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Explicit representations and computation of W – weighted Drazin inverse

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Abstract: In this paper, we establish some new explicit representations of the W – weighted Drazin inverse $A_{d,W}$ in terms of regular inverse. By these new representations we can reduce the cost on computation. Also, we consider a *rank* equation and derive a method for computing W – weighted Drazin inverse.

Key words: W – weighted Drazin inverse; index; *rank*; bordered matrix

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1 Introduction

If A and W are any m by n and n by m complex matrices, respectively, then $X = [(AW)_d]^2A$ is unique solution to the following equations,

$$(AW)^{k+1}XW = (AW)^k,\ XWAWX = X,\ AWX = XWA, \tag{1.1}$$

where $k = \mathrm{Ind}(AW)$, the index of AW, is the smallest nonnegative integer for which $\mathrm{rank}[(AW)^k] = \mathrm{rank}[(AW)^{k+1}]$. The matrix X is called the W – weighted Drazin inverse of A and is written as $X = A_{d,W}$ [2]. In this paper, we present some explicit representations of $A_{d,W}$ which are more condensed than what Wei gave in [4] and better than that of Wang et al. [5]. In Section 3, we consider a *rank* equation and derive a method for computing the W-weighted Drazin inverse.

The following lemmas are needed in what follows.

Lemma 1.1([1]) Let $B, E \in C^{n\times n}$, with $\mathrm{Ind}(B) = k$ and $\mathrm{Ind}(E) = 1$. Then

(a) $R(B_d) = R(B^l)$ and $N(B_d) = N(B^l)$, for all $l \geqslant k$;

(b) $R(E_g) = R(E)$, $N(E_g) = N(E)$;

(c) $EE_g = E_gE = P_{R(E),N(E)}$;

(d) $P_{R(E),N(E)} + P_{N(E),R(E)} = I$.

(e) $(B_d)^l = (B^l)_g$, for all $l \geqslant k$;

Lemma 1.2([4]) Let $A \in C^{m\times n}, W \in C^{n\times m}$ with $Ind(AW) = k_1$ and $Ind(WA) = k_2$. Then

(a) $A_{d,W} = A[(WA)_d]^2 = [(AW)_d]^2A$;

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(b) $A_{d,W}W = (AW)_d, WA_{d,W} = (WA)_d$;

(c) $A_{d,W}WAW = (AW)_dAW = P_{R[(AW)^{k_1}],N[(AW)^{k_1}]} = P_{R[(AW)^k],N[(AW)^k]}$, where $k \geq k_1$.

Lemma 1.3([3]) Let

$$P = \begin{pmatrix} A & C \\ B & D \end{pmatrix},$$

where A is a nonsingular matrix. Then $\text{rank}(P) = \text{rank}(A)$ if and only if $D = CA^{-1}B$.

2 Explicit representations of the W-weighted Drazin inverse

In the following theorem, a result of the W-weighted Drazin $A_{d,W}$ was presented by Yimin Wei.

Theorem 2.1 ([4]) Let $A \in C^{m\times n}$, $W \in C^{n\times m}$, $\text{Ind}(AW) = k_1$, $\text{Ind}(WA) = k_2$, and $\text{rank}((AW)^{k_1}) = \text{rank}((WA)^{k_2}) = r$. Suppose that $U_1 \in C_{n-r}^{n\times(n-r)}$, $V_1^* \in C_{m-r}^{m\times(m-r)}$ are matrices whose columns form bases for $N[(WA)^{k_2}]$ and $N[(AW)^{k_1^*}]$, respectively and $U_2^* \in C_{n-r}^{n\times(n-r)}$, $V_2 \in C_{m-r}^{m\times(m-r)}$ are matrices whose columns form bases for $N[(WA)^{k_2^*}]$ and $N[(AW)^{k_1}]$, respectively. Then

$$D = \begin{pmatrix} WAW & U_1 \\ V_1 & 0 \end{pmatrix}$$

is nonsingular, and its (regular) inverse is

$$D^{-1} = \begin{pmatrix} A_{d,W} & V_2(V_1V_2)^{-1} \\ (U_2U_1)^{-1}U_2 & -(U_2U_1)^{-1}U_2WAWV_2(V_1V_2)^{-1} \end{pmatrix}.$$

By Theorem 2.1, we can easily obtain

$$A_{d,W} = (I \ \vdots \ 0)\begin{pmatrix} WAW & U_1 \\ V_1 & 0 \end{pmatrix}^{-1}\begin{pmatrix} I \\ \cdots \\ 0 \end{pmatrix}.$$

The basic idea of the expression is that a nonsingular bordered matrix is constructed from the original matrix by adjoining to it certain matrices. If we obtain $A_{d,W}$ by computing the nonsingular bordered matrix, the cost will be expensive. In [5], a condensed expression was obtained by Wang et al.. It can be shown that

$$A_{d,W} = ((AW)^{k_1} + U(VU)^{-1}V)^{-1}(AW)^{k_1}A.$$

we note that $(VU)^{-1}$ is employed in the expression. To get $A_{d,W}$, we need compute two regular inverses. The cost of computation is also expensive. In the following results we will develop the condensed explicit representations of the W-weighted Drazin inverse $A_{d,W}$, not involving the inverse of VU.

Theorem 2.2 Let $A \in C^{m\times n}$, $W \in C^{n\times m}$, $\text{Ind}(AW) = k_1$, and $\text{Ind}(WA) = k_2$. Suppose that $U, V \in C_{m-r}^{m\times(m-r)}$ are matrices whose columns form bases for $N((WA)^{k_2})$ and $N(((WA)^{k_2})^*)$ respectively. We define $E = UV^*$. Then

$$A_{d,W} = A((WA)^{k_2+2} + UV^*)^{-1}(WA)^{k_2}. \tag{2.1}$$

Proof. By hypothesis, we deduce that

$$R(E) = R(UV^*) = R(U) = N((WA)^{k_2}), \tag{2.2}$$

$$N(E) = N(UV^*) = N(V^*) = [R(V)] = [N((WA)^{k_2})^*] = R((WA)^{k_2}). \tag{2.3}$$

Thus $(WA)^{k_2}E = 0$, $E(WA)^{k_2} = 0$. Since $\text{rank}(E^2) = \text{rank}(E)$ and $\text{rank}(((WA)^{k_2})^2) = \text{rank}((WA)^{k_2})$, we obtain $\text{Ind}(E) = \text{Ind}((WA)^{k_2}) = 1$.

From lemma1.1(b) combined with (2.2) and (2.3), we have

$$(WA)^{k_2}E_g = 0,\ E((WA)^{k_2})_g = 0,\ \text{and}\ E_g(WA)^{k_2} = 0. \tag{2.4}$$

Using (2.2),(2.3),(2.4) and Lemma1.1(c) and (d), we have

$$((WA)^{k_2+2}+E)((WA)_g^{k_2+2}+E_g) = P_{R((WA)^{k_2}),N((WA)^{k_2})} + P_{N((WA)^{k_2}),R((WA)^{k_2})} = I.$$

Therefore $((WA)^{k_2+2}+E)^{-1} = (WA)_g^{k_2+2}+E_g$, and

$$A((WA)^{k_2+2}+E)^{-1}(WA)^{k_2} = A(WA)_g^{k_2+2}(WA)^{k_2} + AE_g(WA)^{k_2} =$$
$$A((WA)_d)^{k_2+2}(WA)^{k_2} = A(WA)_d^2 = A_{d,W}.$$

In fact, we can show that $A_{d,W} = A((WA)^l + E)^{-1}(WA)^{l-2}$, $l \geqslant k_2+2$ from the proof.

Theorem 2.3 Let $A \in C^{m\times n}$, $W \in C^{n\times m}$, and $\mathrm{Ind}(AW) = k_1$, $\mathrm{Ind}(WA) = k_2$. Suppose that $U, V \in C_{m-r}^{m\times(m-r)}$ are matrices whose columns form bases for $N((AW)^{k_1})$ and $N(((AW)^{k_1})^*)$ respectively. Then $A_{d,W} = ((AW)^l + UV^*)^{-1}(AW)^{l-2}A$, $l \geqslant k_1+2$.

Proof The proof is analogous to that of Theorem 2.2.

3 Computation of the W-weighted Drazin inverse

In the following theorem, a characterization for the W-weighted Drazin $A_{d,W}$ was presented by Yimin Wei.

Theorem 3.1([4]) Suppose that $A \in R^{m\times n}$, $W \in R^{n\times m}$, $\mathrm{Ind}(AW) = k_1$, $\mathrm{Ind}(WA) = k_2$, and $\mathrm{rank}((AW)^{k_1}) = r_1$, $\mathrm{rank}((WA)^{k_2}) = r_2$. Then there exist a unique matrix X such that

$$(AW)^{k_1}X = 0,\ X(AW)^{k_1} = 0,\ X^2 = X, \text{ and } \mathrm{rank}(X) = m - r_1, \tag{3.1}$$

a unique matrix Y such that

$$(WA)^{k_2}Y = 0,\ Y(WA)^{k_2} = 0,\ Y^2 = Y, \text{ and } \mathrm{rank}(Y) = n - r_2, \tag{3.2}$$

and a unique matrix Z such that

$$\mathrm{rank}\begin{pmatrix} WAW & I-Y \\ I-X & Z \end{pmatrix} = \mathrm{rank}(WAW). \tag{3.3}$$

The matrix Z is the W-weighted Drazin inverse $A_{d,W}$ of A. Furthermore, we have

$$X = I - A_{d,W}WAW = I - (AW)_d AW, \tag{3.4}$$

and

$$Y = I - WAWA_{d,W} = (WA)_d(WA). \tag{3.5}$$

By Lemma 1.3 and Theorem 3.1, we can obtain a method for computing the W-weighted Drazin inverse of A. In the following theorem, let $(WAW)[\alpha \mid \beta]$ denote the $r\times r$ sub-matrix of WAW which has the row index set $\alpha = \{i_1, \cdots, i_r\}$ and the column index set $\beta = \{j_1, \cdots, j_r\}$. Let $M = \{1, 2, \cdots m\}$ and $N = \{1, 2, \cdots n\}$. Then we have:

Theorem 3.2. Let $A \in R^{m\times n}$, $W \in R^{n\times m}$, $\mathrm{rank}(WAW) = r$, $\mathrm{Ind}(AW) = k_1$, and $\mathrm{Ind}(WA) = k_2$. Let $(WAW)[\alpha \mid \beta]$ denote the $r\times r$ sub-matrix of WAW. If the matrices X and Y satisfy conditions (3.4) and (3.5), respectively, then

$$A_{d,W} = (I-X)[M \mid \beta]((WAW)[\alpha \mid \beta])^{-1}(I-Y)[\alpha \mid N]. \tag{3.6}$$

Proof Set $P = \begin{pmatrix} (WAW)[\alpha \mid \beta] & (I-Y)[\alpha \mid N] \\ (I-X)[M \mid \beta] & A_{d,W} \end{pmatrix}$. Then we have $\mathrm{rank}(P) \geqslant \mathrm{rank}((WAW)[\alpha \mid \beta]) = r = \mathrm{rank}(WAW)$. It follows from Theorem 3.1 that $\mathrm{rank}(P) \leqslant \mathrm{rank}\begin{pmatrix} WAW & I-Y \\ I-X & Z \end{pmatrix} = \mathrm{rank}(WAW)$.

Therefore, we obtain

$$\mathrm{rank}(P) = \mathrm{rank}((WAW)[\alpha \mid \beta]) = \mathrm{rank}(WAW). \tag{3.7}$$

Lemma 1.3 and (3.7) imply that (3.6) holds.

In terms of Theorem 3.2, we present the following algorithm to compute the W-weighted Drazin inverse of A.

Algorithm Let $A \in R^{m\times n}$, and $W \in R^{n\times m}$.

(1) Compute $\mathrm{rank}(WAW)$, $\mathrm{Ind}(AW)$ and $\mathrm{Ind}(WA)$. Set $k_1 = \mathrm{Ind}(AW)$, $k_2 = \mathrm{Ind}(WA)$ and $r =$

rank(WAW). Determine the index sets α and β and obtain the $r\times r$ matrix $(WAW)[\alpha\mid\beta]$.

(2) Compute $((WAW)[\alpha\mid\beta])^{-1}$.

(3) Construct the nonsingular matrix P_1 such that $P_1^{-1}AWP_1$ is the Jordan form of AW. Compute

$$X = I-(AW)_d(AW) = P_1\begin{pmatrix}0 & 0\\ 0 & I_{m-r_1}\end{pmatrix}P_1^{-1}.$$

Similarly, Construct the nonsingular matrix P_2 such that $P_2^{-1}WAP_2$ is the Jordan form of WA. Compute

$$Y = P_2\begin{pmatrix}0 & 0\\ 0 & I_{n-r_2}\end{pmatrix}P_2^{-1}.$$

(4) Compute the sub-matrices $(I-X)[M\mid\beta]$ and $(I-Y)[\alpha\mid N]$.

(5) Compute the W-weighted Drazin inverse $A_{d,W}$ of A according to (3.6).

Example. Determine the W-weighted Drazin inverse of the following matrix:

$$A=\begin{pmatrix}1&0&1&0&0\\0&1&-1&-1&0\\0&0&0&0&2\\0&0&1&0&0\end{pmatrix}\in R^{4\times5},\ W=\begin{pmatrix}1&0&0&0\\1&1&0&0\\1&0&0&-2\\0&1&0&1\\0&0&1&0\end{pmatrix}\in R^{5\times4},\ WAW=\begin{pmatrix}2&0&0&-2\\2&0&0&-1\\0&0&0&2\\1&0&0&-1\\0&0&2&0\end{pmatrix}.$$

$$\mathrm{Ind}(AW)=1,\ \mathrm{Ind}(WA)=2,\ \mathrm{rank}(WAW)=3,$$

By Step 1, take the index set $\alpha=\{3,4,5\}$ and $\beta=\{1,3,4\}$. We have

$$(WAW)[\alpha\mid\beta]=\begin{pmatrix}0&0&2\\1&0&-1\\0&2&0\end{pmatrix},((WAW)[\alpha\mid\beta])^{-1}=\begin{pmatrix}0.5&1&0\\0&0&0.5\\0.5&0&0\end{pmatrix}.$$

Take $P_1=\begin{pmatrix}2&0&0&0\\1&0&-2&1\\0&4&0&0\\0&0&2&0\end{pmatrix}$ and $P_2=\begin{pmatrix}2&0&0&0&0\\2&1&0&1&0\\0&2&0&0&0\\1&0&0&0&1\\0&0&4&0&0\end{pmatrix}$.

Then $P_1^{-1}=\begin{pmatrix}0.5&0&0&0\\0&0&0.25&0\\0&0&0&0.5\\-0.5&1&0&1\end{pmatrix}$ and $P_2^{-1}=\begin{pmatrix}0.5&0&0&0&0\\0&0&0.5&0&0\\0&0&0&0&0.25\\-1&1&-0.5&0&0\\-0.5&0&0&1&0\end{pmatrix}$.

By Steps 3 and 4, we get

$$X=\begin{pmatrix}0&0&0&0\\-0.5&1&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},\ Y=\begin{pmatrix}0&0&0&0&0\\-1&1&-0.5&0&0\\0&0&0&0&0\\-0.5&0&0&1&0\\0&0&0&0&0\end{pmatrix},$$

$$(I-X)[M\mid\beta]=\begin{pmatrix}1&0&0\\0.5&0&-1\\0&1&0\\0&0&1\end{pmatrix},\ (I-Y)[\alpha\mid N]\begin{pmatrix}0&0&1&0&0\\0.5&0&0&0&0\\0&0&0&0&1\end{pmatrix}.$$

Thus, by (3.6) we get

$$A_{d,W} = \begin{pmatrix} 0.5 & 0 & 0.5 & 0 & 0 \\ 0.25 & 0 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0 \end{pmatrix}.$$

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加W权Drazin逆显式表达式及其计算

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摘要: 给出了几个加权 Drazin 逆的显式表达式. 通过这些表达式可以减少计算量. 同时,通过 一个秩方程,推导出求加权 Drazin 逆的一个计算方法.

关键词: 加权 Drazin 逆; 指标; 秩; 加边矩阵

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