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On the Spectral Radii of Bicyclic Graphs

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Abstract: A graph G of order n is called a bicyclic graph if G is connected and the number of edges of G is $n+1$. Let $\mathcal{B}(n)$ be the set of all bicyclic graphs on n vertices. In this paper, the first three largest spectral radii in the class $\mathcal{B}(n)$ $(n \geq 9)$ together with the corresponding graphs are given.

Key words: bicyclic graph; spectral radius; characteristic polynomial.
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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Let G be a graph with n vertices, and $A(G)$ be the $(0,1)$-adjacent matrix of G. Since $A(G)$ is symmetric and real, its eigenvalues are real. These eigenvalues of $A(G)$ are independent of the ordering of the vertices of G, and consequently, without loss of generality, we can write them in non-increasing order as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and call them the eigenvalues of G. The characteristic polynomial of G is just $\det(xI - A(G))$, and denoted by $P(G,x)$. The largest eigenvalue $\lambda_1(G)$ is called the spectral radius of G, denoted by $\rho(G)$. If G is connected, then $A(G)$ is irreducible and $\rho(G)$ is simple and there exists a unique positive unit eigenvector corresponding to $\rho(G)$ by the Perron-Frobenius Theorem of non-negative matrices. The unique positive unit eigenvector corresponding to $\rho(G)$ is called the Perron vector of G.

The investigation on the spectral radii of graphs is an important topic in the theory of graph spectral. For results on the spectral radii of graphs, one may refer to [1]–[10] and the reference therein. Bicyclic graphs are connected graphs in which the number of edges equals the number of vertices plus one. Yu and Tian[10] determined the graphs with the largest spectral radius among all the bicyclic graphs on n vertices with matching number m.

In this paper, we study the spectral radii of bicyclic graphs, and give the first three largest spectral radii in the class $\mathcal{B}(n)$ $(n \geq 9)$ together with the corresponding graphs.

2. Three types of bicyclic graphs

Definition 2.1 *A graph G of order n is called a bicyclic graph if G is connected and the number of edges of G is $n+1$.*

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It is easy to see from the definition that G is a bicyclic graph if and only if G can be obtained from a tree T (with the same order) by adding two new edges to T.

Let G be a bicyclic graph. The *base* of G, denoted by $\widehat{G}$, is the (unique) minimal bicyclic subgraph of G. It is easy to see that $\widehat{G}$ is the unique bicyclic subgraph of G containing no pendant vertices, while G can be obtained from $\widehat{G}$ by attaching trees to some vertices of $\widehat{G}$.

It is well-known that there are the following three types of bicyclic graphs containing no pendant vertices:

Let $B(p,q)$ be the bicyclic graph obtained from two vertex-disjoint cycles C_p and C_q by identifying vertices u of C_p and v of C_q (see Fig.1).

Let $B(p,l,q)$ be the bicyclic graph obtained from two vertex-disjoint cycles C_p and C_q by joining vertices u of C_p and v of C_q by a new path $uu_1u_2\cdots u_{l-1}v$ with length l ($l \geq 1$) (see Fig.1).

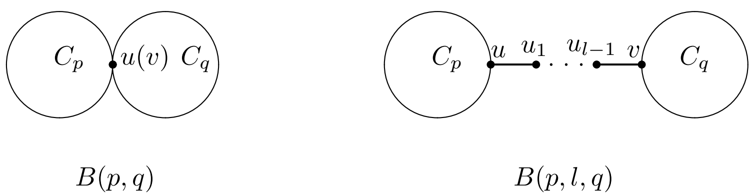


Figure 1 The graphs $B(p,q)$ and $B(p,l,q)$

Let $P(p,q,l)$ ($1 \leq l \leq \min\{p,q\}$) be the bicyclic graph obtained from a cycle C_{p+q-2l} : $v_1v_2\cdots v_{p-l}v_{p-l+1}v_{p-l+2}\cdots v_{p+q-2l}v_1$ by joining vertices v_1 and v_{p-l+1} by a new path $v_1u_1u_2\cdots u_{l-1}v_{p-l+1}$ with length l (see Fig.2).

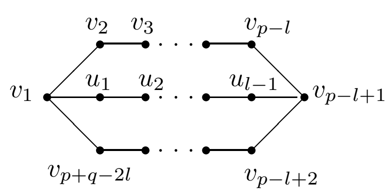


Figure 2 The graph $P(p,q,l)$

Now we can define the following three classes of bicyclic graphs of order n:

$$\mathcal{B}_1(n) = \{G \in \mathcal{B}(n) \mid \widehat{G} = B(p,q) \text{ for some } p \geq 3 \text{ and } q \geq 3\},$$
$$\mathcal{B}_2(n) = \{G \in \mathcal{B}(n) \mid \widehat{G} = B(p,l,q) \text{ for some } p \geq 3, q \geq 3 \text{ and } l \geq 1\},$$
$$\mathcal{B}_3(n) = \{G \in \mathcal{B}(n) \mid \widehat{G} = P(p,q,l) \text{ for some } 1 \leq l \leq \min\{p,q\}\}.$$

It is easy to see that $\mathcal{B}(n) = \mathcal{B}_1(n) \dot{\cup} \mathcal{B}_2(n) \dot{\cup} \mathcal{B}_3(n)$.

For a graph G, let $|G|$, $V(G)$, $E(G)$ denote the order, the vertex set and the edge set of G, respectively. For $v \in V(G)$, let $d(v)$ denote the degree of v and $N(v)$ denote the set of all neighbors of v in G.

Now we quote some basic lemmas which will be used in the proofs of our main results.

Lemma 2.1[2] *Let v be a vertex of degree 1 in the graph G and u be the vertex adjacent to v. Then $P(G,x) = xP(G-v,x) - P(G-u-v,x)$.*

From Lemma 2.1, we have the following corollary:

Corollary 2.1 *Let G be a connected graph with n vertices, and consist of a proper induced subgraph H and $n-|H|$ new pendant edges (not in H) attached to some vertex $v \in V(H)$. Then*

$$P(G,x) = x^{n-|H|}P(H,x) - (n-|H|)x^{n-|H|-1}P(H-v,x).$$

Lemma 2.2[3,6] *Let u be a vertex of the connected graph G and for positive integers k,l ($k \geq l \geq 1$), and let $G_{k,l}$ be the graph obtained from G by adding two pendant paths of length k and l at u. Then $\rho(G_{k,l}) > \rho(G_{k+1,l-1})$.*

Lemma 2.3[3,6] *Let uv be an edge of graph G satisfying $d(u) \geq 2$ and $d(v) \geq 2$, and suppose that two new paths of length k,m ($k \geq m \geq 1$) are attached to G by their end vertices at u and v, respectively, to form $M_{k,m}$. Then we have $\rho(M_{k,m}) > \rho(M_{k+1,m-1})$.*

Lemma 2.4[3] *Let G be a connected graph, and G' be a proper spanning subgraph of G. Then $P(G',x) > P(G,x)$, for all $x \geq \rho(G)$.*

Let $\Delta(G)$ denote the maximum degree of G. From Lemma 2.4, we have $\rho(G) \geq \rho(K_{1,\Delta(G)} \cup (n-\Delta(G)-1)K_1) = \Delta(G) \geq \sqrt{\Delta(G)}$.

Theorem 2.1 *Let G be a connected graph and $u, v_1, v_2, \ldots, v_r$ ($r \geq 1$) be vertices of G. Suppose $V_1, V_2, \ldots, V_r$ are pairwise disjoint vertex subsets of G, which are not all empty, and*

$$V_i = \{v_{i1}, v_{i2}, \ldots, v_{ik_i}\} \subseteq N(v_i)\backslash(N(u) \cup \{u\}) \quad (i = 1,2,\ldots,r).$$

Let $G_u(v_1,\ldots,v_r)$ be the graph with $V(G_u(v_1,\ldots,v_r)) = V(G)$ and

$$E(G_u(v_1,\ldots,v_r)) = (E(G)\backslash \bigcup_{i=1}^{r}\{v_iv_{i1},\ldots,v_iv_{ik_i}\})\bigcup(\bigcup_{i=1}^{r}\{uv_{i1},\ldots,uv_{ik_i}\}).$$

Let X be the Perron vector of G. Suppose we have $X_u \geq \max\{X_{v_1},\ldots,X_{v_r}\}$, where X_u denotes the coordinates of X corresponding to the vertex u, then $\rho(G_u(v_1,\ldots,v_r)) > \rho(G)$.

Proof Let $G' = G_u(v_1,\cdots,v_r)$. From the well-known results of real symmetric matrices in linear algebra, we have $X^TA(G)X = \rho(G)$ and $\rho(G') \geq X^TA(G')X$. So we have

$$\begin{aligned}\rho(G') - \rho(G) &\geq X^TA(G')X - X^TA(G)X = X^T[A(G') - A(G)]X \\ &= \sum_{i=1}^{r}(\sum_{j=1}^{k_i}(X_u - X_{v_i})X_{v_{ij}}) \geq 0. \qquad (2.1)\end{aligned}$$

Next we want to show the strict inequality holds in (2.1). Suppose not, then all the equalities hold in (2.1). In particular, we have $\rho(G') = \rho(G) = X^TA(G')X$, which implies that X is also an eigenvector of $A(G')$ corresponding to $\rho(G')$, namely we also have $A(G')X = \rho(G')X$. Thus

$$(A(G') - A(G))X = (\rho(G') - \rho(G))X = 0. \tag{2.2}$$

Moreover, let α_u be the row vector of $A(G') - A(G)$ corresponding to the vertex u. Since $k_1, \dots, k_r$ are not all 0, it is easy to see that $\alpha_u \geq 0$, $\alpha_u \neq 0$. Then $\alpha_u X > 0$, which contradicts (2.2). □

From Theorem 2.1, we have the following corollary:

Corollary 2.2 *Let G be a connected graph, $v_1, \dots, v_k \in V(G)$ $(k \geq 2)$ are vertices in G adjacent to some pendant vertices. Suppose G has s pendant edges incident to $v_1, \dots, v_k$, G_i $(i = 1, \dots, k)$ be the graph obtained from G by deleting all these s pendant edges and attaching them to v_i. Then $\max\{\rho(G_1), \dots, \rho(G_k)\} > \rho(G)$.*

Lemma 2.5 *Let G be a connected graph which consists of a proper induced subgraph H and a tree T of order $k+1$ $(k \geq 2)$ such that T and H has a unique common vertex v and T is not a star with center v. Let $G' = H + vw_1 + \cdots + vw_k$, where $w_1, w_2, \dots, w_k \in V(T)$ are distinct pendant vertices of G'. Then $\rho(G') > \rho(G)$.*

Proof We use induction on the number q of non-pendant vertices different from v in T.

Since T is not a star with center v, we have $q \geq 1$.

Let x be a non-pendant vertex in T different from v which is furthest from v among all non-pendant vertices in T. Let P be the unique path in T from v to x, and let y be the vertex in P which is adjacent to x. Let $N(x) = \{y, x_1, \dots, x_{d-1}\}$, where $d = d(x)$, then all the vertices $x_1, \dots, x_{d-1}$ are pendant vertices.

Now we use Theorem 2.1 by taking $u = y,\ r = 1,\ v_1 = x$ and $V_1 = \{x_1, \dots, x_{d-1}\}$, then the graph $G_u(v_1)$ in Theorem 2.1 is

$$G_u(v_1) = G - \{xx_1, \dots, xx_{d-1}\} + \{yx_1, \dots, yx_{d-1}\}.$$

On the other hand, if we take $u = x,\ r = 1,\ v_1 = y$ and $V_1 = N(y)\backslash\{x\}$, then it is not difficult to see that the graph $G_{v_1}(u)$ obtained in Theorem 2.1 is isomorphic to $G_u(v_1)$. Let X be the Perron vector of G. Since we either have $X_y \geq X_x$ or $X_x \geq X_y$, so by Theorem 2.1, we have $\rho(G_u(v_1)) > \rho(G)$.

Also it is obvious that the number of non-pendant vertices in the corresponding tree T' in $G_u(v_1)$ different from v is $q-1$. So by induction on $G_u(v_1)$ we have $\rho(G') \geq \rho(G_u(v_1))$ (equality holds when $G' = G_u(v_1)$). Combining the above two inequalities we obtain $\rho(G') > \rho(G)$. □

Definition 2.2 *Let G be a bicyclic graph, $v \in V(\widehat{G})$, if v is adjacent to some vertices not in $V(\widehat{G})$, then we call v a divarication-vertex of G.*

For convenience, in this paper, we denote

$$\mathcal{B}_{ij} = \{G \mid G \in \mathcal{B}_i(n) \text{ and has exactly } j \text{ divarication-vertices},\ 1 \leq i \leq 3,\ j \geq 0\}.$$

We use $\lambda(P(x))$ to denote the largest real root of the equation $P(x) = 0$.

3. The basic strategy

Let $G_1,\ G_2,\ G_3$ be the graphs on $n \geq 9$ vertices as Fig.3.1.

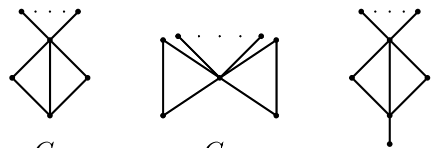


Figure 3.1 The graphs $G_1 - G_3$

For $n \geq 9$, our basic strategy of determining the first three largest spectral radii of $\mathcal{B}(n)$ together with the corresponding graphs is to prove the following results (R1)–(R5) later:

(R1). $\rho(G_2) < \rho(G_1)$.

(R2). $\rho(G_3) < \rho(G_2)$.

(R3). For any $G \in \mathcal{B}_1(n) \setminus \{G_2\}$, we have $\rho(G) < \rho(G_3)$.

(R4). For any $G \in \mathcal{B}_2(n)$, we have $\rho(G) < \rho(G_3)$.

(R5). For any $G \in \mathcal{B}_3(n) \setminus \{G_1, G_3\}$, we have $\rho(G) < \rho(G_3)$.

We will prove the result (R2) in this section, prove the result (R3) in §4, prove the result (R4) in §5, prove the result (R5) in §6, prove the result (R1) and our main results in §7.

Now we prove (R2): $\rho(G_3) < \rho(G_2)$.

Proof Taking $H = B(3,3)$ in Corollary 2.1, we have

$$P(G_2, x) = x^{n-6}[x^6 - (n+1)x^4 - 4x^3 + (2n-5)x^2 + 4x + 5 - n] \triangleq x^{n-6} f_2(x).$$

Next we take H to be the graph obtained from $P(2,2,1)$ with one pendant edge attached to some common vertex of three cycles, then by Corollary 2.1, we have

$$P(G_3, x) = x^{n-4}[x^4 - (n+1)x^2 - 4x + 3n - 13] \triangleq x^{n-6} f_3(x) \tag{3.1}$$

Let

$$g(x) = f_3(x) - f_2(x) = (n-8)x^2 - 4x + n - 5.$$

For $n \geq 9$, we have $g(x) > 0$, when $x = \rho(G_3) \geq \sqrt{n-2}$, i.e., $P(G_2, x) < 0$, which implies $\rho(G_2) > \rho(G_3)$. □

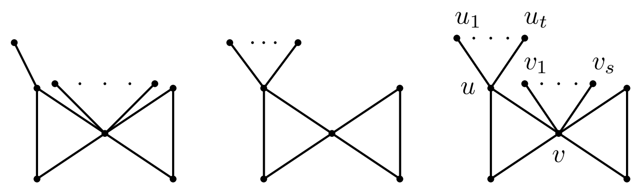
4. The proof of (R3)

Lemma 4.1 *Let $G \in \mathcal{B}_1(n)$ and $\widehat{G} = B(p,q)$. If $p + q \geq 7$, then $\rho(G) < \rho(B_1)$ or $\rho(G) < \rho(B_2)$ (B_1 and B_2 are the graphs shown in Fig.4.1).*

Proof By Lemma 2.5, Corollary 2.2 and Theorem 2.1, we have $\rho(G) < \rho(B_{s,t})$ for some s, t with $s, t \geq 1$, ($B_{s,t}$) is the graph shown in Fig.4.1). Let x be the perron vector of $B_{s,t}$.

If $X_u \geq X_v$. By Theorem 2.1, we have $\rho(B_{s,t}) < \rho(B_2)$.

If $X_u < X_v$. By Theorem 2.1, $\rho(B_{s,t}) < \rho(B_1)$. □



B_1 B_2 $B_{s,t}$

Figure 4.1 The graphs B_1, B_2 and $B_{s,t}$

Lemma 4.2 $\rho(B_1) < \rho(G_3)$.

Proof By Corollary 2.1, we have

$$P(B_1, x) = x^{n-6}[x^6 - (n+1)x^4 - 4x^3 + 3(n-3)x^2 + 6x + 11 - 2n] \triangleq x^{n-6} f_4(x).$$

Combining the above equation and (3.1), we have

$$f_4(x) - f_3(x) = 4x^2 + 6x + 11 - 2n \triangleq p(x).$$

It is easy to see that, $p(x) > 0$, when $x = \rho(B_1) \geq \sqrt{n-2}$, $p(x) > 0$, which implies $\rho(B_1) < \rho(G_3)$. □

The Proof of (R3) By Lemma 4.1, it suffices to consider $\widehat{G} = B(3,3)$. By Theorem 2.1 and Lemma 2.5, we only need to consider $G \in (\mathcal{B}_{11}(n) \backslash \{G_1\}) \cup \mathcal{B}_{12}(n)$. We distinguish the following two cases.

Case 1. $G \in \mathcal{B}_{11}(n) \backslash \{G_2\}$.

Subcase 1.1 The common vertex of the two C_3 in $\widehat{G}$ is the divarication-vertex.

Suppose the unique divarication-vertex is v, and $q(G)$ is the number of non-pendant vertices different from v in the tree attached to $\widehat{G}$. Since $G \neq G_2$, $q(G) > 0$. From the proof of Lemma 2.5, we know that if $q(G) \geq 2$, then there exists a graph $G' \in \mathcal{B}_1(n)$ such that $\widehat{G'} = \widehat{G}$, $q(G') = 1$ and $\rho(G') > \rho(G)$. So it suffices to consider the graph G with $q(G) = 1$. Using u to denote the unique non-pendant vertex different from v in the tree attached to $\widehat{G}$.

B_3 B_3'

Figure 4.2 The graphs B_3 and B_3'

(1) $X_v \geq X_u$. By Theorem 2.1, $\rho(G) \leq \rho(B_3)$ (B_3 is the graph shown in Fig.4.2). By Lemma 2.3, we obtain $\rho(B_3) < \rho(B_1) < \rho(G_3)$, which implies $\rho(G) < \rho(G_3)$.

(2) $X_u > X_v$. By Theorem 2.1, $\rho(G) \le \rho(B_3')$ (B_3' is the graph shown in Fig.4.2). Now we show that $\rho(B_3') < \rho(B_3)$. In Corollary 2.1, let H be the graph obtained from $B(3,3)$ by a path of order 3 attached to v. Then we have

$$\begin{aligned} P(B_3', x) - P(B_3, x) &= (n-7)x^{n-8}[(P(P_2, x))^3 - xP(B(3,3), x)] \\ &= (n-7)x^{n-8}(x-1)(x+1)^2(3x+1) > 0, \end{aligned}$$

where P_2 is the path of order 2. Then we have $\rho(B_3') < \rho(B_3)$, thus, $\rho(G) < \rho(G_3)$.

Subcase 1.2 The common vertex of the two C_3 in $\widehat{G}$ is not the divarication-vertex.

By using Lemma 2.5 and Corollary 2.2, we have $\rho(G) \le \rho(B_2)$. Now we show $\rho(B_2) < \rho(G_3)$. By Corollary 2.1, we have

$$P(B_2, x) = x^{n-6}[x^6 - (n+1)x^4 - 4x^3 + (4n-15)x^2 + (2n-6)x + 5 - n].$$

Combining the above equation and (3.1), we have

$$P(B_2, x) - P(G_3, x) = x^{n-6}[(n-2)x^2 + (2n-6)x + 5 - n] > 0,$$

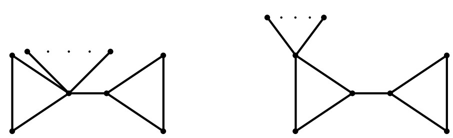
for all $x > 1$, which implies $\rho(B_2) < \rho(G_3)$.

Case 2. $G \in \mathcal{B}_{12}(n)$.

Subcase 2.1 The common vertex of the two C_3 in $\widehat{G}$ is not a divarication -vertex. By Lemma 2.5 and Corollary 2.2, we have $\rho(G) < \rho(B_2) < \rho(G_3)$.

Subcase 2.2 The common vertex of the two C_3 is a divarication -vertex. By Lemma 2.5, we have $\rho(G) \le \rho(B_{s,t}) < \rho(G_3)$. □

5. The proof of (R4)



B_4 B_5

Figure 5.1 The graphs B_4 and B_5

Lemma 5.1 *Let $G \in \mathcal{B}_2(n)$ and $\widehat{G} = B(p, l, q)$. If $p + q + l \ge 8$, then $\rho(G) < \rho(B_4)$ or $\rho(G) < \rho(B_5)$ (B_4, B_5 are the graphs shown in Fig.5.1).*

Proof By Lemma 2.5, Corollary 2.2 and Theorem 2.1, the result follows. □

The Proof of (R4) By Lemma 2.5, Corollary 2.2 and Lemma 5.1 for all graphs in $\mathcal{B}_2(n)$, their spectral radii are no more than $\max\{\rho(B_4), \rho(B_5)\}$. Now we show that $\rho(B_5) < \rho(B_4) < \rho(G_3)$.

By Corollary 2.1, we have

$$P(B_4, x) = x^{n-7}[x^7 - (n+1)x^5 - 4x^4 + (4n-13)x^3 + 2nx^2 + 3(7-n)x + 12 - 2n] \triangleq x^{n-7}f_5(x),$$

$$P(B_5,x)=x^{n-7}[x^7-(n+1)x^5-4x^4+(5n-19)x^3+2nx^2+(27-4n)x+12-2n].$$

Then

$$P(B_5,x)-P(B_4,x)=(n-6)x^{n-6}(x^2-1)>0 \text{ (for } x>1).$$

Thus $\rho(B_4)>\rho(B_5)$. Let $h(x)=f_5(x)-xf_3(x)=nx^3+2nx^2+3(7-n)x+12-2n$. When $x=\rho(B_4)\geq\sqrt{n-3}$, $h(x)>0$, which implies $\rho(B_4)<\rho(G_3)$. □

6. The proof of (R5)

B_6 B_7

Figure 6.1 The graphs B_6 and B_7

Similar to the proof of Lemmas 4.1 and 4.2, we have

Lemma 6.1 *Let $G\in\mathcal{B}_3(n)$ and $\widehat{G}=P(p,q,l)$. If $p+q+l\geq 6$, then $\rho(G)<\rho(B_6)$, $\rho(G)<\rho(B_7)$ or $\rho(G)<\rho(G_3)$ (B_6, B_7 are the graphs shown in Fig.6.1).*

Lemma 6.2 $\rho(B_6)<\rho(G_3)$.

Proof By Corollary 2.1, we have

$$P(B_6,x)=x^{n-6}[x^6-(n+1)x^4-4x^3+(3n-12)x^2+2x+5-n]\triangleq x^{n-6}f_6(x).$$

Combining the above equation and (3.1), we have

$$P(B_6,x)-P(G_3,x)=x^{n-5}[x^2+2x+5-n]\triangleq x^{n-5}q(x).$$

When $x=\rho(B_6)\geq\sqrt{n-2}$, $q(x)>0$, which implies $\rho(B_6)<\rho(G_3)$. □

The Proof of (R5) By Lemma 6.1, it suffices to consider $\widehat{G}=P(2,2,1)$. By Theorem 2.1, we only need to consider $G\in(\mathcal{B}_{31}\setminus\{G_1\})\cup(\mathcal{B}_{32}\setminus\{G_3\})$. We distinguish the following two cases.

Case 1. $G\in\mathcal{B}_{31}(n)\setminus\{G_1\}$.

Subcase 1.1 The divarication-vertex is not a common vertex of three cycles.

By Lemma 2.5, $\rho(G)\leq\rho(B_7)$. Now we show that $\rho(B_7)<\rho(G_3)$. By Corollary 2.1, we have

$$P(B_7,x)=x^{n-5}[x^5-(n+1)x^3-4x^2+(3n-12)x+2n-8].$$

Combining the above equation and (3.1), we have

$$P(B_7,x)-P(G_3,x)=x^{n-5}(x+2n-8)>0, \quad \text{for all } x>0.$$

Thus $\rho(B_7)<\rho(G_3)$.

Subcase 1.2 The divarication-vertex is a common vertex of three cycles.

B_8 B_8'

Figure 6.2 The graphs B_8 and B_8'

Similar to the proof of Subcase 1.2 of (R3), we have $\rho(G) \leq \rho(B_8)$ or $\rho(G) \leq \rho(B_8')$ (B_8 and B_8' are the graphs shown in Fig.6.2).

First we show that $\rho(B_8') < \rho(B_8)$. In Corollary 2.1, let H be the graph obtained from $P(2,2,1)$ by attached a path of order 3 to some common vertex of three cycles, we have

$$\begin{aligned}P(B_8', x) - P(B_8, x) &= (n-6)x^{n-7}[P(P_2, x)P(P_3, x) - xP(P(2,2,1), x)] \\ &= (n-6)x^{n-6}(x+1)^2 > 0,\end{aligned}$$

which implies $\rho(B_8') < \rho(B_8)$.

Next by Lemma 2.3, $\rho(B_8) < \rho(G_3)$. Then $\rho(G) < \rho(G_3)$.

Case 2. $G \in \mathcal{B}_{32}(n) \setminus \{G_3\}$.

Subcase 2.1 Neither the two common vertices of three cycles is a divarication-vertex. By Lemma 2.5 and Corollary 2.2, we have $\rho(G) < \rho(B_7) < \rho(G_3)$.

Subcase 2.2 One of the common vertices of three cycles is a divarication-vertex. We use v to denote the divarication-vertex which is a common vertex of three cycles, and suppose the other divarication-vertex is u.

If $X_u \geq X_v$. By Theorem 2.1, we have $\rho(G) < \rho(B_7) < \rho(G_3)$.

If $X_u < X_v$. By Theorem 2.1, we have $\rho(G) \leq \rho(B_6) < \rho(G_3)$.

Subcase 2.3 The two common vertices of three cycles both are divarication-vertices. By Lemma 2.5 and Theorem 2.1, we have $\rho(G) < \rho(G_3)$.

7. The proof of (R1) and the main results

Now we show (R1): $\rho(G_2) < \rho(G_1)$.

Proof By Corollary 2.1, we have

$$\begin{aligned}P(G_1, x) &= x^{n-4}[x^4 - (n+1)x^2 - 4x + 2n - 8] \stackrel{\Delta}{=} x^{n-4} f_1(x), \\ P(G_2, x) &= x^{n-6}(x-1)(x+1)^2(x^3 - x^2 - nx + x + n - 5) \stackrel{\Delta}{=} x^{n-6}(x^2-1)F(x).\end{aligned}$$

Let $f(x) = f_2'(x) - f_1(x) = x^2 + 3 - n$. Then when $x = \rho(G_2) \geq \sqrt{n-1}$, $f(x) > 0$, which implies $\rho(G_1) > \rho(G_2)$. $\square$

Combining above results (R1)–(R5), we get the following main results.

Theorem 7.1 *If G is a bicyclic graph of order $n \geq 9$, then $\rho(G) \leq r_n$, where r_n is the largest root of the equation $x^4-(n+1)x^2-4x+2n-8=0$, with equality if and only if $G=G_1$.*

Theorem 7.2 *If G is a bicyclic graph of order $n \geq 9$ and $G \neq G_1$, then $\rho(G) \leq s_n$, where s_n is the largest root of the equation $x^3-x^2-(n-1)x+n-5=0$, with equality if and only if $G=G_2$.*

Theorem 7.3 *If G is a bicyclic graph of order $n \geq 9$ and $G \neq G_1, G_2$, then $\rho(G) \leq t_n$, where t_n is the largest root of the equation $x^4-(n+1)x^2-4x+3n-13=0$, with equality if and only if $G=G_3$.*

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关于双圈图的谱半径

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摘要: 如果 G 是连通的并且 G 的边数是 $n+1$, 那么 n 阶图 G 叫做双圈图. 设 $\mathcal{B}(n)$ 是所有的阶为 n 的双圈图构成的集合. 本文给出了 $\mathcal{B}(n)$ $(n \geq 9)$ 中前三大的邻接谱半径以及它们对应的图.

关键词: 双圈图；谱半径；特征多项式.