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球型平移网络逼近周期函数

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摘 要 研究了球型平移网络对周期函数的逼近问题. 文章首先将基函数 e^{imx} 分别表示成为两种球型平移网络. 进一步, 将有关多重 Fourier 级数的 Bochner–Riesz 平均表示成为球型平移网络的形式. 在此基础上构造出了两类球型平移网络序列, 并借助于有关 Bochner–Riesz 平均对 L^p 空间中函数的逼近结果给出了这两类球型平移网络序列在 L^p 空间中的逼近阶.

关键词 周期函数; 球平移网络; Bochner–Riesz 平均
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Approxiamtion of Periodic Functions by Spherical Translation Networks

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Abstract In this paper, we investigate the approximation of periodic functions by translation networks formed by two kinds of spherical translations for a given periodic function. We first express the basis functions e^{imx} as two kinds of spherical translation network sequences respectively. Secondly, the Bochner–Riesz means of multiple Fourier series are also expressed as spherical translation network sequences, with which two kinds of spherical translation network sequences are defined, and thirdly, the orders of approxiamtion in L^p spaces for these two kinds of spherical translation sequences are respectively given by making use of the approximation order of Bochner–Riesz means in L^p spaces.

Keywords periodic function; spherical translation networks; Bochner–Riesz means
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1 引言及定理

稠密性问题的研究及其相应逼近算子的构造是构造性分析所探讨的永恒课题. 小波分析出现之后, 由单个函数的平移和伸缩所生成的网络逼近问题引起了人们的极大兴趣 [1−7]. 设 $q \geq 1$ 为正整数, R^q 为 q- 维欧氏空间, Z^q 表示 R^q 中坐标为整数的向量的集合. 对 $x=(x_1, x_2, \ldots, x_q) \in R^q$, 定义范数 $\|x\| = (\sum_{k=1}^q x_k^2)^{\frac{1}{2}}$. 记 $Q = Q^q = [-\pi, \pi]^q \subset R^q$, $1 \leq p \leq +\infty$, $L^p(Q^q)$ 表示关于

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每个变量均以 2π 为周期的 p- 幂可积函数类, $\phi \in L^p(Q^d)$, $1 \le d \le q$ 为正整数. 记由通常平移 $\tau_h(\phi, x) = \phi(x+h)$, x, $h \in Q^d$, 所生成的网络为

$$\triangle_\phi(x) := \triangle_{\phi, J_{d,q}}(x) = \left\{ \sum_{i=1}^N \phi(A_i x + b_i) : A_i \in J_{d\times q},\, b_i \in R^d \right\}, \quad x \in Q^q,$$

其中 $J_{d\times q}$ 为秩为 d, 元素为整数的 $d \times q$ 阶矩阵的集合. 当 $d = q$ 时, A_i 为单位矩阵, 这时 $\triangle_{\phi, J_{d,q}}(x)$ 为平移网络.

记 $\triangle_\phi(x)$ 的线性扩张为 $\mathrm{Span}\triangle_\phi(x)$. Mhaskar 和 Micchelli 在文 [8] 中证明 $\mathrm{Span}\triangle_\phi(x) = L^p(Q^q)$ 当且仅当 ϕ 的 Fourier 系数 $c_l(\phi) = \frac{1}{(2\pi)^d}\int_{Q^q} \phi(x) \mathrm{e}^{-\mathrm{i}lx}\, dx \ne 0$, $l \in Z^d$ (也见文 [9]), 并且借助于 de la Vallée Poussin 平均构造出了具体的神经网络算子, 给出了逼近阶. 这方面的讨论在文 [10] 中被推广到了非周期的情况, 借助于第一类 Chebyshev 正交多项式的 de la Vallée Poussin 平均也构造出了具体的神经网络算子, 给出了对非周期权函数类 $L^p_W[-1,1]$, $W = (1-x_1^2)^{-\frac{1}{2}} \cdots (1-x_q^2)^{-\frac{1}{2}}$ 的逼近阶. 由周期函数关于维数的转换关系

$$\frac{1}{(2\pi)^q}\int_{[-\pi,\pi]^q} \phi(Ax)\, dx = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \phi(x) dx, \quad A \in J_{d\times q}$$

可知神经网络可以由平移网络转化而得到. 因此, 平移网络才是最本质的.

注意到周期函数的球型平移 $S_t(f, x)$, $t > 0$,

$$S_t(f, x) = \frac{1}{\omega_{q-1}} \int_{\sum_{q-1}} f(x - t\xi)\, d\sigma(\xi), \quad f \in L^1(Q^q), \quad x \in Q^q, \tag{1}$$

(其中 $\omega_{q-1} = \frac{2\pi^{\frac{q}{2}}}{\Gamma(\frac{q}{2})} = \int_{\sum_{q-1}} d\sigma(\xi)$ 为 R^q 中的单位球面 $\sum_{q-1}$ 的面积, $\sigma(\xi)$ 为 $\sum_{q-1}$ 上的面积微元) 和

$$B_t(f, x) = \frac{1}{m(B)t^q} \int_{B_t} f(x+u) dv(u), \quad f \in L^1(Q^q), \quad x \in Q^q, \quad t > 0, \tag{2}$$

(其中 $m(B)$ 为 R^q 中单位球体 B 的体积, $B_t = \{x \in R^q : \|x\| \le t\}$) 在研究多重 Fourier 级数 Bochner–Riesz 平均逼近方面具有明显的优点 [11], 也被用于定义函数的光滑模 [12, 13]. 如果认为 $\tau_t(\phi, x)$ 是沿坐标轴方向的平移, 则 $S_t(\phi, x)$ 和 $B_t(\phi, x)$ 可被认为是沿球面法线方向的伸展, 前后之间的意义有所不同.

因此, 当 ϕ 为周期函数时, 由 ϕ 的球型平移 $S_t(\phi, x)$, $B_t(\phi, x)$, $t > 0$, 所生成的函数类

$$\triangle_\phi^S(x) = \{S_\xi(\phi, x+\zeta), \xi \in R^1, \zeta \in Q^q\} \cup \{1\}, \quad x \in Q^q$$

和

$$\triangle_\phi^B(x) = \{B_\xi(\phi, x+\zeta), \xi \in R^1, \zeta \in Q^q\} \cup \{1\}, \quad x \in Q^q,$$

是否也可以实现对周期函数的逼近是有探讨意义的.

多重 Fourier 级数的 Bochner–Riesz 平均是 Fourier 分析的重要研究内容和经典调和分析的重要工具之一. 由于它在很多方面表现出更适合于作为一元 Fourier 分析理论的推广 [11], 因此, 关于其逼近的研究曾经引起过许多数学家的兴趣 [11, 14–19]. 文中将借助于有关周期函数 Fourier 展开的球型 Bochner–Riesz 平均构造出具体的形如 $\triangle_\phi^S(x)$ 和 $\triangle_\phi^B(x)$ 的平移网络算子并给出逼近阶的估计, 这方面的研究其实也是对 Bochner–Riesz 平均应用的一种探讨.

对于 $1 \le p < +\infty$, 用 $L^p(Q^q)$ 表示关于每一个变量以 2π 为周期并且满足 $\|f\|_{p,Q^q} = \left(\int_{Q^q} |f(x)|^p\, dx\right)^{\frac{1}{p}} < +\infty$, $1 \le p < +\infty$ 的复函数类, 当 $p = +\infty$ 时, 记 $L^{+\infty}(Q^q) = C(Q^q)$, 并定义 $\|f\|_{C,Q^q} = \max_{x\in Q^q} |f(x)|$, $f \in C(Q^q)$.

对于 $f\in L^1(Q^q)$, 其 Bochner–Riesz 球型平均定义为 [11]: $S_R^\alpha(f,\,x)=\sum_{\|m\|<R}\big(1-\frac{\|m\|^2}{R^2}\big)^\alpha\cdot c_m(f)\mathrm{e}^{imx}$, $x\in Q^q$, 其中 $R>0$, Re $\alpha>-1$, $m\in Z^q$. 并记 $S_0^\alpha(f,\,x)=c_0(f)$. 由文 [11, 98–99 页] 知当 $\alpha>\frac{q-1}{2}$ 时

$$\begin{aligned}S_R^\alpha(f,\,x)&=\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(2\pi)^{\frac{q}{2}}}\int_0^{+\infty}S_t(f,\,x)\frac{R^qt^{q-1}J_{\frac{q}{2}+\alpha}(Rt)}{(Rt)^{\frac{q}{2}+\alpha}}\,dt\\&=\frac{1}{(2\pi)^q}\int_{R^q}f(x+y)\frac{2^{\alpha+\frac{q}{2}}R^q\pi^{\frac{q}{2}}\Gamma(\alpha+1)J_{\frac{q}{2}+\alpha}(R\|y\|)}{(R\|y\|)^{\frac{q}{2}+\alpha}}\,dy,\quad x\in Q^q,\end{aligned}\tag{3}$$

其中 $J_v(z)$ 是以 z 为变元的第一类 v 阶 Bessel 函数.

用 $[a]$ 表示 a 的整数部分. 设 $R'>R>0$, $f,\phi\in L^p(Q^q)$, $1\le p\le+\infty$; $c_m(\phi)\neq0$, $m\in Z^q$. 定义

$$\begin{aligned}S_{R,\phi}^\alpha(f,\,x)=c_0(f)+\Big(\frac{1}{2([R']+[R])+1}\Big)^q\sum_{0\le l\le 2([R']+[R])}\sum_{0<\|m\|<R}\frac{\omega_{q-1}c_m(S_R^\alpha(f))\mathrm{e}^{\frac{2lm\pi\mathrm{i}}{2([R']+[R])+1}}}{c_m(\phi)c_q[1-(\frac{\|m\|}{R'})^2]^\alpha}\\\times\Big(\frac{\pi}{\frac{R'}{\|m\|}+1}\Big)\sum_{k\in Z^1}S_{\frac{k\pi}{R'+\|m\|}}\Big(\phi,\,x-\frac{2\pi l}{2([R']+[R])+1}\Big)j_{\frac{q}{2}-1}\Big(\frac{k\pi}{\frac{R'}{\|m\|}+1}\Big),\quad x\in Q^q\end{aligned}\tag{4}$$

及

$$\begin{aligned}S_{R,\phi}^{*,\alpha}(f,x)=c_0(f)+\Big(\frac{1}{2([R']+[R])+1}\Big)^q\sum_{0\le l\le 2([R']+[R])}\sum_{0<\|m\|<R}\frac{m(B)c_m(S_R^\alpha(f))\mathrm{e}^{\frac{2lm\pi\mathrm{i}}{2([R']+[R])+1}}}{c_m(\phi)c_q^*[1-(\frac{\|m\|}{R'})^2]^\alpha}\\\times\Big(\frac{\pi}{\frac{R'}{\|m\|}+1}\Big)\sum_{k\in Z^1}B_{\frac{k\pi}{R'+\|m\|}}\Big(\phi,\,x-\frac{2\pi l}{2([R']+[R])+1}\Big)j_{\frac{q}{2}-1}\Big(\frac{k\pi}{\frac{R'}{\|m\|}+1}\Big),\quad x\in Q^q,\end{aligned}\tag{5}$$

其中 $j_\alpha(z)=\frac{2^\alpha\Gamma(\alpha+1)J_\alpha(z)}{z^\alpha}$, $c_q=\frac{\Gamma(q-2)\Gamma(\frac12)}{\Gamma(\frac{2q-3}{2})}$, $c_q^*=\Gamma\big(\frac12\big)\Gamma\big(\frac{q}{2}\big)\int_0^1\frac{\tau^{\frac{3q}{2}-2}}{(1+\tau^2)^{\frac12}}\,{}_2F_1\big(\frac14,\frac34,\frac{q}{2},\frac{4\tau^2}{(1+\tau^2)^2}\big)\,d\tau$, 而 ${}_2F_1(\alpha,\,\beta,\,\rho,\,z)=\sum_{n=0}^\infty\frac{(\alpha)_n(\beta)_n}{n!(\rho)_n}z^n$, $(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$, $(\alpha)_0=1$, 则 $S_{R,\phi}^\alpha(f,\,x)\in\triangle_\phi^S(x)$, $S_{R,\phi}^{*,\alpha}(f,\,x)\in\triangle_\phi^B(x)$, 且有下列估计成立.

定理 设 $f,\,\phi\in L^p(Q^q)$, $R'\ge R>0$, $1<p<+\infty$, $\alpha>\alpha_p=\frac{q-1}{2}|\frac{2}{p}-1|$ 且 $c_m(\phi)\neq0$, $m\in Z^q$, 则存在常数 $C>0$, 使得

$$\|S_{R,\phi}^\alpha(f)-f\|_{p,Q^q}\le C\Big[\omega_2\Big(f,\,\frac{1}{R}\Big)_{p,Q^q}+\frac{R^\sigma\|f\|_{p,Q^q}}{\Phi_{R,\,R'}^\alpha(\phi)}\,\omega_2\Big(\phi,\,\frac{1}{R'}\Big)_{p,Q^q}\Big]\tag{6}$$

及

$$\|S_{R,\phi}^{*,\alpha}(f)-f\|_{p,Q^q}\le C\Big[\omega_2\Big(f,\,\frac{1}{R}\Big)_{p,Q^q}+\frac{R^\sigma\|f\|_{p,Q^q}}{\Phi_{R,R'}^{*,\alpha}(\phi)}\,\omega_2\Big(\phi,\,\frac{1}{R'}\Big)_{p,Q^q}\Big],\tag{7}$$

其中 $\sigma=q/\min(p,\,2)$,

$$\Phi_{R,R'}^\alpha(\phi)=\frac{\min_{0<\|m\|<R}[1-(\frac{\|m\|}{R'})^2]^\alpha\,|c_m(\phi)|\,c_q}{\omega_{q-1}\int_0^{+\infty}|j_{\frac{q}{2}-1}(t)|\,dt},$$

$$\Phi_{R,R'}^{*,\alpha}(\phi)=\frac{\min_{0<\|m\|<R}[1-(\frac{\|m\|}{R'})^2]^\alpha\,|c_m(\phi)|\,c_q^*}{m(B)\int_0^{+\infty}|j_{\frac{q}{2}-1}(t)|\,dt}.$$

这里及后文中用 C 表示与 $n,\,f$ 均无关的非负常数, 且同一常数符号 C 在多次使用时所代表的值可能不同.

2 一些引理

引理 1 设 $f,\,\phi\in L^1(Q^q)$, 且 $c_m(\phi)\neq 0$, $m\in Z^q$, 则

$$\mathrm{e}^{imx}=\frac{\omega_{q-1}}{(2\pi)^q c_m(f)c_q}\int_0^{+\infty}\bigg(\int_{Q^q}S_{\frac{t}{\|m\|}}(f,\,x-u)\,\mathrm{e}^{imu}\,du\bigg)j_{\frac{q}{2}-1}(t)\,dt,\quad x\in Q^q \tag{8}$$

及

$$\mathrm{e}^{imx}=\frac{m(B)}{(2\pi)^q c_m(f)c_q^*}\int_0^{+\infty}\bigg(\int_{Q^q}B_{\frac{t}{\|m\|}}(f,\,x-u)\,\mathrm{e}^{imu}\,du\bigg)j_{\frac{q}{2}-1}(t)\,dt,\quad x\in Q^q. \tag{9}$$

证明 (8) 式的证明. 显然 $\frac{1}{(2\pi)^q}\int_{S^q}S_t(f,\,x-u)\mathrm{e}^{-imu}\,du=\frac{\mathrm{e}^{-imx}}{(2\pi)^q}\int_{S^q}S_t(f,\,v)\mathrm{e}^{imv}\,dv$. 当 $\|m\|\neq 0$ 时

$$\begin{aligned}\frac{1}{(2\pi)^q}\int_{S^q}S_t(f,\,v)\,\mathrm{e}^{imv}\,dv&=\frac{1}{\omega_{q-1}}\int_{\sum_{q-1}}\bigg(\frac{1}{(2\pi)^q}\int_{S^q}f(v-t\xi)\,\mathrm{e}^{imv}\,dv\bigg)\,d\sigma(\xi)\\&=\frac{c_{-m}(f)}{\omega_{q-1}}\int_{\sum_{q-1}}\mathrm{e}^{imt\xi}\,d\sigma(\xi)=\frac{c_{-m}(f)j_{\frac{q}{2}-1}(t\,\|m\|)}{\omega_{q-1}},\end{aligned}$$

这里用到了等式 (见文 [20] 的 20 页)

$$\int_{\sum_{q-1}}\mathrm{e}^{iux}\,d\sigma(u)=j_{\frac{q}{2}-1}(\|x\|). \tag{10}$$

因此 $\frac{1}{(2\pi)^q}\int_{S^q}S_t(f,\,x-u)\,\mathrm{e}^{-imu}\,du=\frac{c_{-m}(f)j_{\frac{q}{2}-1}(t\|m\|)}{\omega_{q-1}}\,\mathrm{e}^{-imx}$. 上式两边同时乘以 $j_{\frac{q}{2}-1}(t\|m\|)$ 并对 t 积分, 则有

$$\begin{aligned}\mathrm{e}^{imx}&=\frac{\omega_{q-1}\int_0^{+\infty}\big(\int_{S^q}S_t(f,\,x-u)\mathrm{e}^{imu}du\big)j_{\frac{q}{2}-2}(t\|m\|)\,dt}{(2\pi)^q\,c_m(f)\int_0^{+\infty}\big|j_{\frac{q}{2}-1}(t\|m\|)\big|^2\,dt}\\&=\frac{\omega_{q-1}\int_0^{+\infty}\big(\int_{S^q}S_{\frac{t}{\|m\|}}(f,\,x-u)\mathrm{e}^{imu}du\big)j_{\frac{q}{2}-2}(t)\,dt}{(2\pi)^q c_m(f)\int_0^{+\infty}\big|j_{\frac{q}{2}-1}(t)\big|^2\,dt}.\end{aligned}$$

由文 [21, 396 页] 知道, 当 $\Re(\mu+\nu)>0$ 时

$$\int_0^{\infty}\frac{J_\mu(t)\,J_\nu(t)}{t^{\mu+\nu}}\,dt=\frac{\Gamma(\mu+\nu)\,\Gamma(\frac{1}{2})}{2^{\mu+\nu}\,\Gamma(\mu+\nu+\frac{1}{2})\,\Gamma(\mu+\frac{1}{2})\,\Gamma(\nu+\frac{1}{2})}. \tag{11}$$

因此

$$\int_0^{\infty}\big|j_{\frac{q}{2}-1}(t)\big|^2\,dt=\int_0^{\infty}\bigg|\frac{2^{\frac{q}{2}-1}\Gamma(\frac{q}{2})\,J_{\frac{q}{2}-1}(t)}{t^{\frac{q}{2}-1}}\bigg|^2\,dt=\frac{\Gamma(q-2)\,\Gamma(\frac{1}{2})}{\Gamma(\frac{2q-3}{2})}=c_q.$$

故 (8) 式成立.

(9) 式的证明. 记 $S_\tau=\{v\in R^q:\,\|v\|=\tau\}$, 则

$$\begin{aligned}\frac{1}{(2\pi)^q}\int_{Q^q}B_t(f,\,x)\,\mathrm{e}^{-imx}\,dx&=\frac{1}{m(B)t^q}\int_{Q^q}\frac{1}{(2\pi)^q}\bigg(\int_{Q^q}f(x+u)\,\mathrm{e}^{-imx}\,dx\bigg)\,dv(u)\\&=\frac{1}{m(B)t^q}\int_{Q^q}\frac{1}{(2\pi)^q}\bigg(\int_{Q^q}f(v)\,\mathrm{e}^{-imv}\,dv\bigg)\,\mathrm{e}^{imu}\,dv(u)\\&=\frac{c_m(f)}{m(B)t^q}\int_{B_t}\mathrm{e}^{imu}\,dv(u)=\frac{c_m(f)}{m(B)}\int_B\mathrm{e}^{imvt}\,dv(v)\\&=\frac{c_m(f)}{m(B)}\int_0^1\bigg(\int_{S_\tau}\mathrm{e}^{imvt}\,d\sigma(v)\bigg)\,d\tau\\&=\frac{c_m(f)}{m(B)}\int_0^1 j_{\frac{q}{2}-1}(\tau\,t\,\|m\|)\,\tau^{q-1}\,d\tau.\end{aligned}$$

因此

$$\frac{1}{(2\pi)^q}\int_0^{+\infty}\Big(j_{\frac{q}{2}-1}(t\,\|m\|)\int_{Q^q}B_t(f,\,x)\,\mathrm{e}^{-\mathrm{i}mx}\,dx\Big)\,dt$$
$$=\frac{c_m(f)}{m(B)}\int_0^1\bigg(\int_0^{+\infty}j_{\frac{q}{2}-1}(t\,\|m\|)j_{\frac{q}{2}-1}(\tau t\,\|m\|)\,dt\bigg)\tau^{q-1}\,d\tau,$$

即 $\frac{1}{(2\pi)^q}\int_0^{+\infty}\big[j_{\frac{q}{2}-1}(t)\int_{Q^q}B_{\frac{t}{\|m\|}}(f,x)\mathrm{e}^{-\mathrm{i}mx}\,dx\big]\,dt=\frac{c_m(f)}{m(B)}\int_0^1\big[\int_0^{+\infty}j_{\frac{q}{2}-1}(t)j_{\frac{q}{2}-1}(\tau t)\,dt\big]\tau^{q-1}\,d\tau.$

由文 [21, 407 页] 知当 $\Re(2\nu+1)>\Re(\lambda)>-1$ 且 $a,\,b>0$, 时

$$\int_0^{+\infty}\frac{J_\nu(at)\,J_\nu(bt)}{t^\lambda}\,dt=\frac{(ab)^\nu\Gamma(\nu-\frac{\lambda}{2}+\frac{1}{2})}{2^\lambda(a^2+b^2)^{\nu-\frac{\lambda}{2}+\frac{1}{2}}\Gamma(\nu+1)}$$
$$\times\,{}_2F_1\bigg(\frac{2\nu+1-\lambda}{2},\,\frac{2\nu+3-\lambda}{4},\,\nu+1,\,\frac{4a^2b^2}{(a^2+b^2)^2}\bigg).\qquad(12)$$

以 $a=\tau,\,b=1,\,\nu=\frac{q}{2}-1$, 代入 (12) 式知

$$\int_0^{+\infty}j_{\frac{q}{2}-1}(t)\,j_{\frac{q}{2}-1}(\tau t)\,dt=\int_0^{+\infty}\frac{2^{q-2}\Gamma(\frac{q}{2})^2J_{\frac{q}{2}-1}(\tau t)\,J_{\frac{q}{2}-1}(t)}{t^{q-2}}\,dt$$
$$=\frac{\tau^{\frac{q}{2}-1}\Gamma(\frac{1}{2})\Gamma(\frac{q}{2})}{(1+\tau^2)^{\frac{1}{2}}}\,{}_2F_1\bigg(\frac{1}{4},\,\frac{3}{4},\,\frac{q}{2},\,\frac{4\tau^2}{(1+\tau^2)^2}\bigg).$$

因此

$$\frac{1}{(2\pi)^q}\int_0^{+\infty}\bigg[j_{\frac{q}{2}-1}(t)\int_{Q^q}B_{\frac{t}{\|m\|}}(f,\,x)\,\mathrm{e}^{-\mathrm{i}mx}\,dx\bigg]\,dt$$
$$=\frac{c_m(f)}{m(B)}\int_0^1\bigg(\frac{\tau^{\frac{q}{2}-1}\Gamma(\frac{1}{2})\,\Gamma(\frac{q}{2})}{(1+\tau^2)^{\frac{1}{2}}}{}_2F_1\bigg(\frac{1}{4},\,\frac{3}{4},\,\frac{q}{2},\,\frac{4\tau^2}{(1+\tau^2)^2}\bigg)\bigg)\tau^{q-1}\,d\tau.$$

记 $c_q^*=\int_0^1\frac{\tau^{\frac{3q}{2}-1}\Gamma(\frac{1}{2})\Gamma(\frac{q}{2})}{(1+\tau^2)^{\frac{1}{2}}}{}_2F_1(\frac{1}{4},\,\frac{3}{4},\,\frac{q}{2},\,\frac{4\tau^2}{(1+\tau^2)^2})\,d\tau$, 则

$$\frac{1}{(2\pi)^q}\int_0^{+\infty}\bigg[j_{\frac{q}{2}-1}(t)\int_{Q^q}B_{\frac{t}{\|m\|}}(f,\,x-u)\mathrm{e}^{-\mathrm{i}mx}\,dx\bigg]\,dt$$
$$=\frac{1}{(2\pi)^q}\int_0^{+\infty}\bigg[j_{\frac{q}{2}-1}(t)\int_{Q^q}B_{\frac{t}{\|m\|}}(f,\,v)\mathrm{e}^{-\mathrm{i}mv}\,dv\bigg]\,\mathrm{e}^{-\mathrm{i}mu}\,dt=\frac{c_m(f)\,c_q^*\,\mathrm{e}^{-\mathrm{i}mu}}{m(B)}.$$

所以 (9) 式成立.

用 C^q 表示 q- 维的复平面, $\sigma>0$ 为实数, $E_\sigma(C^q)$ 表示定义在 C^q 上且关于每个变量以 σ 为指数的复变函数的集合, 即 $g(z)=g(z_1,z_2,\ldots,z_q)\in E_\sigma(C^q)$ 当且仅当对每个 $\varepsilon>0$ 存在正数 $A=A(\varepsilon)>0$, 使得 $|g(z)|\le A\exp\sum_{j=1}^q(\sigma+\varepsilon)|z_j|$, $z=(z_1,z_2,\ldots,z_q)\in C^q$. $B_\sigma(R^q)$ 表示 $E_\sigma(R^q)$ 中的有界函数, 且记

$$B^q_{\sigma,p}:=B_{\sigma,p}(R^q)=\begin{cases}B_\sigma(R^q)\cap L^p(R^q), & 1\le p<+\infty;\\ B_\sigma(R^q), & p=+\infty,\end{cases}$$

则 $B^q_{\sigma,p}$ 便为通常的 q- 维 Paley–Wiener 空间.

用 H^q_n 表示关于每个变元为阶 $\le n$ 的 q- 维三角多项式的集合.

引理 2 对于 $p\ge1$, 有 [22–25],

$$\int_{R^1}f(x)\,dx=\frac{\pi}{\sigma}\sum_{k\in Z^1}f\Big(\frac{k\pi}{\sigma}\Big),\ f\in B^1_{\sigma,p}\qquad(13)$$

及

$$\frac{1}{(2\pi)^q}\int_{Q^q}T(x)\,dx=\frac{1}{(2n+1)^q}\sum_{0\le l\le 2n}T\Big(\frac{2\pi l}{2n+1}\Big),\ \ T\in H^q_n,\qquad(14)$$

且可以找到两个非负实数 $C_1>0$, $C_2>0$, 使得

$$C_1\|T\|_{p,Q^q}\le\left(\frac{1}{(2n+1)^q}\sum_{0\le l\le 2n}\left|T\left(\frac{2\pi l}{2n+1}\right)\right|^p\right)^{\frac{1}{p}}\le C_2\|T\|_{p,Q^q},\ \ T\in H_n^q \tag{15}$$

及

$$C_1\|f\|_{p,R^1}\le\left(\frac{\pi}{\sigma}\sum_{k\in Z^1}\left|f\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{\frac{1}{p}}\le C_2\|f\|_{p,R^1},\ \ f\in B_{\sigma,p}^1. \tag{16}$$

引理 3 设 $R'>R>0$, $\alpha>\frac{q-1}{2}$. ϕ, $f\in L^1(Q^q)$ 且 $c_m(\phi)\ne 0$, $m\in Z^q$, 则有

$$S_R^\alpha(f,x)=c_0(f)+\left(\frac{1}{2([R']+[R])+1}\right)^q\sum_{0\le l\le 2([R']+[R])}\sum_{0<\|m\|<R}\frac{\omega_{q-1}c_m(S_R^\alpha(f))\mathrm{e}^{\frac{2lm\pi\mathrm{i}}{2([R']+[R])+1}}}{c_m(\phi)c_q[1-(\frac{\|m\|}{R'})^2]^\alpha}$$
$$\times\left(\frac{\pi}{\frac{R'}{\|m\|}+1}\right)\sum_{k\in Z'}S_{\frac{\pi}{R'+\|m\|}}\left(S_{R'}^\alpha(\phi),x-\frac{2\pi l}{2([R']+[R])+1}\right)j_{\frac{q}{2}-1}\left(\frac{k\pi}{\frac{R'}{\|m\|}+1}\right),\ x\in Q^q \tag{17}$$

及

$$S_R^\alpha(f,x)=c_0(f)+\left(\frac{1}{2([R']+[R])+1}\right)^q\sum_{0\le l\le 2([R']+[R])}\sum_{0<\|m\|<R}\frac{m(B)c_m(S_R^\alpha(f))\mathrm{e}^{\frac{2lm\pi\mathrm{i}}{2([R']+[R])+1}}}{c_m(\phi)c_q^*[1-(\frac{\|m\|}{R'})^2]^\alpha}$$
$$\times\left(\frac{\pi}{\frac{R'}{\|m\|}+1}\right)\sum_{k\in Z^1}B_{\frac{\pi}{R'+\|m\|}}\left(S_{R'}^\alpha(\phi),x-\frac{2\pi l}{2([R']+[R])+1}\right)j_{\frac{q}{2}-1}\left(\frac{k\pi}{\frac{R'}{\|m\|}+1}\right),\ x\in Q^q. \tag{18}$$

证明 (17) 式的证明. 由引理 1 知道, 对 $m\in Z^q$, 有

$$\begin{aligned}\mathrm{e}^{\mathrm{i}mx}&=\frac{\omega_{q-1}}{(2\pi)^q c_m(S_{R'}^\alpha(\phi))c_q}\int_0^{+\infty}\left(\int_{Q^q}S_{\frac{t}{\|m\|}}\big(S_{R'}^\alpha(\phi),\,x-u\big)\,\mathrm{e}^{\mathrm{i}mu}\,du\right)j_{\frac{q}{2}-1}(t)\,dt\\&=\frac{\omega_{q-1}}{(2\pi)^q(1-(\frac{\|m\|}{(R')})^2)^\alpha c_m(\phi)c_q}\int_0^{+\infty}\left(\int_{Q^q}S_{\frac{t}{\|m\|}}\big(S_{R'}^\alpha(\phi),\,x-u\big)\,\mathrm{e}^{\mathrm{i}mu}\,du\right)j_{\frac{q}{2}-1}(t)\,dt.\end{aligned}$$

由于 $S_{R'}^\alpha(\phi,\cdot)\in H_{[R']}^q$ 且 $0<\|m\|<R$, 因此, $S_{\frac{t}{\|m\|}}(S_{R'}^\alpha(\phi),\,x-\cdot)\,\mathrm{e}^{\mathrm{i}m\cdot}\in H_{[R']+[R]}^q$. 由引理 2 知

$$\begin{aligned}&\frac{1}{(2\pi)^q}\int_{Q^q}S_{\frac{t}{\|m\|}}(S_{R'}^\alpha(\phi),\,x-u)\mathrm{e}^{\mathrm{i}mu}\,du\\&\quad=\left(\frac{1}{2([R']+[R])+1}\right)^q\sum_{0\le l\le 2([R']+[R])}S_{\frac{t}{\|m\|}}\left(S_{R'}^\alpha(\phi),\,x-\frac{2\pi l}{2([R']+[R])+1}\right)\mathrm{e}^{\frac{2lm\pi\mathrm{i}}{2([R']+[R])+1}}.\end{aligned}$$

又由 $\frac{J_\alpha(z)}{z^\alpha}=\sum_{k=0}^\infty(-1)^k\frac{z^{2k}}{2^{\alpha+2k}k!\Gamma(k+\alpha+1)}\in E_1(C^1)$ 知 $\frac{J_\alpha(R|t|)}{(R|t|)^\alpha}\in E_R(C^1)$. 由 (3) 式知 $S_{R'}^\alpha(\phi,\cdot)\in B_{R,p}(C^q)$, 故 $j_{\frac{q}{2}-1}(\cdot)\,S_{\frac{\cdot}{\|m\|}}(S_{R'}^\alpha(\phi),\,x-u)\in B_{\frac{R'}{\|m\|}+1}(C^1)$. 因此, 由 (13) 式, 有

$$\begin{aligned}\mathrm{e}^{\mathrm{i}mx}&=\frac{\omega_{q-1}}{(2\pi)^q(1-(\frac{\|m\|}{R'})^2)^\alpha c_m(\phi)c_q}\left(\frac{1}{2([R']+[R])+1}\right)^q\\&\quad\times\sum_{0\le l\le 2([R']+[R])}\int_0^{+\infty}\int_{Q^q}S_{\frac{t}{\|m\|}}\left(S_{R'}^\alpha(\phi),\,x-\frac{2\pi l}{2([R']+[R])+1}\right)j_{\frac{q}{2}-1}(t)\,dt\times\mathrm{e}^{\frac{2lm\pi\mathrm{i}}{2([R']+[R])+1}}\\&=\left(\frac{1}{2([R']+[R])+1}\right)^q\frac{\omega_{q-1}}{(1-(\frac{\|m\|}{R'})^2)^\alpha c_m(\phi)c_q}\\&\quad\times\sum_{0\le l\le 2([R']+[R])}\left(\frac{\pi}{\frac{R'}{\|m\|}+1}\right)\sum_{k\in Z^1}S_{\frac{\pi}{\frac{R'}{\|m\|}+1}}\left(S_{R'}^\alpha(\phi),\,x-\frac{2\pi l}{2([R']+[R])+1}\right)\\&\quad\times j_{\frac{q}{2}-1}\left(\frac{k\pi}{\frac{R'}{\|m\|}+1}\right)\mathrm{e}^{\frac{2lm\pi\mathrm{i}}{2([R']+[R])+1}}.\end{aligned}$$

所以 (17) 式成立. 类似证明 (18) 式.

引理 4[11] 设 $q\ge 1$, $1<p<+\infty$, $\alpha>\alpha_p=\frac{q-1}{2}|\frac{2}{p}-1|$, 则对 $f\in L^p(Q^q)$, 有

$$\|S_R^\alpha(f)-f\|_{p,Q^q}\le C_{q,\alpha,p}\,\omega_2\Big(f;\ \frac{1}{R}\Big)_p, \tag{19}$$

其中 $\omega_2(f;t)_p$ 为通常的二阶光滑模.

3 定理证明

(7) 式的证明. 由于

$$\begin{aligned}&|S_{R,\phi}^{*,\alpha}(f,x)-S_R^\alpha(f,x)|\\&\le\Big(\frac{1}{2([R']+[R])+1}\Big)^q\sum_{0\le l\le 2([R']+[R])}\sum_{0<\|m\|<R}\frac{m(B)|c_m(S_R^\alpha(f))|}{c_m(\phi)|c_q^*[1-(\frac{\|m\|}{R'})^2]^\alpha|}\\&\quad\times\Big(\frac{\pi}{\frac{R'}{\|m\|}+1}\Big)\sum_{k\in Z^1}B_{\frac{k\pi}{R'+\|m\|}}\Big(\big|S_{R'}^\alpha(\phi)-\phi\big|,\,x-\frac{2\pi l}{2([R']+[R])+1}\Big)\Big|j_{\frac{q}{2}-1}\Big(\frac{k\pi}{\frac{R'}{\|m\|}+1}\Big)\Big|.\end{aligned}$$

因此

$$\begin{aligned}&\|S_{R,\phi}^{*,\alpha}(f)-S_R^\alpha(f)\|_{p,Q^q}\\&\le\Big(\frac{1}{2([R']+[R])+1}\Big)^q\sum_{0\le l\le 2([R']+[R])}\sum_{0<\|m\|<R}\frac{m(B)|c_m(S_R^\alpha(f))|}{c_m(\phi)|c_q^*[1-(\frac{\|m\|}{R'})^2]^\alpha|}\Big(\frac{\pi}{\frac{R'}{\|m\|}+1}\Big)\\&\quad\times\sum_{k\in Z^1}\Big(\int_{Q^q}\Big|S_{\frac{k\pi}{R'+\|m\|}}\Big(\big|S_{R'}^\alpha(\phi)-\phi\big|,\,x-\frac{2\pi l}{2([R']+[R])+1}\Big)\Big|^p dx\Big)^{\frac{1}{p}}\Big|j_{\frac{q}{2}-1}\Big(\frac{k\pi}{\frac{R'}{\|m\|}+1}\Big)\Big|.\end{aligned}$$

由文 [13] 知道对一切 $t>0$, 有 $\big(\int_{Q^q}\big|S_t(f,x)\big|^p dx\big)^{\frac{1}{p}}\le\|f\|_{p,Q^q}$. 由 (15), (16) 式, 有

$$\begin{aligned}&\|S_{R,\phi}^{*,\alpha}(f)-S_R^\alpha(f)\|_{p,Q^q}\\&\le\Big(\frac{1}{2([R']+[R])+1}\Big)^q\sum_{0\le l\le 2([R']+[R])}\sum_{0<\|m\|<R}\frac{m(B)|c_m(S_R^\alpha(f))|}{c_m(\phi)|c_q^*[1-(\frac{\|m\|}{R'})^2]^\alpha|}\\&\quad\times\Big(\frac{\pi}{\frac{R'}{\|m\|}+1}\Big)\sum_{k\in Z^1}\|S_{R'}^\alpha(\phi)-\phi\|_{p,Q^q}\Big|j_{\frac{q}{2}-1}\Big(\frac{k\pi}{\frac{R'}{\|m\|}+1}\Big)\Big|\\&\le\sum_{0<\|m\|<R}\frac{m(B)|c_m(S_R^\alpha(f))|}{|c_m(\phi)|c_q^*[1-(\frac{\|m\|}{R'})^2]^\alpha}\|S_{R'}^\alpha(\phi)-\phi\|_{p,Q^q}\Big(\frac{\pi}{\frac{R'}{\|m\|}+1}\Big)\sum_{k\in Z^1}\Big|j_{\frac{q}{2}-1}\Big(\frac{k\pi}{\frac{R'}{\|m\|}+1}\Big)\Big|\\&\le C\|S_{R'}^\alpha(\phi)-\phi\|_{p,Q^q}\sum_{0<\|m\|<R}\frac{m(B)|c_m(S_R^\alpha(f))|}{|c_m(\phi)|c_q^*[1-(\frac{\|m\|}{R'})^2]^\alpha}\int_0^{+\infty}\big|j_{\frac{q}{2}-1}(t)\big|dt.\end{aligned}$$

由文 [21, 408 页] 知道 $\int_0^{+\infty}|j_{\frac{q}{2}-1}(t)|\,dt$ 收敛, 记 $\Phi_{R,R'}^{*,\alpha}(\phi)=\frac{\min_{0<\|m\|<R}\big[1-(\frac{\|m\|}{R'})^2\big]^\alpha|c_m(\phi)|c_q^*}{m(B)\int_0^{+\infty}|j_{\frac{q}{2}-1}(t)|\,dt}$, 则由 Bessel 不等式, 有

$$\begin{aligned}\|S_{R,\phi}^{*,\alpha}(f)-S_R^\alpha(f)\|_{p,Q^q}&\le C\frac{\omega_2(\phi,\frac{1}{R'})_{p,Q^q}}{\Phi_{R,R'}^{*,\alpha}(\phi)}\sum_{0<\|m\|<R}|c_m(S_R^\alpha(f))|\\&\le\frac{CR^{\frac{q}{2}}\omega_2(\phi,\frac{1}{R'})_{p,Q^q}}{\Phi_{R,R'}^{*,\alpha}(\phi)}\Big(\sum_{0<\|m\|<R}|c_m(S_R^\alpha(f))|^2\Big)^{\frac{1}{2}}\le\frac{CR^{\frac{q}{2}}\omega_2(\phi,\frac{1}{R'})_{p,Q^q}}{\Phi_{R,R'}^{*,\alpha}(\phi)}\|S_R^\alpha(f)\|_{2,Q^q}\\&\le\frac{CR^\sigma\,\omega_2(\phi,\frac{1}{R'})_{p,Q^q}}{\Phi_{R,R'}^{*,\alpha}(\phi)}\|S_R^\alpha(f)\|_{p,Q^q}\le\frac{CR^\sigma\|f\|_{p,Q^q}}{\Phi_{R,R'}^{*,\alpha}(\phi)}\,\omega_2\Big(\phi,\ \frac{1}{R'}\Big)_{p,Q^q},\end{aligned}$$

其中 $\sigma = q/\min(p,2)$. 这里用到了关于三角多项式的 Nikolskii 不等式 [25]

$$\|T\|_{p,Q^q} \le cn^{q(\frac{1}{p'}-\frac{1}{p})_+}\|T\|_{p',Q^q},\ \ 1\le p,\, p'\le +\infty,$$

其中$(a)_+=\max(a,0)$. 由引理 4, 有

$$\|S^{*,\alpha}_{R,\phi}(f)-f\|_{p,Q^q}\le C\left[\omega_2\left(f,\frac{1}{R}\right)_{p,Q^q}+\frac{R^\sigma\|f\|_{p,Q^q}}{\Phi^{*,\alpha}_{R,R'}(\phi)}\omega_2\left(\phi,\frac{1}{R'}\right)_{p,Q^q}\right].$$

类似证明 (6) 式.

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