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Cylinder Coalgebras and Cylinder Coproducts for Quasitriangular Hopf Algebras

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Abstract: This paper introduces the concepts of cylinder coalgebras and cylinder coproducts for quasitriangular bialgebras, and points out that there exists an anti-coalgebra isomorphism $(H,\overline{\Delta}) \cong (H,\tilde{\Delta})$, where $(H,\overline{\Delta})$ is the cylinder coproduct, and $(H,\tilde{\Delta})$ is the braided coproduct given by Kass. For any finite dimensional Hopf algebra H, the Drinfel'd double $(D(H),\overline{\Delta}_{D(H)})$ is proved to be the cylinder coproduct. Let (H,H,R) be copaired Hopf algebras. If $R \in Z(H \otimes H)$ with inverse R^{-1} and skew inverse $\Re$, then the twisted coalgebra $(H^{\Re})^{R^{-1}}$ is constructed via twice twists, whose comultiplication is exactly the cylinder coproduct. Moreover, for any generalized Long dimodule, some solutions for Yang-Baxter equations, four braid pairs and Long equations are constructed via cylinder twists.

Key words: quasitriangular Hopf algebras; cylinder coalgebras; cylinder coproducts; braided coproducts.
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1. Introduction and preliminaries

In the paper [2], the authors introduced the concept of cylinder forms for coquasitriangular bialgebras and gave a new structure into the representation theory of quantum groups.

In 1999, Takeuchi introduced the concept of cylinder algebras for coquasitriangular bialgebras, which is a generalization of cylinder forms; and introduced cylinder matrices for any Yang-Baxter operator given in [3].

The aim of this paper is to study the theory of cylinder coalgebras and cylinder coproducts for quasitriangular bialgebras.

The paper is organized as follows. In Section 2, we will introduce the concept of cylinder coalgebras for quasitriangular bialgebras. It is easy to see that the finite dual H^0 and the twisted coproduct $k \times_\alpha H$ are cylinder coalgebras for any quasitriangular Hopf algebra (H,R) under certain conditions. Moreover, for any generalized Long dimodule, solutions for Yang-Baxter equations, four braid pairs in [2] and Long equations in [6] are constructed via cylinder twists. In Section 3, we will introduce the concept of cylinder coproducts for quasitriangular bialgebras, and

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prove that the linear map $f : C \to H$ is a cylinder homomorphism if and only if $\overline{\Delta} f = (f \otimes f)\Delta$ and $\varepsilon f = \varepsilon$, and that there exists an anti-coalgebra isomorphism $(H, \overline{\Delta}) \cong (H, \tilde{\Delta})$, where $(H, \overline{\Delta})$ is a cylinder coproduct and $(H, \tilde{\Delta})$ is a braided coproduct. For any finite dimensional Hopf algebra H, the cylinder coproduct $(D(H), \overline{\Delta}_{D(H)})$ is given, where $D(H)$ is the Drinfel'd double given in [1].

Let (H, H, R) be copaired Hopf algebras. If $R \in Z(H \otimes H)$ (the center of $H \otimes H$) with inverse R^{-1} and skew inverse $\Re$, then the twisted coalgebra $(H^{\Re})^{R^{-1}}$ in [8] is constructed via twice twists, whose comultiplication is exactly the cylinder coproduct.

We always work over a fixed field k and follow Montgomery's book for terminologies on coalgebras, comodules and Hopf algebras, see [1].

• (Coalgebras) A $k-$coalgebra is a $k-$space C together with two $k-$linear maps, comultiplication $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to k$, such that the following equalities hold:

$$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta, (\varepsilon \otimes I)\Delta = I = (I \otimes \varepsilon)\Delta,$$

where the comultiplication structure map Δ is written by $\Delta(c) = \Sigma c_1 \otimes c_2$, for $c \in C$.

• (Comodules) For a $k-$coalgebra C, a right $C-$comodule is a $k-$space M with a $k-$linear map $\rho : M \to M \otimes C$, such that

$$(\rho \otimes I)\rho = (I \otimes \Delta)\rho, (I \otimes \varepsilon)\rho = I,$$

where the comodule structure map ρ is written by $\rho(m) = \Sigma m_{(0)} \otimes m_{(1)}$, for $m \in M$.

• (Bialgebras) A $k-$space H is a bialgebra if (H, m, μ) is an algebra, (H, Δ, ε) is a coalgebra, and either of the following equivalent conditions holds:

(1) Δ and ε are algebra morphisms;

(2) m and μ are coalgebra morphisms.

• (Hopf algebras) Let $(H, m, \mu, \Delta, \varepsilon)$ be a bialgebra. Then H is called a Hopf algebra if there exists an element $S \in Hom_k(H, H)$, which is an inverse to I_H under convolution $*$, where S is called an antipode for H.

2. Cylinder coalgebras for quasitriangular bialgebras

A quasitriangular bialgebra as defined in [1] means a pair of a bialgebra H and an invertible element $R = \Sigma R'_i \otimes R''_i \in H \otimes H$ satisfying

(Q1) $\tau\Delta(h) = R\Delta(h)R^{-1}$,

(Q2) $(\Delta \otimes I)R = R^{13}R^{23}$, that is, $\Sigma R'_{i1} \otimes R'_{i2} \otimes R''_i = \Sigma R'_i \otimes r'_i \otimes R''_i r''_i$, $(R = \Sigma r'_i \otimes r''_i)$

(Q3) $(I \otimes \Delta)R = R^{13}R^{12}$, that is, $\Sigma R'_i \otimes R''_{i1} \otimes R''_{i2} = \Sigma R'_i r'_i \otimes r''_i \otimes R''_i$,

where τ denotes the twisted map, $R^{13} = \Sigma R'_i \otimes 1 \otimes R''_i, R^{23} = \Sigma 1 \otimes R'_i \otimes R''_i, R^{12} = \Sigma R'_i \otimes R''_i \otimes 1$.

If H is also a Hopf algebra with antipode S, then R is invertible, whose inverse is given by $R^{-1} = (S \otimes I)R$.

Let (H, R) be a quasitriangular bialgebra. If $R^{-1} = \tau R$, then (H, R) is called a triangular bialgebra.

Definition 2.1 *Let C be a coalgebra and (H, R) a quasitriangular bialgebra. A linear map $f : C \to H$ is called a cylinder homomorphism if it satisfies*

(C1) $\varepsilon f = \varepsilon$,

(C2) $\Delta f(c) = \Sigma R_i'' f(c_1) r_i' \otimes R_i' r_i'' f(c_2)$,

where $R = r = \Sigma r_i' \otimes r_i'' \in H \otimes H$.

A pair of a coalgebra C and a cylinder homomorphism $f : C \to H$ is called a cylinder coalgebra for (H, R). In the following, we denote by (C, f) a cylinder coalgebra.

Note that if (H, R) is a triangular bialgebra, then the condition (C2) is equivalent to

(C2$'$) $R\Delta f(c) = \Sigma f(c_1) r_i' \otimes r_i'' f(c_2)$

Example 2.2 (1) Assume that (H, R) is a quasitriangular bialgebra, and (C, f) is a cylinder coalgebra for (H, R). If $Imf \subseteq Z(H)$ (the center of H) or $R \in Z(H \otimes 1_H)$, then $f : C \to H$ is a coalgebra map, if and only if (H, R) is triangular.

(2) Assuming that (H, R) is a triangular bialgebra, and C is a coalgebra. If $R \in Z(H \otimes 1_H)$ or there exists a map $f : C \to H$ such that $\mathrm{Im} f \subseteq Z(H)$, then f is a cylinder homomorphism if and only if f is a coalgebra map.

In particular, (k, μ_H) is a cylinder coalgebra, where $u_H : k \to H$ is the unit of H.

(3) Let (H, R) be a triangular Hopf algebra, and H^0 denote the finite dual of the Hopf algebra H. Define $\lambda : H^0 \to H, \alpha \mapsto \Sigma\langle \alpha, R_i' \rangle R_i''$. Then (H^0, λ) is a cylinder coalgebra for (H, R) if and only if for any $\alpha \in H^0$,

$$\Sigma\langle \lambda, \alpha_1 \rangle R_i' \otimes R_i'' \langle \lambda, \alpha_2 \rangle = \Sigma R_i' \langle \lambda, \alpha_2 \rangle \otimes R_i'' \langle \lambda, \alpha_1 \rangle.$$

In particular, when H is commutative, (H^0, λ) is a cylinder coalgebra for (H, R).

(4) Let $H = kZ_2 = k\{1, g\}$, and $\mathrm{char} k \neq 2$. Then, by [1], (H, R) is a triangular Hopf algebra with $R^{-1} = R$, where $R = \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)$.

Let $C = k\{x, y\}$ be a coalgebra. Define its comultiplication and counit as follows:

$$\Delta(x) = x \otimes x, \varepsilon(x) = 1,$$
$$\Delta(y) = x \otimes y + y \otimes x, \varepsilon(y) = 0,$$

then for any non-zero linear map $f : C \to H$ with $f(y) \neq 0$, (C, f) are not cylinder coalgebras.

Proof (1) Assume that $f : C \to H$ is a coalgebra map. Then, by Definition 2.1, for any $c \in C$,

$$\Sigma R_i'' f(c_1) r_i' \otimes R_i' r_i'' f(c_2) = \Sigma f(c_1) \otimes f(c_2).$$

It follows from $\mathrm{Im} f \subseteq Z(H)$ or $R \in Z(H \otimes 1_H)$ that

$$\Sigma(R_i'' r_i' \otimes R_i' r_i'')(f(c_1) \otimes f(c_2)) = \Sigma f(c_1) \otimes f(c_2).$$

So $\Sigma R_i'' r_i' \otimes R_i' r_i'' = 1 \otimes 1$, that is, $R^{-1} = \tau R$, (H, R) is triangular.

Conversely, it is straightforward.

(2) It follows from Definition 2.1 that $f : C \to H$ is a cylinder homomorphism if and only if f is a coalgebra map.

Since $u_H : k \to H$ is a coalgebra map, (k, μ_H) is a cylinder coalgebra for (H, R).

(3) Indeed, $\varepsilon\lambda = \varepsilon$, and for any $\alpha \in H^0$,

$$\begin{aligned}\Delta\lambda(\alpha) &= \Sigma\Delta(\langle\alpha, R_i'\rangle R_i'') = \Sigma\langle\alpha, R_i'\rangle R_{i1}'' \otimes R_{i2}'' \\ &\overset{(Q3)}{=} \Sigma\langle\alpha, R_i' r_i'\rangle r_i'' \otimes R_i'' = \Sigma\langle\alpha_1, R_i'\rangle\langle\alpha_2, r_i'\rangle r_i'' \otimes R_i'' \\ &= \Sigma\lambda(\alpha_2) \otimes \lambda(\alpha_1),\end{aligned}$$

so $\Sigma R_i''\lambda(\alpha_1)r_i' \otimes R_i' r_i''\lambda(\alpha_2) = \Delta\lambda(\alpha)$ if and only if

$$\Sigma\langle\lambda, \alpha_1\rangle R_i' \otimes R_i''\langle\lambda, \alpha_2\rangle = \Sigma R_i'\langle\lambda, \alpha_2\rangle \otimes R_i''\langle\lambda, \alpha_1\rangle.$$

When H is commutative, since $\Sigma\langle\lambda, \alpha_1\rangle R_i' \otimes R_i''\langle\lambda, \alpha_2\rangle = \Sigma R_i'\langle\lambda, \alpha_2\rangle \otimes R_i''\langle\lambda, \alpha_1\rangle$, (H^0, λ) is a cylinder coalgebra for (H, R).

(4) Assume that (C, f) is a cylinder coalgebra. Then, for $f(x) = a, f(y) = b$,

$$\begin{aligned}\Delta(b) = \Delta f(y) &= \Sigma R_i'' f(y_1) r_i' \otimes R_i' r_i'' f(y_2) \\ &= \Sigma R_i'' f(x) r_i' \otimes R_i' r_i'' f(y) + R_i'' f(y) r_i' \otimes R_i' r_i'' f(x) \\ &= \Sigma R_i'' a r_i' \otimes R_i' r_i'' b + R_i'' b r_i' \otimes R_i' r_i'' a = a \otimes b + b \otimes a \quad (R^2 = 1 \otimes 1).\end{aligned}$$

However, $\Delta(b) \neq a \otimes b + b \otimes a$, so the condition (C2) is not satisfied and (C, f) is not a cylinder coalgebra.

In the following example, we will prove that for any triangular Hopf algebra (H, R), the twisted coproduct $(k_\alpha(H), \delta)$ is a cylinder coalgebra for (H, R); for any ribbon Hopf algebra (H, R) with ribbon element y, and the map $\lambda : k \to H, 1 \mapsto y$ is a cylinder homomorphism if and only if $(R^{21}R)^2 = 1 \otimes 1$.

• (Crossed coproducts) For the crossed coproduct $C \times_\alpha H$, whose coproduct is given by

$$\Delta_{C\times\alpha H}(c \times h) = \Sigma c_1 \times c_{2(-1)}\alpha_1(c_3)h_1 \otimes c_{2(0)} \times \alpha_2(c_3)h_2.$$

The crossed coproduct $C \times_\alpha H$ is a coalgebra given in [4] if and only if the following conditions hold.

(i) (Cocycle condition):

$\Sigma c_{1(-1)}\alpha_1(c_2) \otimes \alpha_1(c_{1(0)})\alpha_2(c_2)_1 \otimes \alpha_2(c_{1(0)})\alpha_2(c_2)_2 = \Sigma\alpha_1(c_1)\alpha_1(c_2)_1 \otimes \alpha_2(c_1)\alpha_1(c_2)_2 \otimes \alpha_2(c_2)$, for any $c \in C$.

(ii) (Twisted comodule condition):

$\Sigma c_{1(-1)}\alpha_1(c_2) \otimes c_{1(0)(-1)}\alpha_2(c_2) \otimes c_{1(0)(0)} = \Sigma\alpha_1(c_1)c_{2(-1)1} \otimes \alpha_2(c_1)c_{2(-1)2} \otimes c_{2(0)}$, for any $c \in C$.

(iii) (Counit condition): $(I \otimes \varepsilon)\alpha = \mu\varepsilon = (\varepsilon \otimes I)\alpha$.

• (Ribbon Hopf algebras) Let (H, R) be a quasitriangular Hopf algebra. If there exists an element $y \in H$ such that the following conditions hold.

(y1) $y^2 = c$, where $c = uS(u)$ with $u = \Sigma S(R_i'')R_i'$,

(y2) $S(y) = y$,

(y3) $\varepsilon(y) = 1$,

(y4) $\Delta(y)=(R^{21}R)^{-1}(y\otimes y)$, where $R^{21}=\tau R$,

then y is called a quas-ribbon element of (H,R). If $y\in Z(H)$ the center of H, then y is called a ribbon element, in the case (H,R,y) is called a ribbon Hopf algebra as defined in [5].

Example 2.3 (1) Assuming that (H,R) is a triangular bialgebra, and $f:k\to H, 1\mapsto v$ such that $\varepsilon(v)=1$. Then (k,f) is a cylinder coalgebra for (H,R) if and only if $R\Delta(v)=(v\otimes 1)R(1\otimes v)$.

In particular, if (H,R) is a triangular Hopf algebra, then the map $\gamma:k\to H, 1\mapsto t$ is a cylinder homomorphism, and hence (H,γ) is a cylinder coalgebra for (H,R), where $u=\Sigma S(R_i'')R_i'$, and $t=uS(u)$ which is called the Casimir element of (H,R) (see [5]).

(2) Let (H,R) be a triangular Hopf algebra, and $k_\alpha(H)=k\times_\alpha H$ be the twisted coproduct, whose coproduct is given by $\Delta_{k_\alpha(H)}(1\times h)=\Sigma 1\times\alpha_1(1)h_1\otimes 1\times\alpha_2(1)h_2$. Then the twisted coproduct $(k_\alpha(H),\delta)$ is a cylinder coalgebra for (H,R), where $\alpha(1)=R$ and is denoted by $\Sigma\alpha_1(1)\otimes\alpha_2(1)$, and the map $\delta:k_\alpha(H)\to H$ is given by $1\times_\alpha h\mapsto\varepsilon(h)t$.

(3) Let (H,R) be a ribbon Hopf algebra with ribbon element y. Define the map $\lambda:k\to H, 1\mapsto y$, then λ is a cylinder homomorphism if and only if $(R^{21}R)^2=1\otimes 1$.

Proof (1) If $R\Delta(v)=(v\otimes 1)R(1\otimes v)$, then

$$\Sigma R_i''f(1)r_i'\otimes R_i'r_i''f(1)=\Sigma R_i''vr_i'\otimes R_i'r_i''v=\Sigma R_i''r_i'v_1\otimes R_i'r_i''v_2=\Sigma v_1\otimes v_2=\Delta(v).$$

So (k,f) is a cylinder coalgebra.

Conversely, it is obvious.

In particular, if (H,R) is a triangular Hopf algebra, then by the proof of Theorem 10.1.3 in [1], $\Delta(u)R^{21}R=u\otimes u$. Since $R^{-1}=R^{21}$, $\Delta(u)=u\otimes u$ and $\Delta(t)=t\otimes t$.

According to Proposition 10.1.4 in [1], $t\in Z(H)$, so $R\Delta(t)=\Sigma tR_i'\otimes R_i''t$.

By $\varepsilon(t)=1$, it is easy to see that the map $\gamma:k\to H, 1\mapsto t$ is a cylinder homomorphism.

(2) and (3) are straightforward.

Note that in Example 2.2(4) it is easy to see $u=\Sigma S(R_i'')R_i'=g$ and $c=uS(u)=1$. If let $y=g$, then it is easy to show that (kZ_2,R,y) is a ribbon Hopf algebra with $(R^{21}R)^2=1\otimes 1$. So by Example 2.3 we know that the map $\lambda:k\to kZ_2, 1\mapsto g$, is a cylinder homomorphism.

• (Generalized Long dimodules) Let M be both a left $A-$module via " $\cdot$ " and a left C-comodule via "ρ". If for any $a\in A, m\in M$, $\rho(a\cdot m)=\Sigma m_{(-1)}\otimes a\cdot m_{(0)}$, then it is called a generalized $[C,A]$-Long dimodule, which is a generalization of Long dimodules as defined in [6,7].

For examples, if M is a left C-comodule and A is an algebra, then $(M\otimes A,\rho_{M\otimes A},\rightharpoonup)$ is a generalized $[C,A]-$Long dimodule via the following structure maps:

$$\rho_{M\otimes A}(m\otimes a)=\Sigma m_{(-1)}\otimes m_{(0)}\otimes a;\ a\rightharpoonup(m\otimes b)=m\otimes ab.$$

If M is a left A-module and C is a coalgebra, then $(C\otimes M,\rho_{C\otimes M},\rightharpoonup)$ is a generalized $[C,A]$-Long dimodule via the following structure maps:

$$\rho_{C\otimes M}(c\otimes m)=\Sigma c_1\otimes c_2\otimes m;\ a\rightharpoonup(b\otimes m)=b\otimes a\cdot m.$$

In the following, we will construct an $H-$Long dimodule $(M, \cdot, \rho_f)$ via the cylinder homomorphism f, where $(M, \cdot, \rho)$ is is a generalized $[C, H]$-Long dimodule.

Proposition 2.4 *Let (H, R) be a triangular Hopf algebra, and (C, f) a cylinder coalgebra for (H, R). Define*

$$\rho_f : C \to H \otimes C, c \mapsto \Sigma f(c_1) \otimes c_2,$$

then (C, ρ_f) is a left H-comodule if and only if the R-commutative condition holds:

$$\Sigma R_i' f(c_1) \otimes R_i'' f(c_2) = \Sigma f(c_1) R_i' \otimes R_i'' f(c_2). \tag{Rf}$$

In this case, (1) if (C, f) and (D, g) are two cylinder coalgebras for (H, R), and there exists a coalgebra morphism $\alpha : C \to D$, then α is a left H-comodule map if and only if $g \circ \alpha = f$.

(2) Assume that $(M, \cdot, \rho)$ is a generalized $[C, H]$-Long dimodule, then $(M, \cdot, \rho_f)$ is an H-Long dimodule, where the map $\rho_f : M \to H \otimes M$ is given by $m \mapsto \Sigma f(m_{(-1)}) \otimes m_0$.

Proof As a matter of fact, for any $c \in C, (\varepsilon \otimes I)\rho_f(c) = \Sigma \varepsilon f(c_1) c_2 = \Sigma \varepsilon(c_1) c_2 = c$, and

$$(I \otimes \rho_f)\rho_f(c) = \Sigma (I \otimes \rho_f)(f(c_1) \otimes c_2) = \Sigma f(c_1) \otimes f(c_2) \otimes c_3 \tag{A}$$

and

$$(\Delta \otimes I)\rho_f(c) = \Sigma f(c_1)_1 \otimes f(c_1)_2 \otimes c_2 = \Sigma R_i'' f(c_1) r_i' \otimes R_i' r_i'' f(c_2) \otimes c_3. \tag{B}$$

If the condition (Rf) holds, then

$$\text{(B)} = \Sigma R_i'' r_i' f(c_1) \otimes R_i' r_i'' f(c_2) \otimes c_3 = \Sigma f(c_1) \otimes f(c_2) \otimes c_3 = (A).$$

So (C, ρ_f) is a left H-comodule.

Conversely, if (A)=(B), then

$$\Sigma R_i'' f(c_1) r_i' \otimes R_i' r_i'' f(c_2) \otimes c_3 = \Sigma f(c_1) \otimes f(c_2) \otimes c_3.$$

By applying $I \otimes I \otimes \varepsilon$ into the both sides of the above equality, we get

$$\Sigma R_i'' f(c_1) r_i' \otimes R_i' r_i'' f(c_2) = \Sigma f(c_1) \otimes f(c_2).$$

Since $R^{-1} = \tau R$, it is easy to show that $\Sigma R_i' f(c_1) \otimes R_i'' f(c_2) = \Sigma f(c_1) R_i' \otimes R_i'' f(c_2)$.

(1) It is obvious that $g \circ \alpha = f$ implies that α is a left $H-$comodule map. Conversely, if α is a left $H-$comodule map, then for any $c \in C$,

$$\Sigma f(c_1) \otimes \alpha(c_2) = \Sigma g(\alpha(c_1)) \otimes \alpha(c_2).$$

By applying $I \otimes \varepsilon$ to the both sides of the above equality, we get $f(c) = g \circ \alpha(c)$.

(2) Since the condition (Rf) holds, it is easy to see that (M, ρ_f) is a left H-comodule and $(M, \cdot, \rho_f)$ is an H-Long dimodule.

• (Cylinder twists) Assume that (C, f) is a cylinder coalgebra for (H, R), and M is both a left $H-$module and a left C-comodule. Define the map $t_M : M \to M$, which is given by

$t_M(m) = \Sigma f(m_{(-1)}) \cdot m_{(0)}$. If $\varphi : M \to N$ is both a left H-module map and a left C-comodule map, then it is easy to show $\varphi t_M = t_N \varphi$. In the following, t_M will be called a cylinder twist on M.

By cylinder twists, we will construct Long equations, Yang-Baxter equations, and four braid pairs.

• (Yang-Baxter operators and four braid pairs) In [2], Dieck and Oldenburg introduced two concepts of Yang-Baxter operators and four braid pairs. That is, let V be a left A-module, $\zeta : V \otimes V \to V \otimes V$ is a linear map, and $F : V \to V$ is an A-linear automorphism with the following properties:

(1) ζ is a Yang-Baxter operator, that is, ζ satisfies the equation

$$(\zeta \otimes I)(I \otimes \zeta)(\zeta \otimes I) = (I \otimes \zeta)(\zeta \otimes I)(I \otimes \zeta)$$

on $V \otimes V \otimes V$.

(2) With $Y = F \otimes I$, the braid relation $Y\zeta Y\zeta = \zeta Y \zeta Y$ is satisfied.

If (1) and (2) hold, then (ζ, F) is called a four braid pair.

• (Long equations) In [6], the author introduced the concept of Long equations: assume that V is a vector space, and $R : V \otimes V \to V \otimes V$ is satisfied

$$R^{12}R^{13} = R^{13}R^{12};\ R^{12}R^{23} = R^{23}R^{12}. \tag{LE}$$

The above equation (LE) is called the Long equation.

Proposition 2.5 *Assume that (H, R) is a quasitriangular Hopf algebra, and (C, f) a cylinder coalgebra for (H, R), and M a generalized $[C, H]$-Long dimodule, and there exists a map $g : C \to H$ such that $f *_\tau g = g *_\tau f$, that is, for any $c \in C, \Sigma f(c_2)g(c_1) = \Sigma g(c_2)f(c_1)$.*

Define $s_M : M \to M, m \mapsto \Sigma g(m_{(-1)}) \cdot m_{(0)}$, and $R : M \otimes M \to M \otimes M, m \otimes n \mapsto t_M(m) \otimes s_M(n)$. Then

(1) R is a solution of Long equations, and so is a solution of Yang-Baxter equations:

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}.$$

(2) If the map $f : C \to H$ has a skew convolution inverse $g : C \to H$, that is, $f *_\tau g = \mu_H \varepsilon_C = g *_\tau f$, then the cylinder twist t_M has a composition inverse with $t_M^{-1} = s_M$.

(3) Let N be a left H-module. Define $z_{M,N} : M \otimes N \to N \otimes M, m \otimes n \mapsto \Sigma R_i'' \cdot n \otimes R_i' \cdot m$. If for any $h \in H, m \in M$, $\Sigma f(m_{(-1)})h \cdot m_{(0)} = \Sigma h f(m_{(-1)}) \cdot m_{(0)}$, then

$$z_{N,M}(t_N \otimes I_M)z_{M,N}(t_M \otimes I_N) = (t_M \otimes I_N)z_{N,M}(t_N \otimes I_M)z_{M,N}.$$

(4) If $f *_\tau f = f$, then $t_M^2 = t_M$. Furthermore, if the map $f : C \to H$ has a skew convolution inverse $g : C \to H$ and $F : M \to M$ is a generalized $[C, H]$-Long dimodule map, then $(\zeta, Y = F \otimes I)$ is a four braid pair, where $\zeta : M \otimes M \to M \otimes M, m \otimes n \mapsto t_M(m) \otimes s_M(n)$.

Proof (1) For any $m \in M$,

$$\begin{aligned} s_M t_M(m) &= \Sigma s_M(f(m_{(-1)}) \cdot m_{(0)}) = \Sigma g((f(m_{(-1)}) \cdot m_{(0)})_{(-1)}) \cdot (f(m_{(-1)}) \cdot m_{(0)})_{(0)} \\ &= \Sigma g(m_{(0)(-1)}) \cdot (f(m_{(-1)}) \cdot m_{(0)(0)}) = \Sigma g(m_{(-1)2}) f(m_{(-1)1}) \cdot m_{(0)} \\ &= \Sigma f(m_{(-1)2}) g(m_{(-1)1}) \cdot m_{(0)} = t_M s_M(m), \end{aligned}$$

so $s_M t_M = t_M s_M$.

Hence, for any $m, n, p \in M$,

$$\begin{aligned} R^{12}R^{23}(m \otimes n \otimes p) &= R^{12}(m \otimes t_M(n) \otimes s_M(p)) \\ &= t_M(m) \otimes s_M t_M(n) \otimes s_M(p) = t_M(m) \otimes t_M s_M(n) \otimes s_M(p) \\ &= R^{23}R^{12}(m \otimes n \otimes p), \end{aligned}$$

$R^{12}R^{23} = R^{23}R^{12}$. It is obvious that $R^{12}R^{13} = R^{13}R^{12}$, so R is a solution of Long equations.

It is easy to see that R is also a solution of Yang-Baxter equations.

(2) For any $m \in M$, $s_M t_M(m) = \Sigma f(m_{(-1)2}) g(m_{(-1)1}) \cdot m_{(0)} = \Sigma \varepsilon_C(m_{(-1)}) 1_H \cdot m_{(0)} = 1_H \cdot m = m$. In a similar way, we can prove $t_M s_M(m) = m$, so $s_M t_M = I_M = t_M s_M$.

(3) In fact, for any $m \in M, n \in N$,

$$\begin{aligned} & z_{N,M}(t_N \otimes I_M) z_{M,N}(t_M \otimes I_N)(m \otimes n) = z_{N,M}(t_N \otimes I_M) z_{M,N}(t_M(m) \otimes n) \\ &= \Sigma z_{N,M}(t_N \otimes I_M)(R_i'' \cdot n \otimes R_i' \cdot t_M(m)) \\ &= \Sigma z_{N,M}(t_N(R_i'' \cdot n) \otimes R_i' \cdot t_M(m)) \\ &= \Sigma r_i'' R_i' \cdot t_M(m) \otimes r_i' \cdot t_N(R_i'' \cdot n) \quad (R = \Sigma r_i' \otimes r_i''). \end{aligned}$$

Since $\Sigma f(m_{(-1)}) h \cdot m_{(0)} = \Sigma h f(m_{(-1)}) \cdot m_{(0)}$, t_M is a left H-module map. Then

$$\begin{aligned} & (t_M \otimes I_N) z_{N,M}(t_N \otimes I_M) z_{M,N}(m \otimes n) = \Sigma (t_M \otimes I_N) z_{N,M}(t_N(R_i'' \cdot n) \otimes R_i' \cdot m) \\ &= \Sigma (t_M \otimes I_N)(r_i'' R_i' \cdot m \otimes r_i' \cdot t_N(R_i'' \cdot n)) \\ &= \Sigma t_M(r_i'' R_i' \cdot m) \otimes r_i' \cdot t_N(R_i'' \cdot n) \\ &= \Sigma r_i'' R_i' \cdot t_M(m) \otimes r_i' \cdot t_N(R_i'' \cdot n) \end{aligned}$$

and hence $z_{N,M}(t_N \otimes I_M) z_{M,N}(t_M \otimes I_N) = (t_M \otimes I_N) z_{N,M}(t_N \otimes I_M) z_{M,N}$.

(4) Let $f *_\tau f = f$. Then for any $m \in M, t_M^2(m) = \Sigma f(m_{(-1)2}) f(m_{(-1)1}) \cdot m_{(0)} = \Sigma f(m_{(-1)}) \cdot m_{(0)} = t_M(m)$.

If $f *_\tau g = \mu\varepsilon = g *_\tau f$, then $g *_\tau g = f *_\tau g *_\tau g *_\tau g = f *_\tau f *_\tau g *_\tau g *_\tau g = g$.

In a similar way, we can show $s_M^2 = s_M$.

For the defined map $\zeta : M \otimes M \to M \otimes M, m \otimes n \mapsto t_M(m) \otimes s_M(n)$,

$(\zeta \otimes I)(I \otimes \zeta)(\zeta \otimes I)(m \otimes n \otimes p) = t_M^2(m) \otimes s_M t_M s_M(n) \otimes s_M(p) = t_M(m) \otimes s_M t_M(n) \otimes s_M(p)$ and

$(I \otimes \zeta)(\zeta \otimes I)(I \otimes \zeta)(m \otimes n \otimes p) = t_M(m) \otimes t_M s_M t_M(n) \otimes s_M^2(p) = t_M(m) \otimes t_M s_M(n) \otimes s_M(p)$, so $(\zeta \otimes I)(I \otimes \zeta)(\zeta \otimes I) = (I \otimes \zeta)(\zeta \otimes I)(I \otimes \zeta)$, ζ is a Yang-Baxter operator.

It is easy to see $Ft_M = t_M F$, so, for any $m, n \in M$,

$Y\zeta Y\zeta(m \otimes n) = Ft_M Ft_M(m) \otimes s_M^2(n) = F^2 t_M^2(m) \otimes s_M^2(n) = F^2 t_M(m) \otimes s_M(n)$.

In a similar way, we can show $\zeta Y\zeta Y(m\otimes n) = F^2 t_M(m) \otimes s_M(n)$. So $Y\zeta Y\zeta = \zeta Y\zeta Y$, (ζ, F) is a four braid pair.

3. Cylinder coproducts for quasitriangular Hopf algebras

Assume that (H, R) is a quasitriangular bialgebra. If there exists an element $\Re = \Sigma \Re'_i \otimes \Re''_i$ such that

$$\Sigma R'_i \Re'_i \otimes \Re''_i R''_i = 1 \otimes 1 = \Sigma \Re'_i R'_i \otimes R''_i \Re''_i$$

then R is called a skew invertible element. In particular, if H is a quasitriangular Hopf algebra with antipode S, then by Proposition 10.1.8 in [1], R is skew invertible with skew inverse $\Re = (I \otimes S)R$.

Lemma 3.1 *Let (H, R) be a quasitriangular bialgebra with inverse R^{-1}. Then*

(1) $\Sigma \tilde{R}'_i \bar{R}'_i \otimes \tilde{R}''_i \breve{R}'_i \otimes \bar{R}''_i \breve{R}''_i = \Sigma \tilde{R}'_i \bar{R}'_i \otimes \breve{R}'_i \bar{R}''_i \otimes \breve{R}''_i \tilde{R}''_i$,
where $R^{-1} = \Sigma \tilde{R}'_i \otimes \tilde{R}''_i = \Sigma \bar{R}'_i \otimes \bar{R}''_i = \Sigma \breve{R}'_i \otimes \breve{R}''_i$.

(2) If R is skew invertible with skew inverse $\Re$, then

$$\Sigma \Re'_i R'_i \otimes \bar{R}'_i R''_i \otimes \bar{R}''_i \Re''_i = \Sigma R'_i \Re'_i \otimes R''_i \bar{R}'_i \otimes \Re''_i \bar{R}''_i,$$

where $\Re = \Sigma \Re'_i \otimes \Re''_i = \Sigma R'_i \otimes R''_i$, and $R^{-1} = \Sigma \bar{R}'_i \otimes \bar{R}''_i$.

Proof It is well known that R satisfies the following Yang-Baxter equation:

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}$$

that is,

$$\Sigma R'_i r'_i \otimes R''_i s'_i \otimes r''_i s''_i = \Sigma r'_i s'_i \otimes R'_i s''_i \otimes R''_i r''_i, \tag{YBE}$$

where $R = \Sigma R'_i \otimes R''_i = \Sigma r'_i \otimes r''_i = \Sigma s'_i \otimes s''_i$.

In the above equality (YBE), consider $R^{-1} \otimes 1$ acting on $H^{\otimes 3}$ by left multiplication. Then

$$\Sigma r'_i \otimes s'_i \otimes r''_i s''_i = \Sigma \bar{R}'_i r'_i s'_i \otimes \bar{R}''_i R'_i s''_i \otimes R''_i r''_i.$$

So we conclude

$$\begin{aligned}
&\Sigma r'_i \tilde{R}'_i \otimes s'_i \tilde{R}''_i \otimes r''_i s''_i = \Sigma \bar{R}'_i r'_i \otimes \bar{R}''_i R'_i \otimes R''_i r''_i \\
&\Longrightarrow \Sigma \tilde{R}'_i \otimes s'_i \tilde{R}''_i \otimes s''_i = \Sigma \tilde{R}'_i \bar{R}'_i r'_i \otimes \bar{R}''_i R'_i \otimes \tilde{R}''_i R''_i r''_i \\
&\Longrightarrow \Sigma \tilde{R}'_i \bar{R}'_i \otimes s'_i \tilde{R}''_i \otimes s''_i \bar{R}''_i = \Sigma \tilde{R}'_i \bar{R}'_i \otimes \bar{R}''_i R'_i \otimes \tilde{R}''_i R''_i \\
&\Longrightarrow \Sigma \tilde{R}'_i \bar{R}'_i \otimes \tilde{R}''_i \otimes \bar{R}''_i = \Sigma \tilde{R}'_i \bar{R}'_i \otimes \breve{R}'_i \bar{R}''_i R'_i \otimes \breve{R}''_i \tilde{R}''_i R''_i \\
&\Longrightarrow \Sigma \tilde{R}'_i \bar{R}'_i \otimes \tilde{R}''_i \breve{R}'_i \otimes \bar{R}''_i \breve{R}''_i = \Sigma \tilde{R}'_i \bar{R}'_i \otimes \breve{R}'_i \bar{R}''_i \otimes \breve{R}''_i \tilde{R}''_i
\end{aligned}$$

and (1) holds. In a similar way, we can prove (2).

In the following, we always assume that (H, R) is a quasitriangular bialgebra with inverse R^{-1}.

Definition 3.2 *Let (H, R) be a quasitriangular bialgebra with skew inverse $\Re$. Define a coproduct on H as follows:*

$$\overline{\Delta}: H \to H \otimes H, h \mapsto \Sigma \bar{R}_i'' h_1 \Re_i' \otimes \Re_i'' \bar{R}_i' h_2,$$

where $R^{-1} = \Sigma \bar{R}_i' \otimes \bar{R}_i''$, and $\Re = \Sigma \Re_i' \otimes \Re_i''$. This coproduct is called a cylinder coproduct, and denoted by $(H, \overline{\Delta})$.

• (Copaired bialgebras) (H, K, R) are called copaired bialgebras if H and K are two bialgebras such that $R \in H \otimes K$ satisfies the following conditions:

(R1) $(I_H \otimes \varepsilon_K)R = 1$,

(R2) $(\varepsilon_H \otimes I_K)R = 1$,

(R3) $(\Delta_H \otimes I_K)R = \Sigma R_i' \otimes r_i' \otimes R_i'' r_i''$,

(R4) $(I_H \otimes \Delta_K)R = \Sigma R_i' r_i' \otimes R_i'' \otimes r_i''$,

where $R = \Sigma R_i' \otimes R_i'' = \Sigma r_i' \otimes r_i''$.

Furthermore, if R is skew invertible with skew inverse $\Re$, then it is easy to show

($\Re$1) $(I_H \otimes \varepsilon_K)\Re = 1$,

($\Re$2) $(\varepsilon_H \otimes I_K)\Re = 1$,

($\Re$3) $(\Delta_H \otimes I_K)\Re = \Sigma \Re_i' \otimes R_i' \otimes R_i'' \Re_i''$,

($\Re$4) $(I_H \otimes \Delta_K)\Re = \Sigma R_i' \Re_i' \otimes \Re_i'' \otimes R_i''$,

where $\Re = \Sigma \Re_i' \otimes \Re_i'' = \Sigma R_i' \otimes R_i''$.

• (Twisted coalgebras) Assume that (H, K, R) are copaired bialgebras, and C is an (H, K)-bimodule coalgebra. Define

$$\overline{\Delta}_R(c) = \Sigma R_i' \cdot c_1 \otimes c_2 \cdot R_i'',$$

then, by [8], $(C, \overline{\Delta}_R, \varepsilon)$ is a coalgebra and called the twisted coalgebra. In the following, $\overline{\Delta}_R$ is called the twisted coproduct, and $(C, \overline{\Delta}_R, \varepsilon)$ denoted by $(C^R, \Delta_{C^R}, \varepsilon)$.

Let (H, K, R) be copaired bialgebras with inverse R^{-1}. Then, by Proposition 1.2 in [8], the following hold.

($R^{-1}1$) $(I_H \otimes \varepsilon_K)R^{-1} = 1$,

($R^{-1}2$) $(\varepsilon_H \otimes I_K)R^{-1} = 1$,

($R^{-1}3$) $(\Delta_H \otimes I_K)R^{-1} = \Sigma \tilde{R}_i' \otimes \tilde{r}_i' \otimes \tilde{r}_i'' \tilde{R}_i''$,

($R^{-1}4$) $(I_H \otimes \Delta_K)R^{-1} = \Sigma \tilde{R}_i' \tilde{r}_i' \otimes \tilde{r}_i'' \otimes \tilde{R}_i''$,

where $R^{-1} = \Sigma \tilde{R}_i' \otimes \tilde{R}_i'' = \Sigma \tilde{r}_i' \otimes \tilde{r}_i''$.

Furthermore, assume that R is invertible with inverse $R^{-1} \in Z(H \otimes K)$. Define

$$\check{\Delta}_{R^{-1}}(c) = \Sigma \tilde{R}_i'' \cdot c_1 \otimes c_2 \cdot \tilde{R}_i',$$

then it is not difficult to prove that $(C, \check{\Delta}_{R^{-1}}, \varepsilon)$ is a coalgebra, which is denoted by $C^{R^{-1}}$.

Let (H, K, R) be copaired bialgebras, and C an (H, K)-bimodule coalgebra. If $R \in Z(H \otimes K)$ with skew inverse $\Re$, then

(1) $(H, K, \Re)$ are copaired bialgebras, and $(C^{\Re}, \Delta_{\Re}, \varepsilon)$ is a coalgebea with comultiplication

$$\Delta_{\Re}(c) = \Sigma \Re_i' \cdot c_1 \otimes c_2 \cdot \Re_i''.$$

(2) If R is invertible with inverse R^{-1}, then (H, K, R^{-1}) are copaired bialgebras, and $(C^{\Re})^{R^{-1}}$ is the twisted coalgebra, whose comultiplication is given by

$$\Delta_{(C^{\Re})^{R^{-1}}}(c) = \Sigma \tilde{R}_i'' \cdot c_1 \cdot \Re_i' \otimes \tilde{R}_i' \Re_i'' \cdot c_2,$$

where $R^{-1} = \Sigma \tilde{R}_i' \otimes \tilde{R}_i''$.

In particular, if (H, H, R) are copaired Hopf algebras with $R \in Z(H \otimes H)$, then R has an invertible element R^{-1} and a skew-invertible element $\Re$, and $H^{\Re}$ is an (H, H)-bimodule coalgebra, whose actions are given by multiplication m_H. Thus, by the above discussion, we get the twisted coalgebra $(H^{\Re})^{R^{-1}}$ with comultiplication

$$\Delta_{(H^{\Re})^{R^{-1}}}(h) = \Sigma \tilde{R}_i'' h_1 \Re_i' \otimes \Re_i'' \tilde{R}_i' h_2$$

which is exactly the cylinder coproduct given in Definition 3.2.

Theorem 3.3 *The cylinder coproduct $(H, \overline{\Delta}, \varepsilon)$ given in Definition 3.2 is a coalgebra.*

Proof Let $R^{-1} = \Sigma \tilde{R}_i' \otimes \tilde{R}_i''$ and $\Re = \Sigma \Re_i' \otimes \Re_i''$.

According to $\Sigma \varepsilon(\tilde{R}_i') \tilde{R}_i'' = 1 = \Sigma \tilde{R}_i' \varepsilon(\tilde{R}_i'')$ and $\Sigma \varepsilon(\Re_i') \Re_i'' = 1 = \Sigma \Re_i' \varepsilon(\Re_i'')$, then, for any $h \in H$,

$$(I \otimes \varepsilon)\overline{\Delta}(h) = h = (\varepsilon \otimes I)\overline{\Delta}(h),$$

that is, ε is a counit.

By (Q2) and (Q3), one easily obtain the following equalities:

$$\Sigma \Delta(\Re_i') \otimes \Re_i'' = \Sigma \Re_i' \otimes \mathtt{R}_i' \otimes \mathtt{R}_i'' \Re_i'' - (C); \quad \Sigma \Re_i' \otimes \Delta(\Re_i'') = \Sigma \Re_i' \mathtt{R}_i' \otimes \Re_i'' \otimes \mathtt{R}_i'' - (D)$$

$$(\Delta \otimes I)R^{-1} = \Sigma \tilde{R}_i' \otimes \tilde{r}_i' \otimes \tilde{r}_i'' \tilde{R}_i'' - (E); \quad (I \otimes \Delta)R^{-1} = \Sigma \tilde{R}_i' \tilde{r}_i' \otimes \tilde{R}_i'' \otimes \tilde{r}_i'' - (F)$$

Thus, by (C)–(F) and Lemma 3.1, for any $h \in H$, we have

$$\begin{aligned}
(I \otimes \overline{\Delta})\overline{\Delta}(h) &= \Sigma (I \otimes \overline{\Delta})(\bar{R}_i'' h_1 \Re_i' \otimes \Re_i'' \bar{R}_i' h_2) \\
&= \Sigma \bar{R}_i'' h_1 \Re_i' \otimes \tilde{R}_i'' (\Re_i'' \bar{R}_i' h_2)_1 R_i' \otimes R_i'' \tilde{R}_i' (\Re_i'' \bar{R}_i' h_2)_2 \\
&= \Sigma \bar{R}_i'' h_1 \Re_i' \otimes \tilde{R}_i'' \Re_{i1}'' \bar{R}_{i1}' h_2 R_i' \otimes R_i'' \tilde{R}_i' \Re_{i2}'' \bar{R}_{i2}' h_3 \\
&\overset{(D)}{=} \Sigma \bar{R}_i'' h_1 \Re_i' r_i' \otimes \tilde{R}_i'' \Re_i'' \bar{R}_{i1}' h_2 R_i' \otimes R_i'' \tilde{R}_i' r_i'' \bar{R}_{i2}' h_3 \quad (\Re = \Sigma r_i' \otimes r_i'') \\
&\overset{(E)}{=} \Sigma \bar{R}_i'' \breve{R}_i'' h_1 \Re_i' r_i' \otimes \tilde{R}_i'' \Re_i'' \breve{R}_i' h_2 R_i' \otimes R_i'' \tilde{R}_i' r_i'' \bar{R}_i' h_3 \\
&= \Sigma \bar{R}_i'' \breve{R}_i'' h_1 r_i' \Re_i' \otimes \Re_i'' \tilde{R}_i'' \breve{R}_i' h_2 R_i' \otimes R_i'' r_i'' \tilde{R}_i' \bar{R}_i' h_3 \quad \text{(Lemma 3.1)}
\end{aligned}$$

and

$$
\begin{aligned}
(\overline{\Delta}\otimes I)\overline{\Delta}(h) &= \Sigma(\overline{\Delta}\otimes I)(\bar{R}_i''h_1\Re_i'\otimes\Re_i''\bar{R}_i'h_2)\\
&= \Sigma\breve{R}_i''(\bar{R}_i''h_1\Re_i')_1r_i'\otimes r_i''\breve{R}_i'(\bar{R}_i''h_1\Re_i')_2\otimes\Re_i''\bar{R}_i'h_2\\
&= \Sigma\breve{R}_i''\bar{R}_{i1}''h_1\Re_{i1}'r_i'\otimes r_i''\breve{R}_i'\bar{R}_{i2}''h_2\Re_{i2}'\otimes\Re_i''\bar{R}_i'h_3\\
&\overset{(F)}{=} \Sigma\breve{R}_i''\bar{R}_i''h_1\Re_{i1}'r_i'\otimes r_i''\breve{R}_i'\tilde{R}_i''h_2\Re_{i2}'\otimes\Re_i''\bar{R}_i'\tilde{R}_i'h_3\\
&\overset{(C)}{=} \Sigma\breve{R}_i''\bar{R}_i''h_1\Re_i'r_i'\otimes r_i''\breve{R}_i'\tilde{R}_i''h_2R_i'\otimes R_i''\Re_i''\bar{R}_i'\tilde{R}_i'h_3\\
&= \Sigma\breve{R}_i''\tilde{R}_i''h_1\Re_i'r_i'\otimes r_i''\breve{R}_i'\bar{R}_i''h_2R_i'\otimes R_i''\Re_i''\tilde{R}_i'\bar{R}_i'h_3\\
&= \Sigma\bar{R}_i''\breve{R}_i''h_1\Re_i'r_i'\otimes r_i''\tilde{R}_i''\breve{R}_i'h_2R_i'\otimes R_i''\Re_i''\tilde{R}_i'\bar{R}_i'h_3, \quad \text{(by Lemma 3.1)}
\end{aligned}
$$

that is, $(\overline{\Delta}\otimes I)\overline{\Delta}=(I\otimes\overline{\Delta})\overline{\Delta}$. So $(H,\overline{\Delta})$ is a coalgebra.

Let (H,R) and (L,S) be two quasitriangular bialgebras. If there exists a bialgebra map $F:H\to L$ such that $(F\otimes F)(R)=S$, then, by Theorem 3.3, $F:(H,\overline{\Delta}_H)\to(L,\overline{\Delta}_L), h\mapsto F(h)$, is a coalgebra map.

Proposition 3.4 *Let (H,R) be a quasitriangular bialgebra with skew inverse $\Re$. Then the linear map $f:C\to(H,\overline{\Delta})$ is a cylinder homomorphism if and only if $\overline{\Delta}f=(f\otimes f)\Delta$, and $\varepsilon f=\varepsilon$.*

In other words, the cylinder homomorphism is nothing but a coalgebra map $f:C\to(H,\overline{\Delta})$.

Proof Assume that f is a cylinder homomorphism. Then, by (C2), for any $c\in C$,

$$\overline{\Delta}f(c)=\Sigma\bar{R}_i''f(c)_1\Re_i'\otimes\Re_i''\bar{R}_i'f(c)_2=\Sigma f(c_1)R_i'\Re_i'\otimes\Re_i''R_i''f(c_2)=\Sigma f(c_1)\otimes f(c_2).$$

Conversely, if $\overline{\Delta}f=(f\otimes f)\Delta$ and $\varepsilon f=\varepsilon$, then for any $c\in C$,

$$\Sigma\bar{R}_i''f(c)_1\Re_i'\otimes\Re_i''\bar{R}_i'f(c)_2=\Sigma f(c_1)\otimes f(c_2).$$

Hence we can conclude

$$\Sigma\bar{R}_i''f(c)_1\Re_i'R_i'\otimes R_i''\Re_i''\bar{R}_i'f(c)_2=\Sigma f(c_1)R_i'\otimes R_i''f(c_2)$$

$$\Sigma\bar{R}_i''f(c)_1\otimes\bar{R}_i'f(c)_2=\Sigma f(c_1)R_i'\otimes R_i''f(c_2).$$

So $\Sigma f(c)_1\otimes f(c)_2=\Sigma r_i''f(c_1)R_i'\otimes r_i'R_i''f(c_2)$.

Example 3.5 Let H be a finite dimensional Hopf algebra with antipode S. Then, by [1], the multiplication and comultiplication of the Drinfel'd double $D(H)=H^{*COP}\bowtie H$ are given as follows: for $f,f'\in H^*, h,h'\in H$,

$$(f\bowtie h)(f'\bowtie h')=\Sigma f(h_1\twoheadrightarrow f_2')\bowtie(h_2\twoheadleftarrow f_1')h'$$
$$\Delta_{D(H)}(f\bowtie h)=\Sigma(f_2\bowtie h_1)\otimes(f_1\bowtie h_2).$$

By Theorem 10.3.6 in [1], the Drinfel'd double $D(H)$ is a quasitriangular Hopf algebra with antipode

$$S_{D(H)}=\Sigma S(h_2)\rightharpoonup S(f_1)\otimes f_2\rightharpoonup S(h_1).$$

(1) $(D(H),\overline{\Delta}_{D(H)})$ is a cylinder coproduct, whose coproduct is given by

$$\overline{\Delta}_{D(H)}(f \bowtie h) = \Sigma h_i^* f_3 \bowtie h_1 \ell_i \otimes S(\ell_i^*)(S(h_{i2} \twoheadrightarrow f_2) \bowtie (S(h_{i1}) \twoheadleftarrow f_1) h_2,$$

where $\{h_i^*, h_i\}$ and $\{\ell_i^*, \ell_i\}$ are two dual bases of H.

(2) Let $\gamma : H \to D(H), h \mapsto \varepsilon \bowtie h$. Then (H, γ) is not a cylinder coalgebra for $(D(H), R)$, where $R = \Sigma \varepsilon_H \bowtie h_i \otimes h_i^* \bowtie 1_H$.

Proof (1) By the above discussion, we know $R^{-1} = \Sigma S(R_i') \otimes R_i''$ and $\Re = \Sigma R_i' \otimes S(R_i'')$. According to [1], $R = \Sigma \varepsilon_H \bowtie h_i \otimes h_i^* \bowtie 1_H$, it is easy to show $R^{-1} = \Sigma \varepsilon_H \bowtie S(h_i) \otimes h_i^* \bowtie 1_H$ and $\Re = \Sigma \varepsilon_H \bowtie h_i \otimes S(h_i^*) \bowtie 1_H$. Thus, by Definition 3.2, we get easily

$$\overline{\Delta}_{D(H)}(f \bowtie h) = \Sigma h_i^* f_3 \bowtie h_1 \ell_i \otimes S(\ell_i^*)(S(h_{i2} \twoheadrightarrow f_2) \bowtie (S(h_{i1}) \twoheadleftarrow f_1) h_2.$$

(2) It is obvious that $\gamma : H \to (D(H), \overline{\Delta}_{D(H)}), h \mapsto \varepsilon \bowtie h$ is not a coalgebra map, so, by Proposition 3.4, γ is not a cylinder homomorphism for $(D(H), R)$.

• (Braided coproducts) Let (H, R) be a triangular Hopf algebra, and " $\rightharpoonup$ " denote a quantum adjoint action on H: $g \rightharpoonup h = \Sigma g_1 h S(g_2)$. Then H has the second coalgebra structure

$$\tilde{\Delta}(h) = \Sigma h_1 S(R_i'') \otimes R_i' \rightharpoonup h_2.$$

which is called the braided coproduct of H (see Theorem 7.4.2 given in [9]).

Proposition 3.6 *Let (H, R) be a triangular Hopf algebra, and $(H, \overline{\Delta})$ the cylinder coproduct, and $(H, \tilde{\Delta})$ the braided coproduct. Then exists an anti-coalgebra isomorphism:*

$$(H, \overline{\Delta}) \overset{f}{\cong} (H, \tilde{\Delta}),$$

where the map $f : (H, \overline{\Delta}) \to (H, \tilde{\Delta})$ is given by $h \mapsto S(h)$.

Proof We have only to prove that f is an anti-coalgebra map.

As a matter of fact, for any $h \in H$,

$$\begin{aligned}
(f \otimes f)\overline{\Delta}(h) &= \Sigma (f \otimes f)(\bar{R}_i'' h_1 \Re_i' \otimes \Re_i'' \bar{R}_i' h_2) = \Sigma S(\Re_i') S(h_1) S(\bar{R}_i'') \otimes S(h_2) S(\bar{R}_i') S(\Re_i'') \\
&= \Sigma S(\Re_i') S(h_1) S(R_i') \otimes S(h_2) S(R_i'') S(\Re_i'') \quad (R^{-1} = \tau R) \\
&= \Sigma S(\Re_i') S(h_1) R_i' \otimes S(h_2) R_i'' S(\Re_i'') \quad ((S \otimes S)R = R \text{ in } [1]) \\
&= \Sigma S(r_i') S(h_1) R_i' \otimes S(h_2) R_i'' S^2(r_i'') \quad (\Re = (I \otimes S)R) \\
&= \Sigma r_i' S(h_1) R_i' \otimes S(h_2) R_i'' S(r_i'')
\end{aligned}$$

and

$$\begin{aligned}
\tau \tilde{\Delta} f(h) &= \Sigma \tau (S(h_2) S(R_i'') \otimes R_i' \rightharpoonup S(h_1)) = \Sigma R_i' \rightharpoonup S(h_1) \otimes S(h_2) S(R_i'') \\
&= \Sigma R_{i1}' S(h_1) S(R_{i2}') \otimes S(h_2) S(R_i'') \\
&\overset{(Q2)}{=} \Sigma R_i' S(h_1) S(r_i') \otimes S(h_2) S(R_i'' r_i'') \\
&= \Sigma R_i' S(h_1) r_i' \otimes S(h_2) r_i'' S(R_i''),
\end{aligned}$$

so $(f\otimes f)\overline{\Delta}=\tau\tilde{\Delta}f$. It is obvious that $\varepsilon f=\varepsilon$ and hence f is an anti-coalgebra map.

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关于拟三角 Hopf 代数的 Cylinder 余代数和 Cylinder 余积

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摘要: 本文引入两个概念，即，关于拟三角双代数的 cylinder 余代数和 cylinder 余积，并指出存在一个反余代数同构： $(H,\overline{\Delta})\cong(H,\tilde{\Delta})$, 其中 $(H,\overline{\Delta})$ 是 cylinder 余积， $(H,\tilde{\Delta})$ 是辫余积. 对任意有限维 Hopf 代数 H, 我们证明 Drinfel'd 量子偶 $(D(H),\overline{\Delta}_{D(H)})$ 是 cylinder 余积. 设 (H,H,R) 是余配对 Hopf 代数，如果 $R\in Z(H\otimes H)$, 则通过两次扭曲，我们可以构造扭曲余代数 $(H^{\Re})^{R^{-1}}$, 它的余乘法恰是 cylinder 余积. 而且对任意的广义 Long 重模，通过 cylinder 扭曲，我们可以构造 Yang-Baxter 方程，四辫对和 Long 方程.

关键词: 拟三角 Hopf 代数； cylinder 余代数； cylinder 余积；辫余积