Uniqueness in Random-Proposer Multilateral Bargaining

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Abstract

Solution uniqueness is an important property for a bargaining model. Rubinstein's (1982) seminal 2-person alternating-offer bargaining game has a unique Subgame Perfect Equilibrium outcome. Is it possible to obtain uniqueness results in the much enlarged setting of multilateral bargaining with a characteristic function? In an exploratory effort, this paper investigates a model first proposed in Okada (1993) in which each period players have equal probabilities of being selected to make a proposal and bargaining ends after one coalition forms. Focusing on transferable utility environments and Stationary Subgame Perfect Equilibria (SSPE), we find _ex ante_ SSPE payoff uniqueness for large classes of characteristic functions. This study includes as a special case a variant of the legislative bargaining model in Baron and Ferejohn (1989), and our results imply (unrestricted) SSPE payoff uniqueness in this case.

Keywords: random-proposer, multilateral bargaining, unique, coalition

JEL Classification: C72

1 Introduction

Solution uniqueness is an important property for a bargaining model. Rubinstein's (1982) seminal 2-person alternating-offer bargaining game has a unique Subgame Perfect Equilibrium outcome. A considerable literature exists that generalizes Rubinstein (1982) to the n-person dividing-a-pie problem, producing equilibrium uniqueness results of various strengths. Noteworthy studies include Merlo and Wilson (1995) and Krishna and Serrano (1996), and a brief survey can be found in the introduction of Chatterjee and Sabourian (2000). Is it possible to obtain uniqueness results in the still richer setting of bargaining with a characteristic function (i.e. coalitional bargaining), which, unlike the dividing-a-pie problem where only the grand coalition has a positive pie, allows subsets of players to have positive pies as well? This question motivates this exploratory study.

It turns out that there is little in the dividing-a-pie literature that we can build upon, because in our setting, coalition formation plays an important role in determining payoffs from bargaining. Previous studies in this setting either seek to support all core allocations as equilibrium outcomes (see Yan (2003) for a brief review) or focus on the efficiency properties of the proposed bargaining models (Chatterjee et al. (1993), Okada (1993) and Okada (1996)).[1]

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[1] Although Chatterjee et al. (1993) focuses on efficiency, one of their results shows the

This paper studies the same, one-stage stationary stochastic bargaining model as Okada (1993), and reports strong uniqueness results regarding (*ex ante*) equilibrium payoffs for a considerable range of games, including simple games, symmetric games, convex games and strictly monotone games that admit an efficient equilibrium. Naturally, payoff uniqueness restricts the equilibrium pattern of coalition formation. For Okada's (1993) model, except when players are indifferent about which coalition to nominate, equilibrium payoff uniqueness implies a unique profile of (possibly identical) coalitions formed with equal probabilities.

1.1 The Model (Okada (1993))

An unusual feature of Okada's (1993) model is that bargaining ends after one coalition forms, and players excluded from the coalition get their autarky payoffs. This one-stage model is worth studying as a starting point, and it is reasonable when the one-stage property holds, that is, when no coalition and its complement both have strictly positive worths. The multi-stage extension of this model is studied in Okada (1996). As discussed later, the main result of Okada (1996) has positive implication for extending our results to the multi-stage model.

uniqueness of the cutoff values players use when responding to proposals. However, unlike in our paper, the cutoff values typically do not correspond to equilibrium payoffs.

We now describe the model informally. We assume that the underlying cooperative game is of transferable utility, essential, monotone, and zero-normalized. Bargaining proceeds as follows. Each period one player is randomly selected with equal probablities from all n players to make a proposal. In her proposal the selected player nominates a coalition and announces a feasible allocation for that coalition. The coalition members then respond sequentially. The proposal passes if and only if it is accepted by all the members of the nominated coalition. If the proposal passes, the game ends, the proposed allocation is realized and each player excluded from the coalition gets zero payoff. Otherwise, the game continues into the next period with payoffs discounted by $\delta \in [0, 1)$, and the same bargaining procedure is followed as in the previous period. We call this game the *random-proposer game*.

For the rest of the paper we suppress references to the cooperative game, to the properties of which we refer instead as those of the characteristic function, so the term "symmetric game", for instance, refers to random-proposer games with symmetric characteristic functions.

1.2 SSPE Payoff Uniqueness in the Variant Baron-Ferejohn Model

This paper makes an indirect contribution to the legislative bargaining literature building on Baron and Ferejohn (1989) (henceforth BF). When the

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characteristic function assigns worth 1 to all majority coalitions and 0 to the rest the random-proposer game becomes, with nonsubstantive differences[2], a variant of the BF model, for which we obtain a stronger uniqueness result than BF.

More precisely, BF first shows that with 5 or more players any allocation can be supported in a Subgame Perfect Equilibrium (SPE) for high enough δ. This result, typical of multilateral bargaining models, has an obvious analogue (which we omitted) in the random-proposer game. BF, as do we and most of the literature, addresses such multiplicity by restricting attention to Stationary Subgame Perfect Equilibria (SSPE). BF shows payoff uniqueness within the class of SSPE restricted in the following ways: i) players do not play weakly dominated strategies and ii) they play the "simplest" SSPE satisfying i), where simplicity is made precise by Baron and Kalai (1993), whose result implies that of all the SSPE in which players do not play weakly dominated strategies, those in BF require the least number of states if executed by automata.[3] In

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[2] In the original BF model, all players respond to the proposed allocation. The proposal passes if accepted by a majority. This difference in response procedure proves nonsubstantive as the analyses of the two models are similar and they produce identical allocations in equilibrium.

[3] In fact Baron and Kalai (1993) show a stronger result: of all the SPE in which players do not play weakly dominated strategies the BF type SSPE require the fewest automaton states.

this paper, using simple techniques that exploit subgame perfection we show that for every general SSPE there exists a BF type SSPE that generates the same outcome; and this result, Proposition 2, can be easily adapted to the original BF model.[4] Moreover, the proof of Proposition 2 does not depend on the proposing probabilities, the discount factors, or on the size of the minimum winning coalition, consequently it can be extended to obtain unrestricted SSPE payoff uniqueness for the generalized BF model studied in Eraslan (2002).[5]

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[4]The adaptation is straightforward and not included in this paper.

[5]In Eraslan (2002), although the SSPE is defined the same way as the general SSPE in this paper, it is clear from the analysis that only the BF class of SSPE are considered. For instance, an assumption is made in footnote 3 to the effect that a player is to accept a proposal if the offer to her equals her continuation payoff. This assumption effectually prevents a player from conditioning her response on the identity of the proposer, or on offers made to the other players, or on prior responses made by other players. Moreover, it is assumed implicitly that players do not play weakly dominated strategies, as the arguments in Eraslan (2002) ignore the possibility that a player may reject (accept) an offer strictly higher (lower) than her continuation payoff because her vote is not pivotal. Finally, no delay is assumed implicitly, since there is no arguments ruling out those SSPE in which a proposer finds it optimal to make a proposal that she knows will be rejected as she expects to receive higher payoff when other players make proposals.

1.3 Results

We focus on SSPE, and we call the _ex ante_ payoff profile generated by an SSPE an _SSPE payoff profile_.

We first demonstrate the outcome-equivalence between general SSPE and the simple, BF type SSPE, which we call the _cutoff strategy SSPE_. We then establish the existence of (mixed) SSPE in the general random-proposer game, although pure SSPE do not always exist.

Our uniqueness results vary in strength depending on the characteristic function. For simple games we show that all pure SSPE must be payoff equivalent. For symmetric games we show that there is a unique SSPE payoff profile. For convex games we focus on the _inclusive SSPE_, namely those SSPE in which proposers nominate, roughly speaking, the "largest" payoff maximizing coalitions. It is shown that all inclusive SSPE of a convex game must be pure and payoff equivalent. Our last result concerns efficient SSPE. As shown in Okada (1993), an efficient SSPE exists for all $\delta \in [0, 1)$ if and only if equal division is a core allocation. Our last result shows for strictly monotone games that an efficient cutoff strategy SSPE, if it exists, is the unique cutoff strategy SSPE of the game.

1.4 Organization

The rest of the paper is organized as follows: Section 2 describes the model; Section 3 establishes the existence of SSPE and the outcome equivalence between general SSPE and cutoff strategy SSPE; Section 4 contains the main technical results used in the uniqueness proofs; Section 5 presents the uniqueness results; Section 6 discusses extensions; the appendix collects the long and the purely technical proofs.

2 The Random-Proposer Game

Let N denote the set of all players, and let $n = |N|$. A _coalition_ is a nonempty subset of players. A _characteristic function_ v maps each coalition S to an element in $\Re_+$, and $v(S)$ is called the _worth_ of S, which may be interpreted as the size of the surplus available for division among members of S. An _allocation for a coalition S_ is an element in $\Re_+^{|S|}$, written as $w = (w_i)_{i \in S}$; it is _feasible_ if $\sum_{i \in S} w_i \leq v(S)$. $w = (w_i)_{i \in N}$ is _efficient_ if $\sum_{i \in N} w_i = v(N)$.

v is _monotone_ if $v(T) \leq v(S)$ for any $T \subset S$, _strictly monotone_ if the inequality is strict; v is _essential_ if $v(S) > 0$ for some S; v is _0-normalized_ if $v(\{i\}) = 0$ for all i. We maintain throughout that v is monotone, 0-normalized and essential, and we require v to be strictly monotone for the last result,

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Proposition 7.

The random-proposer game is as follows. Bargaining proceeds in periods $1, 2, 3, \ldots$, until an agreement is reached. At the beginning of each period, one player is randomly selected to be the *proposer*. Every player has probability $\frac{1}{n}$ of being selected in any period. Suppose in period t player i is selected. i then makes a proposal that consists in the nomination of a coalition S — we allow $i \notin S$, although $i \in S$ in equilibrium — and the announcement of a feasible allocation for S, $w = (w_j)_{j \in S}$. Let (S, w) denote the proposal. We will sometimes abuse terminology and say that i "nominates j" if $j \in S$. The proposal passes immediately if $S = \{i\}$. If $S \backslash \{i\} \neq \emptyset$, sequentially all the players in $S \backslash \{i\}$ respond by accepting or rejecting the proposal. The exact order of responses is exogenous and, as can be seen shortly, immaterial to our model. If all in $S \backslash \{i\}$ accept, the proposal passes, the game ends, and each $j \in S$ gets $\delta^{t-1} w_j$, where $\delta \in [0, 1)$, while each $j \in N \backslash S$ gets $v(\{j\})$, which equals 0 due to 0-normalization. If at least one player in $S \backslash \{i\}$ rejects, bargaining proceeds into period $t + 1$ and the same bargaining procedure is repeated. If no agreement is ever reached, every player gets zero payoff.

Except Proposition 3 the results in the next two sections have been generalized in Yan (2003),[6] nevertheless we produce the proofs for the reader's

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[6]The results in the next two sections first appeared in the working paper version of this

convenience.

3   The SSPE

A strategy in a random proposer game is called _stationary_ if it is independent

of the history of past periods.[7]   A _Stationary Subgame Perfect Equilibrium_

(SSPE) is a Subgame Perfect Equilibrium in which players use stationary

strategies.   As we will discuss later, an SSPE may be "complicated" despite

its stationarity. Fortunately, Proposition 2 shows that for every SSPE there

exists an outcome-equivalent, "simple" SSPE.[8] Let us first define the "simple"

stationary strategy profiles, which we call the _cutoff strategy profiles_ because

an important feature of these strategy profiles is that each player accepts a

proposal if and only if she herself is offered at least a certain cutoff value.

Formally, let $\mathcal{F}_i = \{S | i \in S\}$, let P_i denote the set of probability distributions

over $\mathcal{F}_i$, and let $P = \Pi_{i \in N} P_i$. Define a (possibly mixed) _cutoff strategy profile_

paper, which has fallen behind Yan (2003) in the publication process. We decide to claim

credits to these results here instead of in Yan (2003).

[7]Here to save space we do not give a formal definition of stationarity.

[8]Here we refer to the complexity or simplicity of a strategy profile rather than that of

a strategy in the same spirit as Baron and Kalai (1993), who measure the complexity of

a strategy profile using the number of automaton states needed when it is executed by an

automaton.

σ_c by a pair (p, x), where $p \in P$ and $x \in \Re_+^n$, such that player i when proposing nominates coalition S with probability $p_i(S)$, offers x_j to each $j \in S \backslash \{i\}$, and when responding i accepts a proposal if and only if the offer to her is at least x_i. p is called the _coalitional profile_ and x the vector of _cutoff values_. For each i and $S \in F_i$, let $g_i(S, x)$ be the payoff to i of nominating S, that is, $g_i(S, x) = v(S) - \sum_{j \in S \backslash \{i\}} x_j$. We will write $g_i(S)$ in place of $g_i(S, x)$ when x is clear. Let π_i be i's expected payoff when selected to propose, termed her _proposer payoff_, given by $\pi_i = \sum_{S \in \mathcal{F}_i} p_i(S) g_i(S)$. Let q_i^j be the probability of i receiving an offer from j conditional on j being selected,

$$q_i^j = \sum_{S \in \mathcal{F}_j} p_j(S) I_i(S)$$

where $j \neq i$, and $I_i(S) = 1$ if $i \in S$ and 0 otherwise. Let q_i be the _ex ante_ probability of i receiving an offer from another player,

$$q_i = \frac{1}{n} \sum_{j \neq i} q_i^j$$

q_i is called i's _nomination probability_. Note that by definition $q_i \leq 1 - \frac{1}{n}$. Let y be the _ex ante_ payoff function that maps each strategy profile to the induced _ex ante_ payoff profile, an element in $\Re_+^n$. It is easy to see that

$$y_i = \frac{1}{n} \pi_i + q_i x_i. \tag{1}$$

Note that because σ_c is stationary, player i's _continuation payoff_ in any period t, which is defined to be her expected payoff (discounted to period t)

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in any subgame that begins with a proposal being rejected by some player in period t, is δy_i.

The equilibrium conditions for a cutoff strategy SSPE are straightforward.

Proposition 1 (Okada (1993)) *$\sigma_c = (p, x)$ is an SSPE if and only if*

i) $x = \delta y$;

ii) $p_i \in \operatorname{argmax}_{\tilde{p}_i \in P_i} \pi_i(\tilde{p}_i, x)$ *for all i.*

Proof: The "only if" direction is obvious. For the "if" direction, the only nontrivial step is to verify that proposers do not prefer being rejected, that is $\pi_i(p_i) \geq \delta y_i$. To see this, substitute δy_i for x_i on the right hand side of (1), we have

$$y_i = \frac{\frac{1}{n}\pi_i}{1 - q_i \delta} \tag{2}$$

Because $1 - q_i \delta \geq \frac{1}{n}$, it follows that $\pi_i \geq y_i \geq \delta y_i$ since we know $\pi_i \geq 0$ as i can get 0 by nominating $\{i\}$. ∎

Before we state Proposition 2, which shows the outcome equivalence between cutoff strategy SSPE and general SSPE, it is useful to note how the general stationary strategy profile is more complicated than the cutoff strategy profile. First, in a (mixed) general stationary strategy profile the proposing

behavior may be given by an arbitrary probability measure over the proposal space $\cup_{S \in 2^N \setminus \emptyset} \{S\} \times \Re_+^{|S|}$, allowing among other things offering the same player different values depending on the coalition nominated as well as nominating a coalition of which the proposer is not a member. Second, our definition of stationarity allows strategies to condition on the history within the current period, so when responding to a proposal a player may condition her response not only on the offer to herself but also on the identity of the proposer, on the coalition nominated, on offers made to the other coalition members, and on responses made by players preceding her. Finally, a player when responding may randomize between acceptance and rejection in a (mixed) general stationary strategy profile.

We now prepare to state Proposition 2. Given a general stationary strategy profile, consider for each player her possibly randomized proposing behavior and focus on the induced marginal distribution over the set of coalitions $2^N \setminus \emptyset$. We call the collection of these marginal distributions the *coalitional profile*, in keeping with the terminology used for cutoff strategy profiles. For each player i, let $\mathcal{F}_{i+}$ denote the set of coalitions player i nominates with positive probabilities, and let C_i be the union of these coalitions.

Proposition 2 *For any SSPE σ with a coalitional profile p the cutoff strategy profile $\sigma_c = (p, \delta y(\sigma))$ generates the same outcome and is an SSPE.*

Proof: See the appendix. ∎

From now on we only consider cutoff strategy SSPE, which we simply call SSPE.

Next we show the existence of (mixed) SSPE for the general random-proposer game. However, pure SSPE do not always exist.[9]

Proposition 3 *The random-proposer game has an SSPE.*

Proof: For any $x \in [0, v(N)]^n$, let $P^*(x) = \Pi_{i \in N} P_i^*(x)$, where $P_i^*(x) = \text{argmax}_{\tilde{p}_i \in P_i} \pi_i(\tilde{p}_i, x)$. P^* is convex-valued and upper-hemicontinuous by the Theorem of Maximum. Consider the correspondence B from $[0, v(N)]^n$,

$$B(x) = \{b | b = \delta y(\sigma_c(p^*, x)), \text{ where } p^* \in P^*(x)\}$$

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[9]Here is an example of inexistence of pure SSPE for δ close enough to 1. Consider a 3-player game in which the characteristic function is zero-valued except for $v(\{1,2,3\}) = 1$ and $v(\{1,2\}) = 0.8$. In pure SSPE a player when selected to propose chooses one coalition with probability 1. It is easy to see that player 1 and 2 must include each other in their chosen coalitions and player 3 must choose N. This leaves us 3 possibilities: both, one or neither of 1 and 2 choose N. The first can be ruled out as follows: the payoff profile, which can be solved for from the simultaneous equations provided by (2), is $y_i = 1/3$ for all i; this implies that 3 is too "expensive" to be nominated when δ is high enough since 3 increases the worth by only $v(\{1,2,3\}) - v(\{1,2\}) = 0.2 < 1/3$. In the other two possibilities player 3 is too "cheap" not to be nominated as her payoffs turn out to be less than 0.2.

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To see $B(x) \in [0, v(N)]^n$, note that for any $i \in N$, $0 \leq \pi_i(p_i^*, x) \leq v(N)$. Hence, $0 \leq b_i(x) = \delta\frac{1}{n}\pi_i(p_i^*, x) + \delta q_i(p^*)x \leq \frac{1}{n}v(N) + \frac{n-1}{n}v(N) = v(N)$.

We will show that B is convex-valued and upper-hemicontinuous. To see convexity, consider any $b^1, b^2 \in B(x)$. By definition there exist $p^1, p^2 \in P^*(x)$ s.t. $b^1 = \delta y(\sigma_c(p^1, x)), b^2 = \delta y(\sigma_c(p^2, x))$. Note that y is linear in p. Hence, for any $\alpha \in [0, 1]$, $\alpha b^1 + (1 - \alpha)b^2 = \delta y(\sigma_c(\alpha p^1 + (1 - \alpha)p^2), x))$. Since $P^*(x)$ is convex, $\alpha p^1 + (1 - \alpha)p^2 \in P^*(x)$. Therefore $\alpha b^1 + (1 - \alpha)b^2 \in B(x)$. Hence convexity is established.

Suppose to the contrary that B is NOT upper-hemicontinuous. Then there exists a sequence $x_n \to \bar{x}$ and a sequence $b^n \to \bar{b}$, s.t. $b^n \in B(x_n)$ for all n, but $\bar{b} \notin B(\bar{x})$. By definition there exists a sequence $\{p^n\}$ s.t. $p^n \in P^*(x_n) \subseteq P$ for all n, and $b^n = \delta y(\sigma_c(p^n, x^n))$. Since P is compact, $\{p^n\}$ has a subsequence $\{p^{n_k}\}$ that converges to a point $\bar{p}$ in P. Since the subsequence $\{x^{n_k}\}$ converges to $\bar{x}$, and since P^* is upper-hemicontinuous, $\bar{p} \in P^*(\bar{x})$. Note that $\bar{b} \neq \delta y(\sigma_c(\bar{p}, \bar{x}))$ since we assumed $\bar{b} \notin B(\bar{x})$. Since y is continuous, for large k, $b^{n_k} \neq \delta y(\sigma_c(p^{n_k}, x^{n_k}))$, a contradiction.

By Katutani's Fixed Point Theorem, $x \in B(x)$ for some $x \in [0, v(N)]^n$. Therefore, there exists $p^* \in P^*(x)$ s.t. $x = \delta y(\sigma_c(p^*, x))$. Then $\sigma_c(p^*, x)$ is an SSPE. ∎

4 Intermediate Results

The fact that we prove the existence of SSPE for general random-proposer games using Katutani's fixed-point theorem might suggest that uniqueness may also be demonstrated through familiar theorems such as the contraction theorem. Instead we find it more fruitful to use tactics often seen in the discrete branch of mathematics. The overall structure of our demonstration is that we first exhibit the pattern that must be obeyed by two payoff-_non_equivalent SSPE in a general random-proposer game, then we exploit the different traits of different classes of characteristic functions and show how in each case the afore-mentioned pattern cannot hold true.

Lemma 1 below describes the pattern followed by two payoff-_non_equivalent SSPE. Note from (2) that a player's SSPE payoff is determined by, and increasing in, her proposer payoff and her nomination probability. Informally, Lemma 1 i) states that players' payoffs and nomination probabilities change from one SSPE to the other in (weakly) opposite directions on the whole. The intuition for this is that if a player has a payoff increase (decrease), she raises (lowers) her cutoff value and hence, other things equal, makes herself less (more) attractive to the proposers. Lemma 1 ii) considers those players whose only source of payoff increase is the increase in their proposer payoffs.

Roughly speaking, it states that their payoff increases do not exceed the payoff decreases of their coalition members, because the increases in their proposer payoffs come from the coalition members' lowering the cutoff values. Lemma 1 iii) is similar to Lemma 1 i) in spirit: for at least one player, her payoff and nomination probability must change in strictly opposite directions.

The following notation scheme is adopted for the rest of the paper when comparing two SSPE σ and σ': let $y = y(\sigma)$, $y' = y(\sigma')$, $\Delta y = y' - y$, and similarly for the other variables, and we also find it useful to define $Y_{++} = \{i | \Delta y_i > 0\}$, $Y_{--} = \{i | \Delta y_i < 0\}$, and $Q_- = \{i | \Delta q_i \leq 0\}$.

To see what contributes to a payoff change, we obtain from (2)

$$\Delta y_i = \frac{\frac{1}{n}\Delta \pi_i + \Delta q_i \delta y_i}{1 - q_i' \delta}. \tag{3}$$

One can see that a player's payoff change has two sources: one is the change in her proposer payoff, $\Delta \pi_i$, and the other is the change in her nomination probability, Δq_i.

Lemma 1 *If two SSPE σ_c and σ_c' are such that $y \neq y'$, we have*

i) $\sum_{i \in N} \Delta q_i \Delta y_i \leq 0$;

ii) if $Y_{++} \cap Q_- \neq \emptyset$, then for any nonempty $T \subseteq Y_{++} \cap Q_-$, we have $\hat{T} \neq \emptyset$, where $\hat{T} = (\cup_{i \in T} C_i') \cap Y_{--}$, and

$$\sum_{i \in T} \Delta y_i < \sum_{j \in \hat{T}} (-\Delta y_j);$$

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iii) $\Delta q_j \Delta y_j < 0$ *for some j.*

Proof: See the appendix. ■

Corollary *For any coalitional profile $p \in P$, there is at most one $y \in R^n$ such that the cutoff strategy profile $(p, \delta y)$ is an SSPE.*

Proof: Suppose to the contrary that $(p, \delta y)$ is an SSPE for two different values of y, then we have two payoff-nonequivalent SSPE and Lemma 1 iii) implies $\Delta q_j \neq 0$ for some j, which is impossible since the two SSPE have a common coalitional profile. ■

Next we define, for want of a better term, the *proposer surplus*, denoted by $\bar{\pi}_i$, which as we shall see is equal to the proposer payoff minus the proposer's own cutoff value. Formally, define $\bar{g}(S) = v(S) - \sum_{j \in S} x_j$, and define $\bar{\pi}_i = \sum_{S \in \mathcal{F}_i} p_i(S) \bar{g}(S)$. It is easy to see that $\bar{g}(S) = g_i(S) - x_i$, and $\bar{\pi}_i = \pi_i - x_i$. We can express y_i in terms of $\bar{\pi}_i$,

$$ y_i = \frac{\frac{1}{n} \bar{\pi}_i}{1 - q_i \delta - \frac{1}{n} \delta} \tag{4} $$

More importantly,

Lemma 2 *If $\sigma_c = (p, \delta y)$ is an SSPE,*

i) $p_i \in \operatorname{argmax}_{\tilde{p}_i \in P_i} \bar{\pi}_i(\tilde{p}_i, \delta y)$ for all i;

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ii) $\bar{\pi}_i \leq \bar{\pi}_j$ *if* $j \in C_i$;

iii) $\bar{\pi}_i \leq \pi_j$ *for any* i *and* j.

Proof:

i) This follows directly from the fact that π_i is maximized in equilibrium.

ii) $j \in C_i$ means that for some $S \in \mathcal{F}_{i+}$, $j \in S$. So just like i, j could also nominate S, hence $\bar{\pi}_j \geq \bar{g}(S) = \bar{\pi}_i$.

iii) Pick any $S \in \mathcal{F}_{i+}$, we have $\bar{\pi}_j \geq \bar{g}(S \cup \{j\})$ since j could nominate $S \cup \{j\}$. Since $v(S \cup \{j\}) \geq v(S)$ by monotonicity, we have $\bar{g}(S \cup \{j\}) \geq \bar{g}(S) - \delta y_j = \bar{\pi}_i - \delta y_j$. Hence $\bar{\pi}_j \geq \bar{\pi}_i - \delta y_j$, or $\pi_j \geq \bar{\pi}_i$.

∎

5   Uniqueness Results

In this section we present the uniqueness results for simple games, symmetric games, convex games and strictly monotone games that admit efficient equilibria.

Call a random-proposer game *simple* if for any $S \subseteq N$, either $v(S) = 1$ or $v(S) = 0$.

Proposition 4 *The pure SSPE of a simple game, if they exist, must be payoff equivalent.*

Proof: We first assert that in a simple game the <u>*ex post*</u> SSPE payoffs always add up to 1. This is because the proposer always nominates a coalition of worth 1, since otherwise she gets zero proposer payoff and hence by (2) zero <u>*ex ante*</u> payoff, contradicting Claim 3 in the proof of Proposition 2, which states that $y_i > 0$ for all i. It then follows that the <u>*ex ante*</u> payoffs which are simply averages of the <u>*ex post*</u> payoffs must add up to 1 as well. We now invoke a technical result, which, shown in two steps, Claim 1 and Claim 2, states that if two pure SSPE produce different <u>*ex ante*</u> payoff profiles, then the two payoff profiles must add up to different sums. Incidentally, Claim 1 and 2 are valid for general random-proposer games and their proofs below do not use properties of simple games.

Claim 1 *Given two pure SSPE σ_c and σ'_c such that $y(\sigma_c) \neq y(\sigma'_c)$, if $Y_{++} \cap Q_{--} \neq \emptyset$, we must have $\sum_{i \in N} y_i(\sigma_c) < \sum_{i \in N} y_i(\sigma'_c)$.*

Proof of Claim 1: See the appendix. ■

Claim 2 *If $y(\sigma_c) \neq y(\sigma'_c)$ for two pure SSPE σ_c and σ'_c, we must have $\sum_{i \in N} y_i(\sigma_c) \neq \sum_{i \in N} y_i(\sigma'_c)$.*

Proof of Claim 2: Recall Lemma 2 iii): $\Delta y_i \Delta q_i < 0$ for some i. So either $Y_{++} \cap Q_{--} \neq \emptyset$ or $Y_{--} \cap Q_{++} \neq \emptyset$. In the first case, Claim 1 applies immediately. In the second case, if we reverse the labelings of σ_c and σ'_c (so that Y_{--} becomes Y_{++} and Q_{++} becomes Q_{--}), Claim 1 still applies. ∎

The proposition then follows immediately. ∎

Next we turn to symmetric games. Call two players i and j _symmetric_ if for any $S \subseteq N \backslash \{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$; call the random-proposer game _symmetric_ if all players are pairwise symmetric. It follows from the definition that in a symmetric game $v(S) = v(S')$ if $|S| = |S'|$.

Proposition 5

i) *If players i and j are symmetric, $y_i(\sigma_c) = y_j(\sigma_c)$ in any SSPE σ_c;*

ii) *if the random-proposer game is symmetric, there exists a unique SSPE payoff profile.*

Proof:

i) Suppose to the contrary $y_i > y_j$, then

   Claim 1 $\bar{\pi}_j \geq \bar{\pi}_i$

Proof of Claim 1: This follows directly from Lemma 2 ii) if $j \in C_i$. If $j \notin C_i$, then fix any $S \in \mathcal{F}_{i+}$. By Lemma 2 i) and the symmetry of i and j, we have $\bar{\pi}_j \geq \bar{g}((S \backslash \{i\}) \cup \{j\}) = \bar{g}(S) + \delta y_i - \delta y_j > \bar{g}(S) = \bar{\pi}_i$. ∎

Claim 2 $q_j \geq q_i$

Proof of Claim 2: First, consider any $k \neq i, j$. For any $S \in \mathcal{F}_{k+}$, $i \in S$ implies $j \in S$ because i and j are symmetric and $\delta y_i > \delta y_j$. Hence $q_j^k \geq q_i^k$. Next we will show that $q_j^i \geq q_i^j$. Suppose to the contrary that $q_j^i < q_i^j$. Then $q_j^i < 1$ and $q_i^j > 0$. So there exist $S_i \in \mathcal{F}_{i+}$, $S_j \in \mathcal{F}_{j+}$, such that $i \in S_j$ but $j \notin S_i$. We have $g_i(S_i) \geq g_i(S_j) = g_j(S_j) - \delta y_j + \delta y_i > g_j(S_j)$. On the other hand, $g_j(S_j) \geq g_j[(S_i \backslash \{i\}) \cup \{j\}]$ and the symmetry of i and j implies that $g_j[(S_i \backslash \{i\}) \cup \{j\}] = g_i(S_i)$, so $g_j(S_j) \geq g_i(S_i)$, a contradiction. Hence $q_j^i \geq q_i^j$. It follows that $q_j \geq q_i$. ∎

Claims 1 and 2 together imply that $y_i \leq y_j$, contradicting our assumption. Therefore $y_i = y_j$.

ii) Suppose to the contrary that σ_c and σ_c' are two payoff-nonequivalent SSPE. Since the game is symmetric, it follows from i) that payoffs are symmetric for all players in either SSPE. Assume w.l.g. $y_i' > y_i$ for all i. Fix any i. We have $\Delta \pi_i \leq \sum_{j \neq i} q_j^{i'}(-\delta \Delta y_j) \leq 0$, where the first

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inequality is by (7) in the appendix. Since $\Delta y_i > 0$, by (3) we must have $\Delta q_i > 0$. So $\Delta q_i \Delta y_i > 0$. Since this is true for all i, Lemma 1 iii) is violated. The proof is then complete by contraposition.

∎

Proposition 5 has implication for the uniqueness of equilibrium coalition size in symmetric games. Call an SSPE _inclusive_ if for any i and $S \in \mathcal{F}_{i+}$, $S \subset T$ implies $g_i(T) < g_i(S)$. Intuitively, in inclusive SSPE players when proposing nominate, roughly speaking in general but literally for symmetric games, the largest payoff-maximizing coalitions. If we restrict attention to inclusive SSPE, then, since the equilibrium payoff profile is unique and symmetric by Proposition 5 and hence so is the cutoff value profile, the equilibrium coalition size must also be unique.

Next we consider convex games, for which we focus on inclusive SSPE. Formally, the random-proposer game is _convex_ if for any $S, T \subseteq N$, $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$. As shown below, convexity together with inclusiveness induces structure in the equilibrium coalitional profile, which we exploit to show payoff uniqueness. However, inclusive SSPE do not always exist: an inclusive SSPE must be pure for a convex game and as mentioned before pure SSPE may not exist.

Proposition 6 *In a convex game, if σ_c is an inclusive SSPE,*

   i) σ_c *is pure;*

   ii) for each player i let S_i denote the coalition nominated by i when i pro-

       poses,

       a. for any i and j such that $j \in S_i$, we have $S_j \subseteq S_i$ and $y_j \geq y_i$;

       b. $\cap_{i \in N} S_i \neq \emptyset$, and for any $j \in \cap_{i \in N} S_i$, $S_j = \cap_{i \in N} S_i$;

   iii) if σ_c' is another inclusive SSPE, $y(\sigma_c) = y(\sigma_c')$.

Proof: First note that convexity implies that for any $S, T \subseteq N$, $\bar{g}(S \cup T) + \bar{g}(S \cap T) \geq \bar{g}(S) + \bar{g}(T)$.

   i) σ_c being pure means that $\mathcal{F}_{i+}$ is a singleton for all i. Suppose to the contrary $\{S^1, S^2\} \subseteq \mathcal{F}_{i+}$ for some i. Assume w.l.g. $S^2 \not\subseteq S^1$. Since $\bar{g}(S^1 \cup S^2) + \bar{g}(S^1 \cap S^2) \geq \bar{g}(S^1) + \bar{g}(S^2)$, it follows that $\bar{g}(S^1 \cup S^2) - \bar{g}(S^1) \geq \bar{g}(S^2) - \bar{g}(S^1 \cap S^2) \geq 0$, where the second "$\geq$" follows from the optimality of nominating S^2. On the other hand, $\bar{g}(S^1 \cup S^2) - \bar{g}(S^1) = g_i(S^1 \cup S^2) - g_i(S^1) < 0$, where the inequality follows from $S^1 \subset S^1 \cup S^2$ and the inclusiveness of σ_c, a contradiction. Therefore σ_c must be pure.

   ii)   a. First, fix any $j \in S_i$, we will show $S_j \subseteq S_i$ by an argument similar to that in i). Suppose to the contrary $S_j \not\subseteq S_i$. Since $\bar{g}(S_i \cup$

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$S_j) + \bar{g}(S_i \cap S_j) \geq \bar{g}(S_i) + \bar{g}(S_j)$, we have $\bar{g}(S_i \cup S_j) - \bar{g}(S_i) \geq$
$\bar{g}(S_j) - \bar{g}(S_i \cap S_j) \geq 0$, where the second inequality follows from
the optimality of nominating S_j. On the other hand, $\bar{g}(S_i \cup S_j) - $
$\bar{g}(S_i) = g_i(S_i \cup S_j) - g_i(S_i) < 0$, where the inequality follows from
$S_i \subset S_i \cup S_j$ and the inclusiveness of σ_c, a contradiction. So we
must have $S_j \subseteq S_i$.

It follows that for any i and j such that $j \in S_i$, $i \in S_k$ implies
$j \in S_k$ for any k. Therefore $q_j \geq q_i$. Since in addition $\bar{\pi}_j \geq \bar{\pi}_i$ by
Lemma 2 ii), we must have, recalling (4), $y_j \geq y_i$.

b. We will show $\cap_{i \in N} S_i \neq \emptyset$ first.

Note first that for any i, $y_i > 0$ by Claim 3 in the proof of Propo-
sition 2, and hence $\bar{g}(S_i) = \bar{\pi}_i > 0$.

To show $\cap_{i \in N} S_i \neq \emptyset$, we do induction on the number of players.
Suppose to the contrary that for some i and j, $S_i \cap S_j = \emptyset$, then
$\bar{g}(S_i \cup S_j) \geq \bar{g}(S_i) + \bar{g}(S_j) > \bar{g}(S_j)$, which contradicts the optimality
of S_j. Hence we conclude for any two players i and j, $S_i \cap S_j \neq \emptyset$.
Suppose we have shown for any m-players $i_1, i_2, \ldots, i_m$, $\cap_{j=1}^{m} S_{i_j} \neq$
$\emptyset$. Consider any $m + 1$ players $l_1, l_2, \ldots, l_{m+1}$. $\cap_{j=1}^{m} S_{l_j} \neq \emptyset$ by
assumption. Fix any $k \in \cap_{j=1}^{m} S_{l_j}$. By Statement a., $S_k \subseteq \cap_{j=1}^{m} S_{l_j}$.
Since $S_{l_{m+1}} \cap S_k \neq \emptyset$, we have $\cap_{j=1}^{m+1} S_{l_j} \neq \emptyset$. Therefore $\cap_{i \in N} S_i \neq \emptyset$.

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To see the second part of Statement b., note that for any $k \in \cap_{i \in N} S_i$, $S_k \subseteq \cap_{i \in N} S_i$ by Statement a. Since obviously $\cap_{i \in N} S_i \subseteq S_k$, we have $S_k = \cap_{i \in N} S_i$.

iii) Suppose to the contrary $y(\sigma_c) \neq y(\sigma'_c)$ and apply the notation scheme introduced in Section 4, that is, let $\Delta y = y' - y$, $Y_{++} = \{i | \Delta y_i > 0\}$ and so on. By Lemma 1 iii), $\Delta y_i \Delta q_i < 0$ for some $i \in N$. Assume w.l.g. that $\Delta y_i > 0$ and $\Delta q_i < 0$. Note for later use that $\Delta q_i < 0$ implies $\Delta q_i \leq -\frac{1}{n}$ due to the pureness of the two SSPE. Consider any $j \in \cap_{k \in N} S'_k$, we will show $\Delta \bar{\pi}_j > 0$ and $\Delta y_j > 0$. By definition $j \in S'_i$, hence $\bar{\pi}'_i \leq \bar{\pi}'_j$ by Lemma 2 ii). It follows that $\Delta \bar{\pi}_i - \Delta \bar{\pi}_j = (\bar{\pi}'_i - \bar{\pi}'_j) + (\bar{\pi}_j - \bar{\pi}_i) \leq \bar{\pi}_j - \bar{\pi}_i \leq \pi_i - \bar{\pi}_i = \delta y_i$, where the last inequality is due to Lemma 2 iii). Hence $\Delta \bar{\pi}_i \leq \Delta \bar{\pi}_j + \delta y_i$. Now we want to show $\Delta \bar{\pi}_i > \delta y_i$. To see this, note first for any $k \in N$ we can derive from $(3)^{10}$,

$$\Delta y_k = \frac{\frac{1}{n}\Delta\bar{\pi}_k + \Delta q_k \delta y_k}{1 - q'_k \delta - \frac{1}{n}\delta} \tag{5}$$

It follows that $\Delta y_i = \frac{\frac{1}{n}\Delta\bar{\pi}_i + \Delta q_i \delta y_i}{1 - q'_i \delta - \frac{1}{n}\delta} \leq \frac{\frac{1}{n}\Delta\bar{\pi}_i - \frac{1}{n}\delta y_i}{1 - q'_i \delta - \frac{1}{n}\delta}$. Since $\Delta y_i > 0$ by assumption, $\frac{1}{n}\Delta\bar{\pi}_i - \frac{1}{n}\delta y_i > 0$, hence $\Delta\bar{\pi}_i > \delta y_i$, as desired. Since we proved earlier that $\Delta\bar{\pi}_i \leq \Delta\bar{\pi}_j + \delta y_i$, we must have $\Delta\bar{\pi}_j > 0$. Note in addition $j \in \cap_{k \in N} S'_k$ implies $\Delta q_j \geq 0$. Hence $\Delta y_j > 0$ by (5). So

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[10] This step uses the fact that $\Delta\bar{\pi}_k = \Delta\pi_k - \delta\Delta y_k$ and that $d = \frac{a}{b}$ implies $d = \frac{a+cd}{b+c}$ for any a, b, c, d such that $b + c \neq 0$.

we have shown $\Delta\bar{\pi}_j > 0$ and $\Delta y_j > 0$ for all $j \in \cap_{k \in N} S'_k$. Fix some

$j \in \cap_{k \in N} S'_k$. Note that $S'_j = \cap_{k \in N} S'_k$ by Statement b. of ii). Hence

$\Delta\bar{\pi}_j = \Delta\pi_j - \delta\Delta y_j \leq \sum_{k \in S'_j}(-\delta\Delta y_k) < 0$, where the first inequality

follows from (7) in the appendix, a contradiction. The proof is then

complete by contraposition.

∎

Our last result concerns efficient SSPE in strictly monotone games. An

SSPE σ_c is _efficient_ if $\sum_{i \in N} y_i(\sigma_c) = v(N)$. We show that in a strictly

monotone game, an efficient cutoff strategy SSPE, if it exists, is the unique

cutoff strategy SSPE of the game. Note that uniqueness of the cutoff strat-

egy SSPE implies that the general SSPE is unique up to off-equilibrium-path

response behavior.

Proposition 7 _In a strictly monotone game, an efficient cutoff strategy SSPE,_

if it exists, is the unique cutoff strategy SSPE of the game.

Proof: Let σ_c denote the efficient SSPE. Since the game is strictly monotone,

efficiency implies that every player when proposing nominates N with prob-

ability 1, because otherwise the _ex ante_ payoffs, being averages of _ex post_

payoffs, would not add up to $v(N)$. Hence by (4) we have $y_i = \frac{\bar{g}(N)}{1-\delta}$ for all

i. It follows that $y_i = \frac{v(N)}{n}$ for all i. Next we will show that if σ'_c is also an

SSPE, we must have $y(\sigma'_c) = y(\sigma_c)$. Suppose to the contrary $y(\sigma_c) \neq y(\sigma'_c)$ and apply the notation scheme introduced in Section 4, that is, let $\Delta y = y' - y$, $Y_{++} = \{i | \Delta y_i > 0\}$ and so on. We will show that $\sum_{i \in N} \Delta q_i \Delta y_i > 0$, which contradicts Lemma 1 i). Since obviously $\Delta q_i \leq 0$ for all i, we only need to show $\sum_{Y_{++}} |\Delta q_i||\Delta y_i| < \sum_{Y_{--}} |\Delta q_j||\Delta y_j|$. For this we need the following technical observation:

Claim 1 We have $\sum_{i \in I} a_i b_i < \sum_{j \in J} a_j b_j$, where I, J are finite sets and a_i, b_i, a_j, b_j are nonnegative for all $i \in I$ and $j \in J$, if there exists a correspondence $B : I \to J$ such that:

a) $a_i < a_j$ for any $j \in B(i)$;

b) $\sum_T b_i < \sum_{B(T)} b_j$ for any $T \subseteq I$.

Proof of Claim 1: See the appendix. ∎

Note that since $\Delta q_i \leq 0$ for all i, by Lemma 1 iii) $Y_{++} \neq \emptyset$. Let B be a correspondence from Y_{++} to Y_{--} such that $B(i) = C'_i \cap Y_{--}$. $B(i) \neq \emptyset$ by Lemma 1 ii).

Claim 2

i) $|\Delta q_i| < |\Delta q_j|$ for any i and j such that $j \in B(i)$;

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ii) $\sum_T |\Delta y_i| < \sum_{B(T)} |\Delta y_j|$ for any $T \subseteq Y_{++}$.

Proof of Claim 2: See the appendix. ∎

It follows immediately $\sum_{Y_{++}} |\Delta q_i||\Delta y_i| < \sum_{Y_{--}} |\Delta q_j||\Delta y_j|$, contradicting Lemma 1 i). So $y(\sigma'_c) = y(\sigma_c)$ by contraposition. Hence σ'_c is also efficient. Hence in σ'_c every player when proposing must also nominate N. So σ'_c and σ_c have the same cutoff values and coalitional profile. Therefore $\sigma'_c = \sigma_c$. ∎

As shown in Okada (1993), the payoff profile generated by an efficient SSPE is equal division, namely that each player gets $\frac{1}{n}v(N)$, and an efficient SSPE exists for all $\delta \in [0,1)$ if and only if equal division is in the core, or equivalently if and only if $\frac{v(S)}{|S|} \le \frac{v(N)}{n}$ for all $S \subset N$.[11]

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[11] It may be interesting to note the efficiency conditions obtained in Okada (1996) and Chatterjee et al. (1993), both studying multi-stage bargaining models in which after the first coalition forms and leaves bargaining continues until at most one player is left. Okada (1996) studies the multi-stage random-proposer game, for which equal division being in the core is still necessary but no longer sufficient for the existence of efficient SSPE. Okada (1996) provides a sufficient condition: there exists an SSPE for all $\delta \in [0,1)$ in which at every stage of the game all remaining players form a single coalition without delay if and only if for every subset of players S equal division of $v(S)$ cannot be blocked by any subset of S. In Chatterjee et al. (1993), players follow a deterministic order in making proposals and counterproposals, and it is shown that there exists an efficient SSPE for all $\delta \in [0,1)$ and regardless of the order of proposal making if and only if equal division is in the core.

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6　Extension

Naturally one would like to extend the uniqueness results in this paper to the multi-stage random-proposer game. Okada (1996) shows for superadditive characteristic functions that in an SSPE of the multi-stage game there is no delay in any subgame. In the literature, delay is often associated with multiplicity, in view of which Okada's (1996) result is encouraging.

Another direction for extension is to allow asymmetric proposing probabilities. Yan (2003) explores this direction and shows, interestingly, that each core allocation can be realized as the unique SSPE payoff profile of the random-proposer game with the proportionate proposing probabilities.

7　Appendix

Proof of Proposition 2:　We proceed by proving a few claims about σ.

Claim 1　Following any history that ends with a proposal (S, w) being made by player j, the outcome prescribed by σ is such that

a) the proposal passes if $w_i > \delta y_i(\sigma)$ for all $i \in S \backslash \{j\}$;

b) the proposal is rejected if $w_i < \delta y_i(\sigma)$ for some $i \in S \backslash \{j\}$.

Proof of Claim 1:

a) Given such a proposal, following a history in which all but the last player in $S\setminus\{j\}$ have responded and all responses were acceptances, the last player will accept. Suppose we have shown that following a history in which all but the last m players in $S\setminus\{j\}$ have accepted the proposal, the last m players will all accept. Then following a history in which all but the last $m+1$ players have accepted, the first of the last $m+1$ players will accept. By mathematical induction, in the subgame where no one has responded, all in $S\setminus\{j\}$ will accept.

b) If in the subgame that begins with i accepting the proposal there is a positive probability according to σ of the proposal being passed, then i's strategy must prescribe that i reject the proposal. So if i accepts the proposal, with probability 1 some subsequent player must reject it.

■

Claim 2

a) *Any proposal that passes with positive probability must offer to all the nominated players except the proposer exactly their continuation payoffs;*

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b) if $y_i(\sigma) > 0$, there is no delay in the subgame where i is selected to propose; moreover, $i \in S$ for any $S \in \mathcal{F}_{i+}$.

Proof of Claim 2:

a) It follows directly from Claim 1.

b) Let y_i^r denote i's expected payoff conditional on her being selected. It follows from a) that player i's expected payoff conditional on her not being selected is no greater than δy_i. Hence, $y_i \leq \frac{1}{n}y_i^r + (1 - \frac{1}{n})\delta y_i = \frac{1}{n}(y_i^r - \delta y_i) + \delta y_i$, or

$$y_i \leq \frac{\frac{1}{n}(y_i^r - \delta y_i)}{1 - \delta} \tag{6}$$

Since $y_i > 0$, $y_i^r > \delta y_i$. This implies that any proposal i makes with positive probability brings her strictly higher payoff than δy_i. This in turn implies that any proposal i makes with positive probability must pass, because δy_i would be what she gets if her proposal were rejected. It follows that $i \in S$ for any $S \in \mathcal{F}_{i+}$ because otherwise i gets $v(\{i\}) = 0$ implying $y_i^r = 0 \leq \delta y_i$, a contradiction.

■

Claim 3 $y_i(\sigma) > 0$ *for all* i.

Proof of Claim 3: Obviously $\sum_{i \in N} y_i(\sigma) \leq v(N)$. Hence $\sum_{i \in N} \delta y_i(\sigma) < v(N)$, so any player can get a positive payoff by nominating N and offering each player ϵ more than her continuation payoff. $\blacksquare$

It follows from Claims 2 and Claim 3 that in σ a proposer always includes herself in the nominated coalition and offers the other coalition members their continuation payoffs, and that her proposal always passes. Hence σ produces the same outcome as $\sigma_c = (p, \delta y(\sigma))$. To see that σ_c is an SSPE, note that condition i) of Proposition 1 holds obviously and it follows from Claims 1, 2 and 3 that p must be optimal with respect to the cutoff values $\delta y(\sigma)$ and hence condition ii) of Proposition 1 is also satisfied. This completes the proof. $\blacksquare$

Proof of Lemma 1:

i) For any $S \in \mathcal{F}_{i+}$, $S' \in \mathcal{F}'_{i+}$, we have $g_i(S, \delta y) = \pi_i \geq g_i(S', \delta y)$ and $g_i(S', \delta y') = \pi'_i \geq g_i(S, \delta y')$. It follows that $\Delta \pi_i \geq g_i(S, \delta y') - g_i(S, \delta y) = \sum_{j \neq i} I_j(S)(-\delta \Delta y_j)$. Since this is true for all $S \in \mathcal{F}_{i+}$, $\Delta \pi_i \geq \sum_{S \in \mathcal{F}_{i+}} p_i(S) \sum_{j \neq i} I_j(S)(-\delta \Delta y_j) = \sum_{j \neq i} q_j^i(-\delta \Delta y_j)$. Similarly,

$$\Delta \pi_i \leq \sum_{j \neq i} q_j^{i'}(-\delta \Delta y_j) \tag{7}$$

So $\sum_{j \neq i} q_j^{i'}(-\delta \Delta y_j) \geq \sum_{j \neq i} q_j^i(-\delta \Delta y_j)$, or $\sum_{j \neq i} \Delta q_j^i(-\delta \Delta y_j) \geq 0$. Since

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this is true for all i,

$$0 \leq \frac{1}{n}\sum_{i=1}^{n}\sum_{j\neq i}\Delta q_j^i(-\delta\Delta y_j) = \sum_{j=1}^{n}\sum_{i\neq j}\frac{1}{n}\Delta q_j^i(-\delta\Delta y_j)$$

$$= \sum_{j=1}^{n}(-\delta\Delta y_j)\sum_{i\neq j}\frac{1}{n}\Delta q_j^i = \sum_{j=1}^{n}(-\delta\Delta y_j)\Delta q_j \leq \sum_{j=1}^{n}(-\Delta y_j)\Delta q_j$$

therefore $\sum_{j=1}^{n}\Delta y_j\Delta q_j \leq 0$.

ii) It follows from (3) that for any $i \in T$, since $\Delta q_i \leq 0$ by assumption, we

have $0 < \Delta y_i \leq \frac{\frac{1}{n}\Delta\pi_i}{1-q_i'\delta}$. It follows that[12]

$$\Delta y_i \leq \frac{\frac{1}{n}\Delta\pi_i + \left(\sum_{j\in T\setminus\{i\}}\frac{1}{n}q_i^{j'}\right)\delta\Delta y_i}{1 - q_i'\delta + \left(\sum_{j\in T\setminus\{i\}}\frac{1}{n}q_i^{j'}\right)\delta}$$

Note that

$$1 - q_i'\delta + \left(\sum_{j\in T\setminus\{i\}}\frac{1}{n}q_i^{j'}\right)\delta = 1 - \left(\sum_{j\in N\setminus\{i\}}\frac{1}{n}q_i^{j'}\right)\delta + \left(\sum_{j\in T\setminus\{i\}}\frac{1}{n}q_i^{j'}\right)\delta$$

$$= 1 - \left(\sum_{j\notin T}\frac{1}{n}q_i^{j'}\right)\delta \geq 1 - \left(\sum_{j\notin T}\frac{1}{n}\right) = \frac{|T|}{n}. \quad \text{Together with (7) this}$$

implies that

$$\Delta y_i \leq \frac{\frac{1}{n}\sum_{j\neq i}q_j^{i'}(-\delta\Delta y_j) + \sum_{j\in T\setminus\{i\}}\frac{1}{n}q_i^{j'}\delta\Delta y_i}{\frac{|T|}{n}}$$

Summing this over all $i \in T$,

$$\sum_{i\in T}\Delta y_i \leq \frac{\sum_{i\in T}\sum_{j\neq i}\frac{1}{n}q_j^{i'}(-\delta\Delta y_j) + \sum_{i\in T}\sum_{j\in T\setminus\{i\}}\frac{1}{n}q_i^{j'}\delta\Delta y_i}{\frac{|T|}{n}}$$

$$= \frac{\sum_{i\in T}\sum_{j\neq i}\frac{1}{n}q_j^{i'}(-\delta\Delta y_j) + \sum_{i\in T}\sum_{j\in T\setminus\{i\}}\frac{1}{n}q_j^{i'}\delta\Delta y_j}{\frac{|T|}{n}}$$

$$= \frac{\sum_{i\in T}\sum_{j\notin T}q_j^{i'}(-\delta\Delta y_j)}{|T|}$$

[12] This step uses the following fact: $d \leq \frac{a}{b}$ implies $d \leq \frac{a+cd}{b+c}$ for any nonnegative a, b, c

and d.

Note that $\sum_{i\in T}\Delta y_i > 0$ implies that $\frac{\sum_{i\in T}\sum_{j\notin T}q_j^{i'}(-\delta\Delta y_j)}{|T|}$ is strictly positive. This term can be positive only if $\hat{T}$ is non-empty, because $\hat{T}$ consists precisely of those j for which $q_j^{i'}(-\delta\Delta y_j)$ is positive. Having thus established that $\hat{T}$ is non-empty,

$$
\begin{aligned}
\sum_{i\in T}\Delta y_i \;&\leq\; \frac{\sum_{i\in T}\sum_{j\in\hat{T}}q_j^{i'}(-\delta\Delta y_j)}{|T|} \\[2mm]
&\leq\; \frac{\sum_{i\in T}\sum_{j\in\hat{T}}(-\delta\Delta y_j)}{|T|} \\[2mm]
&=\; \frac{|T|\sum_{j\in\hat{T}}(-\delta\Delta y_j)}{|T|} \\[2mm]
&<\; \sum_{j\in\hat{T}}(-\Delta y_j)
\end{aligned}
$$

iii) First we claim that $\Delta q_i \Delta y_i > 0$ for some i. Suppose to the contrary $\Delta q_i \Delta y_i \leq 0$ for all i. Assume w.l.g. that $Y_{++}\neq\emptyset$. We have

$$
\sum_{i\in Y_{++}}\Delta y_i = \sum_{i\in Y_{++}\cap Q_-}\Delta y_i < \sum_{j\in\left(\cup_{i\in Y_{++}\cap Q_-}C_i'\right)\cap Y_{--}}(-\Delta y_j) \leq \sum_{j\in Y_{--}}(-\Delta y_j)
$$

where the strict inequality is by ii). From this we draw two conclusions: $Y_{--}\neq\emptyset$, and $\sum_{i\in N}\Delta y_i < 0$ or equivalently $\sum_{i\in N}y_i' < \sum_{i\in N}y_i$. Since $Y_{--}\neq\emptyset$, by a symmetric argument (through redefining $\Delta y_i = y_i - y_i'$, $\Delta q_i = q_i - q_i'$ and so on) we can show $\sum_{i\in N}y_i < \sum_{i\in N}y_i'$, a contradiction. Therefore, the claim is true. Then iii) follows from i).

∎

Proof of Claim 1 in the proof of Proposition 4: We focus on pure SSPE in this proof and need some notation to utilize this fact. Given a pure SSPE, let S_i denote the coalition player i nominates with probability 1 when proposing and let $\hat{q}_i$ be the number of other players nominating i. Note that $\hat{q}_i = \sum_{j \in N \setminus \{i\}} q_i^j = n q_i$.

Using $\hat{q}_i$ instead of q_i we rewrite two previous results. (4) is rewritten as

$$y_i = \frac{\bar{\pi}_i}{n - \hat{q}_i \delta - \delta} \tag{8}$$

Note for later use that for any $i \in N$, $n - \hat{q}_i \delta - \delta > 0$ since $\hat{q}_i \leq n - 1$ by definition, and hence $\bar{\pi}_i > 0$ since $y_i > 0$ by Claim 3 in the proof of Proposition 2.

Next, equation (3) is rewritten and used in three equivalent forms,

$$\Delta y_i = \frac{\Delta \pi_i + \Delta \hat{q}_i \delta y_i}{n - \hat{q}_i' \delta} = \frac{\Delta \bar{\pi}_i + \Delta \hat{q}_i \delta y_i}{n - \hat{q}_i' \delta - \delta} = \frac{\Delta \bar{\pi}_i + \Delta \hat{q}_i \delta y_i'}{n - \hat{q}_i \delta - \delta} \tag{9}$$

We prove Claim 1 in three steps.

Step 1 Assume w.l.g. that $1 \in Y_{++} \cap Q_{--}$. Then $\Delta \hat{q}_1 \leq -1$.

Obvious, since $\hat{q}_i$ and $\hat{q}_i'$ are integers for all i.

Step 2 $\Delta y_i \leq 0$ implies $\Delta \hat{q}_i \leq 0$ for any $i \in N$, and $\Delta y_i \leq 0$ implies $\Delta \hat{q}_i < 0$ if $i \in S_j'$ for some $j \in Y_{++} \cap Q_{--}$.

To show the first part of the statement, consider any $j \in Y_{++} \cap Q_{--}$. We

have $0 < \Delta y_j = \frac{\Delta \bar{\pi}_j + \Delta \hat{q}_j \delta y_j}{n - \hat{q}'_j \delta - \delta} \leq \frac{\Delta \bar{\pi}_j - \delta y_j}{n - \hat{q}'_j \delta - \delta}$. It follows that

$$\Delta \bar{\pi}_j - \delta y_j > 0 \tag{10}$$

By Lemma 2 iii), for all $i \neq j$, $\Delta \bar{\pi}_j = \bar{\pi}'_j - \bar{\pi}_j \leq (\bar{\pi}'_i + \delta y'_i) - (\bar{\pi}_i - \delta y_j) = $

$\Delta \bar{\pi}_i + \delta y'_i + \delta y_j$. Hence,

$$\Delta \bar{\pi}_j - \delta y_j \leq \Delta \bar{\pi}_i + \delta y'_i \tag{11}$$

So $\Delta \bar{\pi}_i + \delta y'_i > 0$. Hence, recalling (9), $\Delta \hat{q}_i \geq 1$ implies $\Delta y_i > 0$, or equiv-

alently, $\Delta y_i \leq 0$ implies $\Delta \hat{q}_i \leq 0$. To see $\Delta q_i < 0$ for $i \in S'_j$, note that

$\bar{\pi}'_j \leq \bar{\pi}'_i$ by Lemma 2 ii) and $\bar{\pi}_j = \pi_j - \delta y_j \geq \bar{\pi}_i - \delta y_j$ by Lemma 2 iii). Then

$\Delta \bar{\pi}_j \leq \bar{\pi}'_i - (\bar{\pi}_i - \delta y_j) = \Delta \bar{\pi}_i + \delta y_j$. So $\Delta \bar{\pi}_i \geq \Delta \bar{\pi}_j - \delta y_j > 0$. Hence $\Delta \hat{q}_i \geq 0$

implies $\Delta y_i > 0$; or equivalently, $\Delta y_i \leq 0$ implies $\Delta \hat{q}_i < 0$.

Step 3 $\sum_{i \in N} \Delta y_i < 0$.

We will first show

$$\sum_{i \in N} \Delta \hat{q}_i y_i < 0$$

We proceed as follows. Since $1 \in Y_{++} \cap Q_{--}$ by assumption, $S'_1 \cap Y_{--} \neq \emptyset$

by Lemma 1 ii). For any $j \in S'_1 \cap Y_{--}$, $j \in Q_{--}$ by Step 2. Hence,

$$Y_{--} \cap Q_{--} \neq \emptyset \tag{12}$$

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Since in addition $Y_{--} \cap Q_{++}$ is empty by Step 2, it follows that $\sum_{i \in Y_{--}} \Delta\hat{q}_i \Delta y_i > 0$. Recalling Lemma 1 i), which implies $\sum_{i \in N} \Delta\hat{q}_i \Delta y_i \leq 0$, we have

$$\sum_{i \in Y_{++}} \Delta\hat{q}_i \Delta y_i = \sum_{i \in N} \Delta\hat{q}_i \Delta y_i - \sum_{i \in Y_{--}} \Delta\hat{q}_i \Delta y_i < 0 \tag{13}$$

From this we will show $\sum_{i \in Y_{++}} \Delta\hat{q}_i y_i < 0$. Note from (9) that $\Delta\bar{\pi}_i + \Delta\hat{q}_i \delta y_i = (n - \hat{q}_i' \delta - \delta)\Delta y_i$ for any $i \in N$. Pick

$$k \in \mathrm{argmax}_{i \in Y_{++} \cap Q_{--}} \Delta\bar{\pi}_i + \Delta\hat{q}_i \delta y_i$$

Then for any $i \in Y_{++} \cap Q_{--}$,

$$\Delta y_i = \frac{\Delta\bar{\pi}_i + \Delta\hat{q}_i \delta y_i}{n - \hat{q}_i' \delta - \delta} \leq \frac{\Delta\bar{\pi}_k + \Delta\hat{q}_k \delta y_k}{n - \hat{q}_i' \delta - \delta} = \frac{n - \hat{q}_k' \delta - \delta}{n - \hat{q}_i' \delta - \delta}\Delta y_k \leq \frac{n - \hat{q}_k' \delta - \delta}{n - \hat{q}_i \delta}\Delta y_k \tag{14}$$

where the second inequality follows from the obvious fact that $\hat{q}_i' \leq \hat{q}_i - 1$ since $i \in Q_{--}$.

If $Y_{++} \cap Q_{++} = \emptyset$, obviously $\sum_{i \in Y_{++}} \Delta\hat{q}_i y_i < 0$. So suppose $Y_{++} \cap Q_{++} \neq \emptyset$. For any $i \in Y_{++} \cap Q_{++}$, $\Delta\hat{q}_i \geq 1$, and hence by (11) and (9)

$$\Delta\bar{\pi}_i + \Delta\hat{q}_i \delta y_i' \geq \Delta\bar{\pi}_i + \delta y_i' \geq \Delta\bar{\pi}_k - \delta y_k \geq \Delta\bar{\pi}_k + \Delta\hat{q}_k \delta y_k = (n - \hat{q}_k' \delta - \delta)\Delta y_k.$$

Therefore,

$$\Delta y_i = \frac{\Delta\bar{\pi}_i + \Delta\hat{q}_i \delta y_i'}{n - \hat{q}_i \delta - \delta} \geq \frac{n - \hat{q}_k' \delta - \delta}{n - \hat{q}_i \delta - \delta}\Delta y_k \tag{15}$$

Then it follows from (13), (14), and (15) that

$$0 > \sum_{i \in Y_{++}} \Delta\hat{q}_i \Delta y_i = \sum_{i \in Y_{++} \cap Q_{--}} \Delta\hat{q}_i \Delta y_i + \sum_{i \in Y_{++} \cap Q_{++}} \Delta\hat{q}_i \Delta y_i$$

$$\geq \sum_{i \in Y_{++} \cap Q_{--}} \frac{\Delta \hat{q}_i}{n - \hat{q}_i \delta} (n - \hat{q}'_k \delta - \delta) \Delta y_k +$$

$$\sum_{i \in Y_{++} \cap Q_{++}} \frac{\Delta \hat{q}_i}{n - \hat{q}_i \delta - \delta} (n - \hat{q}'_k \delta - \delta) \Delta y_k$$

$$= \left\{ \sum_{i \in Y_{++} \cap Q_{--}} \frac{\Delta \hat{q}_i}{n - \hat{q}_i \delta} + \sum_{i \in Y_{++} \cap Q_{++}} \frac{\Delta \hat{q}_i}{n - \hat{q}_i \delta - \delta} \right\} (n - \hat{q}'_k \delta - \delta) \Delta y_k$$

Since both $(n - \hat{q}'_k \delta - \delta)$ and Δy_k are positive,

$$\sum_{i \in Y_{++} \cap Q_{--}} \frac{\Delta \hat{q}_i}{n - \hat{q}_i \delta} + \sum_{i \in Y_{++} \cap Q_{++}} \frac{\Delta \hat{q}_i}{n - \hat{q}_i \delta - \delta} < 0 \tag{16}$$

Consider $m \in \text{argmax}_{i \in N}\{\bar{\pi}_i\}$. For any $j \in N$, $y_j = \frac{\bar{\pi}_j}{n - \hat{q}_j \delta - \delta} \leq \frac{\bar{\pi}_m}{n - \hat{q}_j \delta - \delta}$. Note also $\pi_j \geq \bar{\pi}_m$ by Lemma 2 iii), hence $y_j = \frac{\pi_j}{n - \hat{q}_j \delta} \geq \frac{\bar{\pi}_m}{n - \hat{q}_j \delta}$. Then multiplying both sides of (16) by $\bar{\pi}_m$, which must be positive as argued at the beginning of the proof of Claim 1, we have

$$0 > \sum_{j \in Y_{++} \cap Q_{--}} \frac{\Delta \hat{q}_j}{n - \hat{q}_j \delta} \bar{\pi}_m + \sum_{j \in Y_{++} \cap Q_{++}} \frac{\Delta \hat{q}_j}{n - \hat{q}_j \delta - \delta} \bar{\pi}_m$$

$$\geq \sum_{j \in Y_{++} \cap Q_{--}} \Delta \hat{q}_j y_j + \sum_{j \in Y_{++} \cap Q_{++}} \Delta \hat{q}_j y_j = \sum_{j \in Y_{++}} \Delta \hat{q}_j y_j$$

Since $Y_- = Y_- \cap Q_-$ by Step 2, $\sum_{i \in N} \Delta \hat{q}_i y_i = \sum_{i \in Y_{++}} \Delta \hat{q}_i y_i + \sum_{i \in Y_-} \Delta \hat{q}_i y_i < 0$, which ends the first part of Step 3.

To complete Step 3 we will show that $n \sum_{i \in N} \Delta y_i \leq \delta \sum_{i \in N} \Delta \hat{q}_i y_i$.

Note that for any $i \in N$,

$$g_i(S_i, \delta y) = \pi_i \geq g_i(S'_i, \delta y)$$

It follows that

$$\sum_{j \in S'_i \setminus S_i} \delta y_j \geq v(S'_i) - v(S_i) \tag{17}$$

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Summing (17) over all i,

$$\sum_{i \in N} \sum_{j \in S_i' \setminus S_i} \delta y_j \geq \sum_{i \in N} v(S_i') - \sum_{i \in N} v(S_i) \tag{18}$$

The left hand side of (18) equals $\delta \sum_{i \in N} \Delta \hat{q}_i y_i$, which is negative since we showed $\sum_{i \in N} \Delta \hat{q}_i y_i < 0$, and the right hand side equals $n \sum_{i \in N} y_i' - n \sum_{i \in N} y_i = n \sum_{i \in N} \Delta y_i$. Thus we conclude $\sum_{i \in N} \Delta y_i < 0$. $\blacksquare$

Proof of Claim 1 in the proof of Proposition 7: Assume w.l.g. that $I = \{1, 2, ..., |I|\}$ and for any $m, n \in I$, $a_m \geq a_n$ if $m > n$. For notational convenience let $a_0 = 0$. If $a_0 = a_{|I|}$, the claim holds obviously. Suppose $a_0 < a_{|I|}$. It follows from condition ii) that

$$\sum_{i=1}^{|I|}(a_i - a_{i-1})\sum_{h=i}^{|I|} b_h < \sum_{i=1}^{|I|}(a_i - a_{i-1}) \sum_{j \in B(\{h | h \geq i\})} b_j \tag{19}$$

The left hand side of (19) may be rewritten as

$$\sum_{i=1}^{|I|}\sum_{h=i}^{|I|}(a_i - a_{i-1})b_h = \sum_{h=1}^{|I|}\sum_{i=1}^{h}(a_i - a_{i-1})b_h = \sum_{h=1}^{|I|} b_h \sum_{i=1}^{h}(a_i - a_{i-1}) = \sum_{h=1}^{|I|} b_h a_h$$

For notational convenience, let $a_{\max(\emptyset)} = 0$. The right hand side of (19) may be rewritten as

$$\sum_{i=1}^{|I|} \sum_{j \in B(\{h | h \geq i\})} (a_i - a_{i-1})b_j = \sum_{j \in J} \sum_{i=1}^{\max(B^{-1}(j))} (a_i - a_{i-1})b_j$$
$$= \sum_{j \in J} b_j \sum_{i=1}^{\max(B^{-1}(j))} (a_i - a_{i-1})$$
$$= \sum_{j \in J} b_j a_{\max(B^{-1}(j))}$$
$$< \sum_{j \in J} b_j a_j$$

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where the last inequality is due to condition i). ∎

Proof of Claim 2 in the proof of Proposition 7:

i) Since B is defined on Y_{++}, $\Delta y_i > 0$, which implies $\frac{1}{n}\Delta\bar{\pi}_i + \Delta q_i \delta y_i > 0$ by (5) in the proof of Proposition 6. It follows that, since $y_i = \frac{1}{n}v(N)$, $|\Delta q_i| < \frac{\Delta\bar{\pi}_i}{\delta v(N)}$. Similarly, one can show $|\Delta q_j| > \frac{\Delta\bar{\pi}_j}{\delta v(N)}$ since by assumption $\Delta y_j < 0$. Note that $\bar{\pi}_i = \bar{g}(N) = \bar{\pi}_j$ and that $j \in C_i'$ implies $\bar{\pi}_j' \geq \bar{\pi}_i'$ by Lemma 2 ii). It follows that $\Delta\bar{\pi}_j \geq \Delta\bar{\pi}_i$. Therefore $|\Delta q_i| < |\Delta q_j|$.

ii) It follows directly from Lemma 1 ii).

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