] is sufficient condition for the solvability of Problem 1.

References:

[1] HOCHSTAD H. *On the construction of a Jacobi matrix from spectral data* [J]. Linear Algebra Appl., 1974, **8**: 435–446.

[2] HALD O H. *Inverse eigenvalue problems for Jacobi matrix* [J]. Linear Algebra Appl., 1976, **14**: 63–85.
[3] BOOR C, GOLUB G H. *The numerical stable reconstruction of a Jacobi matrix from spectral data* [J]. Linear Algebra Appl., 1978, **21**: 245–260.
[4] XU Shu-fang. *On the Jacobi matrix inverse eigenvalue problem with mixed given data* [J]. SIAM J. Matrix Anal. Appl. 1996, **17**: 632–639.
[5] LIAO An-ping, BAI Zhong-zhi. *On the construction of positive definite Jacobian matrix from two eigenpairs* [J]. J. Numer. Methods Comput. Appl., 2002, **23**(2): 131–138.
[6] RAM Y M, ELHAY S. *Constructing the shape of a rod from eigenvalues* [J]. Comm. Numer. Methods Engrg., 1998, **14**: 597–608.
[7] LU Lin-zhang, SUN Wei-wei, *On necessary conditions for reconstruction of a specially structured Jacobi matrix from eigenvalues* [J]. Inverse Problems, 1999, **15**: 977–987.
[8] LU Lin-zhang, MICHAEL K. Ng, *On sufficient and necessary conditions for the Jacobi matrix inverse eigenvalue problem* [J]. Numer. Math., 2004, **98**(1): 167–176.

关于 Jacobi 矩阵的特征值反问题可解的充分必要条件的一个注记

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摘要: 本文讨论一类具有特殊结构的 Jacobi 矩阵的特征值反问题，该问题由描述变截面杆的微分方程离散化得到. 我们得到了这个问题有解的一些必要条件，并且通过一些数值例子，说明了 L.Lu 和 K.Michael 给出的充分条件和算法在矩阵的阶数高于 3 的时候是错误的.

关键词: 反问题；特征值反问题； Jacobi 矩阵.