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A Numerical Analysis of A Class of Stiff Delay Differential Equations

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Abstract: We study the step criteria of numerical methods for delay differential equations. Based on the criteria, we define the concept of stiffness of a class of delay differential equations. We also give a numerical example which illustrates the difficulty for solving stiff delay differential equations with large rate r_D.

Key words: delay differential equation; asymptotic stability; step criterion; stiff

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1 Introduction

Apply a numerical method to the following delay differential equation:

$$U'(t) = aU(t) + bU(t-\tau),\ t > 0, \tag{1}$$

$$U(t) = \varphi(t),\ t \leqslant 0, \tag{2}$$

$U(t)$ is the unknown function to be solved for $t > 0, \tau > 0$ denotes a constant delay, $a, b \in C$ (C is the complex number set) are given constants, $\varphi(t)$ is a given continuous function ($t \leqslant 0$).

Definition 1.1[2] A numerical method for DDEs is said to be P-stable if for all coefficients of equations (1),(2), in which

$$|b| < -\mathrm{Re}(a), \tag{3}$$

$\varphi(t)$ is a continuous function, the numerical solution $\{U_n\}$ at the mesh points $t_n = nh (n \geqslant 0, h > 0)$ satisfies

$$\lim_{n\to\infty} U_n = 0, \tag{4}$$

for every step size $h > 0$ such that $mh = \tau$, where m is a positive integer.

Definition 1.2[2] A numerical method for DDEs is called GP-stable if under condition (3), (4) holds for every stepsize $h > 0$.

Many papers discuss the P-stability and the GP-stability of the θ-method, linear multistep method

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and the Runge-Kutta method[3,4,5,6,9,11,12,13]. It has been proved that these methods are GP-stable if and only if they are A-stable[4,9,12]. As we know, the maximum accessible order of the linear multistep method that is A-stable is 2. Although there is no obstacle to the order of the Runge-Kutta method, not all implicit Runge-Kutta methods are A-stable [8]. Additionally, in many practical cases, e. g. when dealing with real-time control computations, we hope the numerical method is fast and allows to parallel computations. Then the explicit linear multistep method or the Runge-Kutta method will be considered firstly.

For the explicit method, there exists a step restricted problem. Though some papers have discussed the step restricted problem for DDEs, many problems need to be investigated for a class of stiff delay differential equations. The purpose of this paper is to investigate the step criteria of numerical methods for DDEs. Based on the criteria, we define the concept of stiffness of a class of delay differential equations, and point out that we have to use the numerical method that is GP-stable or the method whose stability is better.

2 The step criteria of the numerical method

Definition 2.1[2] The delay differential equation (1) is said to be asymptotically stable if every solution $U(t)$ of (1) satisfies $\lim_{t\to\infty} U(t) = 0$.

The characteristic equation of (1) is:

$$f(\zeta) = \zeta - a - be^{-\zeta\tau} = 0. \tag{5}$$

Now we introduce some known results:

Theorem 2.1[2] If the coefficients $a, b \in C$ of (1) satisfy the condition (3), then all zero points of equation (5) have negative real parts. Thus (1) is asymptotically stable.

Theorem 2.2[2] The following three statements are equivalent:

$$(A)\ |b| < -\mathrm{Re}(a), \quad (B)\ \mathrm{Re}(a + b\xi) < 0, \forall\ \xi \in C, |\xi| \leqslant 1,$$

$$(C)\ \mathrm{Re}(a + b\xi) < 0, \forall\ \xi \in C, |\xi| = 1.$$

Theorem 2.3[2] For any $a, b \in C$ that satisfy the condition (3), the characteristic equation (5) has countably infinite zero points $\xi_1, \xi_2, \cdots, \xi_n, \cdots$, and there exists a positive number σ such that

$$\mathrm{Re}(\xi_i) \leqslant -\sigma < 0, i = 1, 2, \cdots \tag{6}$$

Besides, there exists a subsequence $\{\xi_{ik}\}$, that is denoted by $\{\xi_i\}$, satisfies

$$\lim_{i\to\infty} \mathrm{Re}(\xi_i) = -\infty. \tag{7}$$

Corollary 2.4 For zero points $\{\xi_i\}$ of the characteristic equation (5), we have

$$\sup_i |\mathrm{Re}(\xi_i)| / \inf_i |\mathrm{Re}(\xi_i)| = \infty.$$

We apply the following linear multistep method to (1) and (2):

$$\sum_{j=0}^{k} \alpha_j U_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{8}$$

where $f_n = f(t_n, U_n)$, U_n is the approximation of $U(t_n)$, $t_n = nh$, $h > 0$, we get

$$\sum_{j=0}^{k} \alpha_j U_{n+j} = h \sum_{j=0}^{k} \beta_j [aU_{n+j} + bU_{n+j-m+\delta}], \tag{9}$$

where $(m - \delta)h = \tau$, $0 \leqslant \delta < 1$, $m \geqslant 1$ is a natural number. If $\delta \neq 0$, (9) can not be calculated, so we have to interpolate $U_{n+j-m+\delta}$:

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$$U_{n+j-m+\delta}=\sum_{p=-r}^{s}L_p(\delta)U_{n+j-m+p},\tag{10}$$

$$L_p(\delta)=\prod_{k=-r,k\neq p}^{s}\frac{\delta-k}{p-k},\ m\geqslant s+1.\tag{11}$$

Substituting (10) into (9), we get

$$\sum_{j=0}^{k}\alpha_j U_{n+j}=h\sum_{j=0}^{k}\beta_j[aU_{n+j}+b\sum_{p=-r}^{s}L_p(\delta)U_{n+j-m+p}],\tag{12}$$

In order to obtain the characteristic equation of (12), let $U_n=z^n$ and substitute it into the above formula. Then we get the characteristic polynomial

$$P_m(z)=Q(z)z^{m+r}-p(z,\delta),\tag{13}$$

where

$$Q(z)=\rho(z)-\bar{a}\sigma(z),\ p(z,\delta)=\bar{b}\gamma(z,\delta)\sigma(z),\ \rho(z)=\sum_{j=0}^{k}\alpha_j z^j,$$

$$\sigma(z)=\sum_{j=0}^{k}\beta_j z^j,\ \gamma(z,\delta)=\sum_{p=-r}^{s}L_p(\delta)z^{p+r},$$

$\bar{a}=ha,\bar{b}=hb$. Obviously, the difference equation (12) is stable, or we can say that the linear multistep method (8) is GP-stable for DDEs if and only if $P_m(z)$ is a *Schur* polynomial, that is

$$P_m(z)=0\Rightarrow |z|<1.$$

Lemma 2.5[10] (i) Whenever $|z|=1$ and $0\leqslant\delta<1$, $|\gamma(z,\delta)|\leqslant 1$ holds if and only if the relationship $r\leqslant s\leqslant r+2$ is valid. (*ii*) If $s+r>0, r\leqslant s\leqslant r+2, |z|=1, 0<\delta<1$, then $|\gamma(z,\delta)|=1$ if and only if $z=1$.

Rewrite $P_m(z)$ as

$$P_m(z)=\rho(z)-(\bar{a}+\bar{b}\gamma(z,\delta)z^{-m-r})\sigma(z)=\rho(z)-(\bar{a}+\bar{b}R(z,\delta))\sigma(z),\tag{14}$$

where

$$R(z,\delta)=\gamma(z,\delta)z^{-m-r}.$$

By lemma 2.5, when $|z|=1$, for any $0\leqslant\delta<1$, we have $|R(z,\delta)|\leqslant 1$. Since $m\geqslant\{s+1\}$, we know

$$R(\infty,\delta)=\lim_{z\to\infty}R(z,\delta)=0.$$

By themaximum modulus principle about unbounded region for analytic functions, for any $z\in C$, when $|z|\geqslant 1$, we have

$$|R(z,\delta)|\leqslant 1,\ \forall\ z\in C,|z|\geqslant 1,\ \forall\ 0\leqslant\delta<1.\tag{15}$$

As is known to all, the stable region of the linear multistep method (8) for the initial problem of ODEs is defined as

$$S_{O,LM}=\{\bar{h}\in C:\mathrm{Re}(\bar{h})<0,\rho(z)-\bar{h}\sigma(z)\text{ is a Schur } polynomial\}$$

So we have

Theorem 2.6 Let

(a) $\mathrm{Re}(a+b\xi)<0,\quad \forall\ \xi\in C,|\xi|\leqslant 1,$

(b) $h(a+b\xi)\in S_{O,LM},\quad \forall\ \xi\in C,|\xi|\leqslant 1.$

Then $P_m(z)$ defined by (14) is a *Schur* polynomial.

The stable region of the Runge-Kutta method for DDEs is defined as

$$S_{O,RK}=\{\bar{h}\in C:\mathrm{Re}(\bar{h})<0,|r(\bar{h})|<1\}$$

where

$$r(\bar{h})=1+\bar{h}\,b^T(I-\bar{h}A)^{-1}e,$$

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A, b are corresponding coefficients of the method, $e = (1, 1, \cdots, 1)^T$.

Theorem 2.7 Let

$$\text{(A)} \quad \mathrm{Re}(a + b\xi) < 0, \qquad \forall \xi \in C, |\xi| \leqslant 1,$$

$$\text{(B)} \quad h(a + b\xi) \in S_{O,RK}, \qquad \forall \xi \in C, |\xi| \leqslant 1.$$

then the Runge-Kutta method for DDEs (1) and (2) is asymptotically stable, i.e. $U_n \to 0 (n \to \infty)$.

Remark The stable region of many well—known methods has been given. Under the unknown conditions, $S_{O,LM}, S_{O,RK}$ can be obtained by the tracking descriptive method, so theorem 2.6 and theorem 2.7 are practical step criteria. Generally speaking, $S_{O,LM}, S_{O,RK}$ are irregular regions, so it is difficult to determine the stepsize h by the condition (b) in theorem 2.6 or the condition (B) in theorem 2.7. Now we consider the regular region contained in $S_{O,LM}$ or $S_{O,RK}$, such as a circular disc, an elliptic disc and a rectangle. In this paper, we only consider circular discs, other cases can be solved similarly.

Let $S_{O,LM}$ or $S_{O,RK}$ contain a circular disc C_R: $|z + R| < R$, where $R > 0$ is the radius of the circular disc. We want to prove $h(a + b\xi) \in C_R$, $\forall \xi \in C, |\xi| \leqslant 1$, that is

$$|h(a + b\xi) + R| < R, \ \forall \xi \in C, \ |\xi| \leqslant 1. \tag{16}$$

Let $b = |b| e^{i\theta}, \xi = re^{i\varphi}, 0 \leqslant r \leqslant 1, \ 0 \leqslant \varphi \leqslant 2\pi$. (16) is equivalent to

$$(h\mathrm{Re}(a) + h|b| r\cos(\theta + \varphi) + R)^2 + (h\mathrm{Im}(a) + h|b| r\sin(\theta + \varphi))^2 < R^2.$$

Unwinding it and applying the Schwartz inequality to $\{\mathrm{Re}(a)\cos(\theta + \varphi) + \mathrm{Im}(a)\sin(\theta + \varphi)\}$, we get $\mathrm{Re}(a)\cos(\theta + \varphi) + \mathrm{Im}(a)\sin(\theta + \varphi) \leqslant |a|$. Then the inequality of h is obtained

$$h < \frac{-2R(\mathrm{Re}(a) + |b|)}{(|a| + |b|)^2}. \tag{17}$$

This shows that when $|a|$ and $|b|$ is large and $|Re(a) + |b||$ is small, the step size h will be restricted severely.

3 The concept of stiffness of delay differential equations

As we all know, for *ODEs*

$$U'(t) = AU(t), \mathrm{Re}(\lambda_A) < 0, \tag{18}$$

where $A \in C^{d \times d}, U(t) = (U_1(t), U_2(t), \cdots, U_d(t))^T$. The stiff rate of (18) is defined as

$$r = \max_i |\mathrm{Re}(\lambda_i(A))| / \min_i |\mathrm{Re}(\lambda_i(A))|, \tag{19}$$

where $\lambda_i(A)$ denotes the eigenvalue of A, which is a zero point of the characteristic equation of (18)

$$\det[\xi I - A] = 0. \tag{20}$$

Obviously, if we define the stiff rate for *DDEs* (1) as that in (19), then by the result of corollary 2.4, they are all $+\infty$. But we analyze from (19) that if $r \gg 1$, the general solution of (18) contains not only a fast decreasing component, but also a slow decreasing component, whose corresponding eigenvalues are denoted by λ_f and λ_s respectively. To ensure the stability of the numerical method, let $h\lambda_f \in S_{O,LM}(S_{O,RK})$. If $S_{O,LM}(S_{O,RK})$ is a small bounded region in the left semi-plane, the step size h will be restricted severely. On the other hand, to obtain the stable solution of the initial problem, we let $e^{\lambda_s t} \sim 0$. Since $|\mathrm{Re}(\lambda_s)|$ is relatively very small, the steps of calculation will be excessive. In the next section we will explain it further with a numerical example.

Next, we consider the conditions that coefficients a, b of equation (1) are real numbers, and $a < 0, b > 0, a + b < 0$. Consider the characteristic equation (5) of (1) again, and let $\xi = x + yi$.

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$$\xi - a - be^{-\xi\tau} = 0 \tag{21}$$

is equivalent to

$$x - a = be^{-\tau x}\cos(-\tau y), \quad y = be^{-\tau x}\sin(-\tau y), \tag{22}$$

Obviously $y = 0$ satisfies the second formula of (22). The corresponding x must satisfy

$$x - a = be^{-\tau x}. \tag{23}$$

Let $g(x) = x - a - be^{-\tau x}$. Note that $a < 0, b > 0, a + b < 0$. By the intermediate value theorem, we know $g(x)$ has an unique zero point on the left real axis. From (23), we know $x = a + be^{-\tau x} \geqslant a + b$. By this, we get there exists a zero point ξ_0 of characteristic equation (22) such that $\xi_0 \geqslant a + b$. If $|a + b| \sim 0$, then $e^{\xi_0 t}$ is a slow component. And if $|a - b| \gg 1$, the stepsize h will be restricted severely. This follows from theorem 2.6, theorem 2.7 and (17).

Definition 3.1 For DDE (1), coefficients a, b are real numbers, and $a < 0, b > 0, a + b < 0$. Besides, we define $r_D = \sup\limits_{|\xi| \leqslant 1} |a + b\xi| / \inf\limits_{|\xi| \leqslant 1} |a + b\xi| = |a - b| / |a + b|$, If $r_D \gg 1$, the equation (1) is said to be stiff.

By definition 3.1, if (1) is stiff, it is difficult to calculate the initial problem of (1) just as that of stiff ODEs. Now we should use the numerical method which is *GP*-stable. In the next section, the numerical example will further explain the above analysis.

4 A numerical example

Consider the delay differential equation

$$U'(t) = -1000U(t) + 999.9U(t-1), \ t > 0, \ U(t) = e^{\xi_0 t}, \ t \leqslant 0, \tag{24}$$

where ξ_0 is the solution of (23). So the true solution of (24) is $e^{\xi_0 t}$, $t \in (-\infty, \infty)$. Applying the explicit *Euler* method to (24), we get

$$U_{n+1} = U_n + h[aU_n + bU_{n-m}], \tag{25}$$

where $mh = 1$. The characteristic equation of (24) is

$$z^{m+1} = z^m + haz^m + hb. \tag{26}$$

It is too strict to determine the step size h with (17), since the reasoning of this formula is about complex coefficients a and b. Thus we can rewrite (26) as

$$p_m(z) = z^m(z - (1 + ah)) - hb. \tag{27}$$

The polynomial $p_m(z)$ defined by (27) is a *Schur* polynomial if and only if[9] (1) $z - (1 + ha)$ is a Schur polynomial, (2) $hb \leqslant |z - (1 + ha)|$, $\forall\ z \in C, |z| = 1$, (3) $p_m(z) \neq 0$, $\forall\ z \in C, |z| = 1$.

Supposing that a, b satisfy the conditions in definition 3.1, we can verify that condition (1)~(3) hold if and only if

$$1 + ha \geqslant 0. \tag{28}$$

Therefore, we get $h \leqslant -\dfrac{1}{a} = 0.001$. Using the implicit Euler method, which is *GP*-stable (see [9] and [12]), step size h is an arbitrary positive number. We let $h = 0.1$ and the result of calculation is listed in the following table ($\xi_0 = 0.99905 \times 10^{-4}$).

method	step size	number of iteration	numerical solution	true solution
explicit Euler	0.001	2.3×10^7	0.10025	0.10048
implicit Euler	0.1	2.3×10^5	0.10049	0.10048

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From the table we see that when the stiff rate $r_D = |a-b|/|a+b| = 19999$, it is difficult to use the explicit *Euler* method, and the number of iterations reaches 2.3×10^7. While the step size h of the implicit Euler method can be chosen arbitrarily, computational formulas can use variant step formats, and the number of iterations can be reduced. What more important is that when we apply the explicit formula to stiff delay differential equations, step size h should be very small, since $mh = \tau$, for large $\tau, m \gg 1$. From the formula (25), we see that while calculating U_{n+1}, we must use U_{n-m}, which makes the attenuation of the iteration much slower.

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一类刚性延时微分方程的数值分析

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摘要：研究了数值求解延时微分方程的步长准则. 据此，提出了延时微分方程具有刚性的概念. 最后以一个数值例子分析了求解刚性延时微分方程的困难性.

关键词：延时微分方程；渐近稳定；步长准则；刚性