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Iterative Approximation of Solutions to Nonlinear Equations of Lipschitzian and Strongly Accretive Operators

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Abstract: In this paper, we investigate the Ishikawa iteration process converges strongly to the unique solution of the equation $Tx = f$ in case T is a Lipschitzian and strongly accretive operator from X into X, or to the unique fixed point of T in case T is a Lipschitzian and strictly pseudocontractive mapping from a bounded closed convex subset into itself. Our results improve and extend some recent results.

Key words: iterative approximation; Lipschitzian; Strongly accretive operator; pseudocontractive mapping.

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1 Introduction

Let X be a real Banach space. In 1967, BROWDER[3] and KATO[2] independently introduced the accretive operators, that is, an operator T with domain $D(T)$ and range $R(T)$ in X is said to be accretive if for all $x,y \in D(T)$ and $r > 0$, there holds the inequality

$$\| x - y \| \leqslant \| x - y + r(Tx - Ty) \| . \tag{1.1}$$

An early fundamental result in the theory of accretive operators, due to Browder, states that the initial valued problem

$$du/dt + Tu = 0,\ u(0) = u_0, \tag{1.2}$$

is solvable if T is locally Lipschitzian and accretive on X. See BARBU[1] for more details of the theory of accretive operators.

By X^* we denote the dual space of X. Let C be a nonempty subset of a Banach space X. Recall that a mapping $A: C \to X$ is said to be strongly accretive if there exists a real number $k > 0$ such that for each $x,y \in C$,

$$\langle Ax - Ay, j \rangle \geqslant k \| x - y \|^2 \tag{1.3}$$

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holds for some $j \in J(x-y)$, where

$$J(x) = \{x^* \in X^* : \langle x; x^* \rangle = \|x\|^2 = \|x^*\|^2\},\ x \in X \quad (1.4)$$

is the normalized duality mapping of X and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and X^*. The class of strongly accretive mappings has been investigated by many authors (e g, [4～7]). It is known that T is a (strict) pseudocontraction if and only if $(I-T)$ is an (strongly) accretive operators (see, e.g., [4～7]).

Recently, TAN and XU[4] studied both the Mann and the Ishikawa iteration process in a p-uniformly smooth Banach space X. They proved that the two processes converge strongly to the unique solution of the equation $Tx = f$ in case T is a Lipschitzian and strongly accretive operator from X into X, or to the unique fixed point of T in case T is a lipschitzian pseudocontractive mapping from a bounded closed convex subset C of X into itself. Therefore, TAN and XU[4] gave affirmative answers to problems 1 and 2 of CHIDUME[7], respectively, and also extended all the results of CHIDUME[7] to the p-uniformly smooth Banach space setting. On the other hand, by the Ishikawa iteration process, DENG and DING[5] gave the iterative sequence which converges strongly to the unique fixed point of a Lipschitzian strictly pseudocontractive mapping in a uniformly smooth Banach space X and a related result on the problem that the Ishikawa iteration process converges strongly to a solution of the equation $Tx = f$ in case T is a Lipschitzian and strongly accretive operator of X into itself and thus extended the result of CHIDUME[7] to the uniformly smooth Banach space setting. Further, ZENG[6] extended the result of TAN and XU[4] to the cases of the Lipschitzian and local strongly accretive operators, and the Lipschitzian and local strongly accretive operators, and the Lipschitzian and local strictly pseudocontractive mappings in the p-uniformly smooth Banach space X.

In this paper, we investigate the Ishikawa iteration process in a p-uniformly smooth Banach space X. We prove that the Ishikawa iteration process converges strongly to the unique solution of the equation $Tx = f$ in case T is a Lipschitzian and strongly accretive operator from X into X or to the unique fixed point of T when T is a Lipschitzian and strictly pseudocontractive mapping from a bounded closed convex subset C of X into itself. Our results improve and extend Theorems 4.1 and 4.2 of TAN and XU[4] by removing the restriction $\lim_{n\to\infty}\beta_n = 0$ or $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$ and Theorems 1 and 2 of DENG and DING[5] by removing the restriction $\sum_{n=0}^{\infty}\alpha_n^s < \infty\ (s > 1)$.

2 Preliminaries

Let C be a nonempty subset of a Banach space X. A mapping $T: C \to X$ is said to be strictly pseudocontractive if there exists $t > 1$ such that the inequality

$$\|x-y\| \leqslant \|(1+r)(x-y) - rt(Tx-Ty)\| \quad (2.1)$$

holds for all x, y in C and $r > 0$. If, in the above definition, $t = 1$, then T is said to be a pseudocontractive mapping.

For $1 < p < \infty$, the mapping $J_p: X \to 2^{X^*}$ defined by

$$J_p(x) = \{x^* \in X^* : \langle x; x^* \rangle = \|x^*\| \cdot \|x\|;\ \|x^*\| = \|x\|^{p-1}\},\ x \in X$$

is called the duality mapping with gauge function $\varphi(t) = t^{p-1}$. It is the well-known fact that $J_p(x) = \|x\|^{p-1}J(x)$ for all x in $X\backslash\{0\}$ and $1 < p < \infty$. An operator $T: C \to X$ is said to be accretive if for each x, y in C there exists $j \in J(x-y)$ such that

$$\langle Tx - Ty,\ j_p \rangle \geqslant 0;$$

or equivalently, for each x, y in C there exists $j_p \in J(x-y)$, such that

$$\langle Tx - Ty,\ j \rangle \geqslant 0\ . \tag{2.2}$$

An operator $T: C \to X$ is said to be strongly accretive if for every x, y in C there exists $j_p \in J_p(x-y)$ such that

$$\langle Tx - Ty,\ j_p \rangle \geqslant k \| x - y \|^p \tag{2.3}$$

for some real constant $k > 0$, without loss of generality, we assume that $k \in (0,1)$.

Let X be a Banach space. The modulus of smoothness $\rho_x(\cdot)$ of X is defined by

$$\rho_x(\tau) = \sup\{(\| x + y \| + \| x - y \|)/2 - 1:\ x, y \in X,\ \| x \| = 1,\ \| y \| = \tau\},\ \tau > 0$$

and X is said to be uniformly smooth if $\lim_{\tau \to 0} \rho_x(\tau) = 0$. Recall that for a real number $1 < \rho \leqslant 2$ a Banach space X is said to be p-uniformly smooth if $\rho_x(\tau) \leqslant d\tau^p$ for $\tau > 0$, where $d > 0$ is a constant. It is known that for a Hilbert space H, $\rho_H(\tau) = (1 + \tau^2)^{1/2} - 1$ and hence H is 2-uniformly smooth. It is also known that if $1 < p < 2$, L_p (or l_p) is p-uniformly smooth; while if $2 < p < \infty$, L_p (or l_p) is 2-uniformly smooth.

Lemma 2.1[4] Let X be a smooth Banach space, and p be a fixed number in (1.2). Then X is p-uniformly smooth if and only if there exists a constant $d_p > 0$ such that

$$\| x + y \|^p \leqslant \| x \|^p + p\langle y,\ J_p(x) \rangle + d_p \| y \|^p \tag{2.4}$$

for all x, y in X, where $J_p(x)$ is the subdifferentiable at x of the functional $p^{-1} \| \ \|$.

When X is an L_p (or l_p) space, the constant d_p in (2.4) has been calculated.

Lemma 2.2[4] Let $X = L_p$ (or l_p), $1 < p < \infty$ and x, y belong to X. We have

(1) if $1 < p < 2$, then

$$\| x + y \|^p \leqslant \| x \|^p + p\langle y, J_p(x) \rangle + d_p \| y \|^p, \tag{2.5}$$

where $d_p = \dfrac{1 + b_p^{p-1}}{(1 + b_p)^{p-1}}$, b_p being the unique solution of the equation

$$(p-2)b^{p-1} + (p-1)b^{p-2} - 1 = 0,\ 0 < b < 1;$$

(2) if $p \geqslant 2$, then

$$\| x + y \|^2 \leqslant \| x \|^2 + 2\langle y, J(x) \rangle + (p-1) \| y \|^2. \tag{2.6}$$

BROWDER[3] proved that if $T: C \to X$ is local Lipschitzian and accretive then T is m-accretive; i e, the mapping $(I + T)$ where I denotes the identity mapping of X, is subjective. This result was subsequently generalized by MARTIN[8] to continuous accretive operators. It can be seen that the following lemma is an immediate consequence of Martin's result.

Lemma 2.3 If $T: X \to X$ is continuous and strongly accretive, then T maps X onto X, that is, for each $f \in X$ the equation $Tx = f$ has a solution in X.

3 Main results

In this section, we discuss the Ishikawa iteration process, and prove that if X is a p-uniformly smooth Banach space and $T: X \to X$ is a Lipschitzian and strongly accretive mapping then the Ishikawa iteration process converges strongly to the unique solution of the equation $Tx = f$. Further we present a related result on the problem that the Ishikawa iteration sequence converges strongly to the unique fixed point of T in case T is a Lipschitzian and strictly pseudocontractive mapping from a nonempty closed convex subset C of X into itself.

Theorem 3. 1 Let X be a p-uniformly smooth Banach space with $1 < p \leqslant 2$ and $T: X \to X$ be a Lipschitzian and strongly accretive operator with Lipschitzian constant L. Define $S: X \to X$ by $Sx = f - Tx + x$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two sequences of real numbers in $[0,1]$ satisfying

(1) $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$,

(2) if $1 < p < 2$, then

$$0 \leqslant \beta_n \leqslant \min\{t_p, \frac{k}{2pL_0(1+L_0^p)^{1/p}\min(2,p^2)}\} \quad \text{for each} \quad n \geqslant 0;$$

if $p = 2$, then

$$0 \leqslant \beta_n \leqslant \min\{t_p, \frac{k^2}{4pL_0(1+L_0^p)^{1/p}} \quad \text{for each} \quad n \geqslant 0;$$

where L_0 is the Lipschitzian constant of S with $L_0 \leqslant 1 + L$, t_p is the (smaller) solution of the equation

$$f(t) = p(p-1)(1-k)t - (1 + d_p L_0^p)t^{p-1} + k/p = 0 \ (t > 0), \tag{3. 1}$$

and $k \in (0,1)$, d_p are the constants appearing in (2. 3) and (2. 4), respectively. Then for each x_0 in X, the Ishikawa sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n \text{ and } y_n = (1-\beta_n)x_n + \beta_n S x_n, \ n \geqslant 0$$

converges strongly to the unique solution of the equation $Tx = f$.

Remark 3. 1 If $p = 2$, then the solution of Eq. (3. 1) is

$$t_2 = \frac{k}{2(d_2 L_0^2 + 2k - 1)}.$$

Furthermore, if $X = L_p$ (or l_p) for $p \geqslant 2$, then X is 2-uniformly smooth and $d_2 = p - 1$ by Lemma 2. 2 (2); if $1 < p < 2$, then the function $f(t)$ in (3. 1) is strictly convex on $(0,\infty)$. Also, since $f(0) = k/p > 0$ and $f(\infty) = \infty$, the only three possibilities for the existence of solutions of Eq. (3. 1) are, as illustrated by the figures below, (a) it has no solution so that $f(t) > 0$ for all $t \geqslant 0$; (b) it has exactly one solution; (c) it has two solutions.

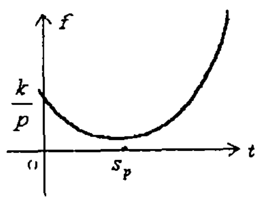

Fig 1

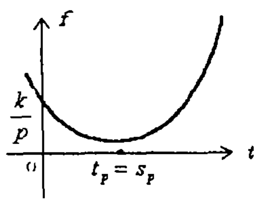

Fig 2

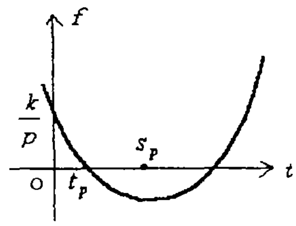

Fig 3

The zero of the derivative $f'(t)$ of $f(t)$ is

$$s_p = \frac{1 + d_p L_0^p}{(p(1-k))^{1/(2-p)}}.$$

and the value of f at s_p is

$$f(s_p) = -(2-p)(1 + d_p L_0^p)^{1/(2-p)}(p(1-k))^{-(p-1)/(2-p)} + k/p.$$

It follows that $f(s_p) \leqslant 0$ (and hence Eq. (3. 1) has at least one solution) if $k > 0$ is small enough.

Throughout this paper, we always make this hypothesis without explicitly specified since otherwise, $f(t) > 0$ for all $t \geqslant 0$ and Theorem 3.1 holds for any $t_p > 0$.

Proof of Theorem 3.1 We first observe that the equation $Tx = f$ has a unique solution which is denoted by q. Indeed, the existence follows from Lemma 2.3 and the uniqueness from the strong accretiveness of T. We also observe that for $x, y \in X$,

$$\langle Sx - Sy, J_p(x-y)\rangle = -\langle Tx - Ty, J_p(x-y)\rangle + \|x-y\|^p =$$
$$-\|x-y\|^{p-2}\langle Tx - Ty, J(x-y)\rangle + \|x-y\|^p - k\|x-y\|^{p-2}\|x-y\|^2 +$$
$$\|x-y\|^p = (1-k)\|x-y\|^p.$$

It then follows that

$$\|x_{n+1} - q\|^p = \|(1-\alpha_n)(x_n - q) + \alpha_n(Sy_n - q)\|^p \leqslant$$
$$(1-\alpha_n)^p\|x_n - q\|^p + p\alpha_n(1-\alpha_n)^{p-1}\langle Sy_n - q, J_p(x_n - q)\rangle + d_p\alpha_n^p\|Sy_n - q\|^p.$$

Since

$$\|Sy_n - q\|^p \leqslant L_0^p\|y_n - q\|^p,$$
$$\langle Sx_n - q, J_p(x_n - q)\rangle \leqslant (1-k)\|x_n - q\|^p,$$
$$\|y_n - q\|^p = \|(1-\beta_n)(x_n - q) + \beta_n(Sx_n - q)\|^p \leqslant$$
$$(1-\beta_n)^p\|x_n - q\|^p + p\beta_n(1-\beta_n)^{p-1}\langle Sx_n - q, J_p(x_n - q)\rangle + d_p\beta_n^p\|Sx_n - q\|^p =$$
$$t_p\|x_n - q\|^p,$$

where

$$t_n = (1-\beta_n)^p + p(1-k)\beta_n(1-\beta_n)^{p-1} + d_pL_0^p\beta_n^p,$$
$$\|y_n - x_n\|^p = \beta_n\|x_n - Sx_n\|^p = \beta_n^p\|(x_n - q) + (q - Sx_n)\|^p \leqslant$$
$$2^p\beta_n^p(\|x_n - q\|^p + \|Sx_n - q\|^p) \leqslant$$
$$2^p(1 + L_0^p)\beta_n^p\|x_n - q\|^p,$$
$$\langle Sy_n - Sx_n, J_p(x_n - q)\rangle \leqslant L_0\|y_n - x_n\|\,\|x_n - q\|^{p-1} \leqslant 2L_0\beta_n(1 + L_0^p)^{1/p}\|x_n - q\|^p$$

and

$$\langle Sy_n - q, J_p(x_n - q)\rangle = \langle Sy_n - Sx_n, J_p(x_n - q)\rangle + \langle Sx_n - q, J_p(x_n - q)\rangle \leqslant$$
$$[2L_0\beta_n(1 + L_0^p)^{1/p} + (1-k)]\|x_n - q\|^p$$

We obtain from (3.2)

$$\|x_{n+1} - q\|^p \leqslant [(1-\alpha_n)^p + p\alpha_n(1-\alpha_n)^{p-1}(1-k+2L_0\beta_n(1+L_0^p)^{1/p}) + d_pL_0^p\alpha_n^pt_n]\|x_n - q\|^p$$

Since $1 < p \leqslant 2$, $(1-t)^p \leqslant 1 - pt + t^p$, and $(1-t)^{p-1} \leqslant 1 - (p-1)t$ for $0 \leqslant t \leqslant 1$, we obtain

$$t_n = (1-\beta_n)^p + p(1-k)\beta_n(1-\beta_n)^{p-1} + d_pL_0^p\beta_n^p \leqslant$$
$$1 - pk\beta_n - p(p-1)(1-k)\beta_n^2 + (1 + d_pL_0^p)\beta_n^p. \tag{3.3}$$

Since $\beta_n \leqslant t_p$ for all $n \geqslant 0$, we have from (3.1)

$$p(p-1)(1-k)\beta_n^2 - (1 + d_pL_0^p)\beta_n^p \geqslant -k\beta_n/p.$$

Hence it follows that

$$t_n \leqslant 1 - k\beta_n/p \text{ for each } n \geqslant 0.$$

On the other hand, since $\lim_{n\to\infty}\alpha_n = 0$, there exists a positive integer N such that

$$0 \leqslant \alpha_n \leqslant t_p \text{ for each } n \geqslant N.$$

This implies that

$$t_n = (1-\alpha_n)^p + p(1-k)\alpha_n(1-\alpha_n)^{p-1} + d_pL_0^p\alpha_n^p \leqslant 1 - k\alpha_n/p \text{ for each } n \geqslant N.$$

Therefore, we obtain for each $n \geqslant N$,

$$\| x_{n+1} - q \|^p \leqslant$$

$$[(1-\alpha_n)^p + p\alpha_n(1-\alpha_n)^{p-1}(1-k) + p\alpha_n(1-\alpha_n)^{p-1}2L_0\beta_n(1+L^p)^{1/p} + d_pL_0^p\alpha_n^p(1-k\beta_n/p)]\| x_n - q \|^p \leqslant$$

$$[t_n + p(\alpha_n - (p-1)\alpha_n^2)2L_o\beta_n(1+L_0^p)^{1/p} - kd_dL_0^p\alpha_n^p\beta_n/p]\| x_n - q \|^p \leqslant$$

$$[1 - k\alpha_n/p + 2pL_0(1+L_0^p)^{1/p}\alpha_n\beta_n - 2p(p-1)L_0(1+L_0^p)^{1/p}\alpha_n^2\beta_n - kd_pL_0^p\alpha_n^p\beta_n/p]\| x_n - q \|^p \leqslant$$

$$[1 - k\alpha_n/p + 2pL_0(1+L_0^p)^{1/p}\alpha_n\beta_n]\| x_n - q \|^p =$$

$$[1 - k\alpha_n/p + 2pL_0(1+L_0^p)^{1/p}\beta_n\alpha_n]\| x_n - q \|^p. \tag{3.4}$$

If $1 < p < 2$, then (3.4) and Condition (2) imply

$$\| x_{n+1} - q \|^p \leqslant$$

$$1 - k\alpha_n/p + 2pL_0(1+L_0^p)^{1/p}\frac{k}{2pL_0(1+L_0^p)^{1/p}\cdot\min(2,p^2)}\alpha_n \| x_n - q \|^p \leqslant$$

$$\left[1 - \left(\frac{1}{p} - \frac{1}{\min(2,p^2)}\right)k\alpha_n\right]\| x_n - q \|^p \leqslant$$

$$\exp\left(-\left(\frac{1}{p} - \frac{1}{\min(2,p^2)}k\alpha_n\right)\right)\| x_n - q \|^p \leqslant$$

$$\exp\left(-\left(\frac{1}{p} - \frac{1}{\min(2,p^2)}k\sum_{j=N_0}^{n}\alpha_j\right)\right)\| x_{n_0} - q \|^p$$

This immediately implies the strong convergence of $\{x_n\}$ to q since $\sum_{n=0}^{\infty}\alpha_n$ diverges.

If $p = 2$, then (3.4) and Condition (2) imply

$$\| x_{n+1} - q \|^p \leqslant$$

$$[1 - k\alpha_n/p + 2pL_0(1+L_0^p)^{1/p}\frac{k^2}{4pL_0(1+L_0^p)^{1/p}}\alpha_n]\| x_n - q \|^p \leqslant$$

$$\left[1 - \left(\frac{1}{p} - \frac{k}{2}\right)k\alpha_n\right]\| x_n - q \|^p =$$

$$\left[1 - \frac{1}{2}(1-k)k\alpha_n\right]\| x_n - q \|^p \leqslant$$

$$\left[\exp\left(-\frac{1}{2}(1-k)k\alpha_n\right)\right]\| x_n - q \|^p \leqslant$$

$$\left[\exp\left(-\frac{1}{2}k(1-k)\sum_{j=N_0}^{n}\alpha_j\right)\right]\| x_{N_0} - q \|^p.$$

This implies the strong convergence of $\{x_n\}$ to q since $\sum_{n=0}^{\infty}\alpha_n$ diverges. The proof is complete.

Remark 3.2 If Condition (2) in Theorem 3.1 is replaced by the condition "if $1 < p < 2$, then

$$0 \leqslant \beta_n \leqslant \min\left\{t_p, \frac{k}{2pL_0(1+L_0^p)^{1/p}\min(2,p^2)}\right\} \text{ for each } n \geqslant N_1,$$

where N_1 is some positive integer; if $p = 2$, then

$$0 \leqslant \beta_n \leqslant \min\left\{t_p, \frac{k^2}{4pL_0(1+L_0^p)^{1/p}}\right\} \text{ for each } n \geqslant N_2,$$

where N_2 is some positive integer", then, Theorem 3.1 is still true. From this it is easily seen that Theorem 3.1 is the improvements and extension of Theorem 4.1 of TAN and XU[4], and Theorem 2 of DENG and DING[5].

Reviewing the proof of Theorem 3.1, we can also obtain the result relative to the Lipschitzian and strictly pseudocontractive mappings.

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Lipschitz 强增生算子的非线性方程解的迭代逼近

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摘要：研究了 p—一致光滑 Banach 空间中 Lipschitz 强增生算子方程解的 Ishikawa 的迭代过程的收敛性，改进与推广了一些最近结果.

关键词：Ishikawa 迭代；Lipschitz 强增生算子；伪压缩映像