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Generalized Nash Equilibrium and Switched Control in Game Theory

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Abstract: A new concept of generalized Nash equilibrium is introduced by coupling it with each player's incentive. There are two type of incentive: "maximize his/her own expected utility payff" and "maximize the opponent's utility payoff". If each player is rational and intends to maximize his/her own expected utility payoff, then it will yield a conventional Nash equilibrium (also the frst type of generalized Nash equilibrium). On the other hand, if each player is wise and intends to maximize the opponent's utility payoff, then the second type of generalized Nash equilibrium is obtained. This extension may be an effcient way to apply the concept of Nash equilibrium to the cooperative game. Furthermore, the concept of switched control is used to help each player determine his/her incentive.

Key words: game theory; generalized Nash equilibrium; switched control
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广义 Nash 平衡点和切换控制在对策论中的应用

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摘　　要：通过把平衡点和决策者的动机耦合的方法，提出了广义纳什平衡点这一新概念。决策者的动机通常有两类：一是最大化自己的利益，另一则是最大化对手的利益。如果每一个决策者的动机都是第一类，一个理性的群体就会形成，整个系统最终会达到第一类平衡点(也就是经典的纳什平衡点)。如果每一个决策者的动机都是第二类，一个有智慧的群体就会形成，整个系统最终会达到第二类平衡点。同时，切换控制被用来帮助决策者确定他们的动机。

关　键　词：对策论；广义纳什平衡点；切换控制

1　Introduction

Game theory has been widely studied and applied in the fields of economics, network and control engineering[1~3]. There are two main branches of game theory: cooperative and noncooperative game theory. Noncooperative game theory deals largely with how intelligent player interacts with other players in an effort to maximize his/her payoff, while cooperative game theory provides strategies which will maximize the payoff of each coalition. A very important concept in noncooperative game theory is the Nash equilibrium, which provides a measure of how purely rational people might be have. The concept of Nash equilibrium is based on the assumption that each player intelligently understands his/her environment and rationally acts to maximize his/her own expected utility payoff[1]. Noncooperative solutions often seem to be descriptive of real outcomes. Since cooperative solutions can in principle be converted to a win-win game, it is desirable to obtain a cooperative solution by generalizing the Nash equilibrium to obtain a cooperative solution when considering economic progress in a globalized world.

A class of two-player games is studied in this paper, where each player has enough power to affect the desired allocation between them[4~6]. Two contributions of this paper are listed as below.

①The concept of Nash equilibrium is generalized by coupling it with each player's incentive. There are two

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types of incentive: "maximize his/her own expected utility payoff" and "maximize the opponent's expected utility payoff". For example, if each player is wise and intends to maximize the opponent's utility payoff, the second type of generalized Nash equilibrium is obtained. The motivation for each player to maximize the opponent's utility payoff is to eventually maximize his/her own utilization payoff. Meanwhile it yields a perfect cooperation by the second type of generalized Nash equilibrium. Thus this extension may be an efficient way to the concept of Nash equilibrium to the cooperative game. Although the cooperation is pursued by the second type of generalized Nash equilibrium, players are not prevented from the competition by it. The motivation for each player to improve himself/herself is to guarantee that he/she has the capability to affect the desired allocation.

②Provide a new application of switched control[7]. Consider a game between players 1 and 2. Suppose that their incentive is the same. However, their incentive is neither fixed as the first type nor fixed as the second type. It is determined by their relative powers. If one player is much stronger than the other, he/she should choose the first type of incentive to maximize his/her utilization payoff. If both players have almost the same power, they should choose the second type of incentive to maximize their utilization payoff. Clearly, the concept of switched control[7] can be used to achieve these objectives.

The rest of this paper is organized as follows. The concept of generalized Nash equilibrium is proposed in the following section. The concept of switched control is used to help each player determine his/her incentive in section 3. Finally, some concluding remarks are given in section 4.

2 The Concept of Generalized Nash Equilibrium

In this section the normal form is adopted to describe a class of 2-player games[5]. For simplicity, it is assumed that there are two basic strategies $\hat{U}_s$ and $\hat{U}_c$ where $\hat{U}_s$ corresponds to "behave selfshly" and $\hat{U}_c$ means "contribute to the common good". The strategies of players 1 and 2, $U_1(k)$ and $U_2(k)$ are is the set $\{\hat{U}_s, \hat{U}_c\}$. The utility payoff functions $f_i\,(i=1,2)$ have the following properties:

$$f_1(\hat{U}_c,\hat{U}_s)\leqslant f_1(\hat{U}_s,\hat{U}_s)\leqslant f_1(\hat{U}_c,\hat{U}_c)\leqslant f_1(\hat{U}_s,\hat{U}_c) \quad (1)$$

$$f_2(\hat{U}_s,\hat{U}_c)\leqslant f_2(\hat{U}_s,\hat{U}_s)\leqslant f_2(\hat{U}_c,\hat{U}_c)\leqslant f_2(\hat{U}_c,\hat{U}_s) \quad (2)$$

An example of a lunch parther is used to illustrate to above model.

Example 1 Two players work at the same company, have agreed that they will always split the bill evenly. There is a restaurant that offers a luxurious $ 20 meal and a simpler $ 10 meal. Suppose each player, if he/she had to pay only his/her own bill, would prefer the $ 10 meal. $\hat{U}_s$ and $\hat{U}_c$ correspond to "choose $ 20 meal" and "choose $ 10 meal", respectively. The utility payoff ma trix is given in Table 1.

Table 1 Two players' utility payoffs

	(player 1, $ 10)	(player 1, $ 20)
(player 2, $ 10)	(4,4)	(5,0)
(player 2, $ 20)	(0,5)	(3,3)

The concept of generalized Nash equilibrium is then defined as below.

Definition 1 Suppose that each player maintains the same incentive, a strategy pair (U_1, U_2) is a generalized Nash equilibrium with respect to the incentive if player 1 (or 2) has no incentive to deviate from U_1 (or U_2) provided that the opponent maintains U_2 (or U_1).

It can be shown from Definition 1 that a type of generalized Nash equilibrium is associated with and determined by a corresponding type of incentive. There are two types of incentives as in Table 2.

Table 2 Two types of incentives

Type	Incentive
1	maximize his/her own utility payoff
2	maximize the opponent's utility payoff

The first type of incentive is the conventional one that is widely used in the existing game theory[1]. It can be shown from (1) and (2) that:

$$f_1(U_1,\hat{U}_s)\leqslant f_1(\hat{U}_s,\hat{U}_s);\forall\, U_1$$

$$f_2(\hat{U}_s,U_2)\leqslant f_2(\hat{U}_s,\hat{U}_s);\forall\, U_2$$

In other words, each player has no incentive to deviate from $\hat{U}_s$ provided that the opponent maintains $\hat{U}_s$. Thus, $(\hat{U}_s, \hat{U}_s)$ is the first type of generalized Nash equilibrium[6], that is based on the assumption that each player is rational. An example is given as follows:

Example 2 Consider Example 1, and suppose that each player's incentive is to maximize his/her own utilization payoff. The dominant strategy is to "choose $ 20 meal", and the first type of generalized Nash equilibrium will be obtained if both players choose the $ 20 meal. However, this equilibrium leads to a lower utility payoff for each player, (3,3), than that of ("choosing $ 10 meal", "choosing $ 10 meal"), (4,4).

On the other hand, it can be shown that:

$f_1(\hat{U}_c, \hat{U}_c) \geq f_1(\hat{U}_c, U_2)$;$\forall\ U_2$

$f_2(\hat{U}_c, \hat{U}_c) \geq f_2(U_1, \hat{U}_c)$;$\forall\ U_1$

Thus, if each player's incentive is to maximize the opponent's utility payoff that is called the second type of incentive, each player has no incentive to deviate from $\hat{U}_c$ provided that the opponent maintains $\hat{U}_c$. $(\hat{U}_c, \hat{U}_c)$ is thus the second type of generalized Nash equilibrium. i.e., a generalized Nash equilibrium with respect to the second type of incentive. It can be verified that:

$$f_1(\hat{U}_s, \hat{U}_s) + f_2(\hat{U}_s, \hat{U}_s) \leq f_1(\hat{U}_c, \hat{U}_c) + f_2(\hat{U}_c, \hat{U}_c)$$

The second type of generalized Nash equilibrium is better than the first type of generalized Nash equilibrium when both players are regarded as a whole. Thus, a perfect cooperation is obtained when each player tries to maximize the opponent's utility payoff. Meanwhile, it can also be verified that:

$f_i(\hat{U}_s, \hat{U}_s) \leq f_i(\hat{U}_c, \hat{U}_c)$;$\forall\ i$

Therefore, the objective of the second type of incentive is to eventually maximize each player's own utilization payoff at the equilibrium. This is illustrated considering the case that inequalities (1) and (2) are satisfied. Two examples are given as below.

Example 3 Consider Example 1 again, and suppose that each player's incentive is to maximize the opponent's utilization payoff. The dominant strategy is to "choose \$ 10 meal", and the second type of generalized Nash equilibrium will be obtained if both players choose the \$ 10 meal. Obviously, this equilibrium leads to a higher utility payoff for each player than that of ("choosing \$ 20 meal", "choosing \$ 20 meal").

Example 4 Consider an example given in [8]. There are two players A and B who have identical strategy space with elements "high" and "low". The corresponding utility payoff matrix is given in Table 3.

Table 3 Utilization payoff matrix

	high	low
high	(5,5)	(0,10)
low	(10,0)	(0, 0)

Consider the case that each player's incentive is to maximize the opponent's utilization payoff. The generalized Nash equilibrium is ("high", "high") and the corresponding utilization payoff is (5,5). Clearly, this is also a Pareto optimal point[6].

Also consider the case where inequalities (1) and (2) are not satisfied. Two examples are given.

Example 5 There are two players, Bob and Susan. Their strategies are A and B, and a and b, respectively. The corresponding utility payoff matrix is given in Table 4.

Table 4 Utilization payoff matrix

	a	b
A	(1,2)	(3,1)
B	(0,-2)	(2,-3)

Assume that each player's incentive is to maximize the opponent's utilization payoff, the generalized Nash equilibrium is (A,b) and the corresponding utilization payoff is (3,1). Obviously, this is a Pareto optimal point[6].

Example 6 Chicken and Volunter's Dilemma

There is a game called chicken, in which two people drive very fast cars toward each other from opposite ends of a long straight road. There are two strategies for them: swerve and drive straight. If one of them swerves before the other, he is called a chicken. Clearly, they will crash if neither swerves. The utility payoff matrix is as given in Table 5[9].

Table 5 Utilization payoff matrix

	Swerve	Drive Straight
Swerve	(2,2)	(1,3)
Drive Straight	(3,1)	(0,0)

If each player's incentive is to maximize the opponent's utilization payoff, then the generalized Nash equilibrium is (Swerve, Swerve) and the corresponding utilization payoff is (2,2). It can be easily shown that this is a Pareto optimal point[6].

If all players' incentives belong to the second type, they will share more and more resources including information, knowledge, technologies, and so on. Their utilization payoffs will be increased according to three facts:

①The utilization efficency of natural resources can be kept increasing. ②The working effecency of human resources can be kept improving. ③The humanty's desire is without limitation.

Clearly, the second type of incentive can be employed in the development of Internet, communication and transportation policies, and possibly global economic policies.

3 Determination of Incentive

Since both players are assumed to have the same incentive, players' incentive is also the incentive of a game.

For simplicity, consider the case that both players know the opponent's power. The concept of switched control is used in this section to determine the incentive of the game[7].

Normally, the player with stronger power has the right to determine the incentive of the game according to the opponent's power. The objective is to maximize his/her own utilization payoff in the end. Assume that player 1's power is stronger than player 2's. The incentive of the game is then determined by

$$\max_{U_1 \in \{\hat{U}_s, \hat{U}_c\}} \{f_1(U_1, U_1)\} \tag{3}$$

When $U_1 = \hat{U}_s$, the incentive of the game is the first type. Otherwise, it is the second type. Player 2 can then choose the same incentive to maximize his/her utilization payoff actually. However, player 2 can keep improving himself/herself such that he/she almost has the same power as player 1. As a result, player 1 has to choose the second type of incentive.

4 Conclusion

Two types of generalized Nash equilibria are provided by coupling them with each player's incentive. The first type of generalized Nash equilibrium is well adopted in the noncooperative game theory. The second type of generalized Nash equilibrium is based on the assumption that each player is wise. This extension proves to be an effective way to apply the concept of concept of Nash equilibrium to the cooperative game. There are several interesting problems related to the application of control theory listed as follows:

①In practice, it is desirable to design a strategy for player 1 (or 2) such that a cooperate solution is provided if player 2 (or 1) contributes to the common good, else a noncooperative solution is finally obtained if player 2 (or 1) behaves selfishly. This will be studied in future researches by using the concept of switched control.

②Another interesting topic is to study the case that each player cannot know the opponent's strategy and power exactly. Robust control may be used to solve this problem.

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