Equivalence between the Existence Theorems on Nash Equilibrium, Core and Hybrid Equilibrium*

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Summary: This paper provides the equivalence between the existence theorems on Nash equilibrium, core, and hybrid equilibrium. It completes the equivalence by showing that Scarf's core theorem implies Kakutani's fixed point theorem. Such equivalence leaves with us a long list of open problems for future research, one of which is to derive Kakutani's fixed point theorem (1941) directly from Nash's theorem (1951).

Keywords: Fixed point theorem, core, Nash equilibrium, hybrid equilibrium.

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1. Introduction

Binding contracts are the touchstone in identifying possible solutions for a game. In a normal form game, the availability of binding contracts determines three fundamental types of solutions: *i)* if no binding contract is available, players will behave strategically and choose *Nash equilibrium* [5] as the solution; *ii)* if binding contacts are available to the grand coalition,[1] players will behave cooperatively and choose *core* (i.e., the a- or b-*core*, [2], [8], and [11]) or its refinements as the solution; and *iii)* if binding contracts are available only to coalitions in a given partition (or coalitional structure), players will behave cooperatively within each coalition and in the same time strategically across the coalitions in the partition, and they will choose the *hybrid equilibrium* for the partition [10] as the solution. Because "no binding contract" and "binding contracts for grand coalition" are two limiting cases of the general situation in which binding contracts are available to coalitions in a coalitional structure, the core and the Nash equilibrium are two limiting cases of the hybrid equilibrium.[2]

This paper shows that the existence theorems on Nash equilibrium (Nash, 1951), core (Scarf, 1967), and hybrid equilibrium (Zhao, 1992) are equivalent to each other. It establishes the equivalence by showing that Scarf's core theorem implies Kakutani's fixed point theorem. Because the concepts of core, Nash equilibrium, and hybrid equilibrium describe three fundamental types of social institutions (i.e., with no binding contract, binding contracts for the grand coalition, and binding contracts for coalitions in a given partition), the equivalence theorem suggests that the outcomes in each institution could be theoretically

[1] When binding contracts are available to a given coalition, it is implicitly assumed that binding contracts are also available to all of its subcoalitions.

[2] See Allen [1] for more discussions of hybrid equilibrium.

implemented as an outcome in each of the other two institutions.

Section 2 below states the equivalence problems, Section 3 states and proves the equivalence theorem, and Section 4 provides a list of open problems for future research.

2. The Equivalence Problems

Let *N = {1, ..., n}* be a finite set, a subset *S* ¹ Æ Ì *N* will be called a coalition of players. Given *S* Ì *N*, let $\boldsymbol{R}^S$ denote the Euclidean space whose dimension is the number of players in *S* and whose coordinates are the players in *S*. Let e Î $\boldsymbol{R}^n$ be the vector of ones, *0* be the vector of zeros, *Arg-Max { f(z) | z* Î *Z }* be the optimal set for

Max { f(z) |z Î *Z }.*

For any *x, y* Î $\boldsymbol{R}^n$, *x* ³ *y* ⇔ x_i ³ y_i, all *i*; *x > y* ⇔ *x* ³ *y*, *x* ¹ *y*; *x >> y* ⇔ $x_i > y_i$, all *i*; x_S be the projection of *x* on $\boldsymbol{R}^S$, and (x_S, y_{-S}) be the column vector *z* Î $\boldsymbol{R}^n$ such that $z_i = x_i$ if *i* Î *S*, $z_i = y_i$ if *i* Ï *S*.

For simplicity of exposition, we will state the next four theorems in their simplest forms. First, Kakutani's fixed point theorem is given below:

(i) (Kakutani, [4]) Let X ¹ Æ Ì $\boldsymbol{R}^n$ *be compact and convex, f: X*®2^X *be a correspondence. Assume: (a) f is upper semicontinuous; (b) for each x* Î *X, f(x)* ¹ Æ Ì *X is closed and convex. Then there exists at least one x* Î *X such that x* Î *f(x).*

Second, consider the Nash equilibrium for a normal form game:

$G = \{N, X_i, u_i\}$,

where for each player $i \in N$, $X_i \neq \emptyset \subset \mathbf{R}^{m(i)}$ is i's choice set, and u_i: $X = \Pi X_k \to \mathbf{R}$ is i's payoff function. A choice profile $x \in X$ is a ***Nash equilibrium*** if for each $i \in N$,

$$x_i \in Arg\text{-}Max\ \{u_i(y_i, x_{-i}) \mid y_i \in X_i\};$$

or in other words if each x_i is player i's best response to the complementary choice $x_{-i} = \{x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$.

(ii) (Nash, [5]) For each $i \in N$, assume: (a) X_i is compact and convex; (b) $u_i(x)$ is continuous in x and quasi-concave in x_i. Then G has at least one Nash equilibrium.

Next, consider the core for a coalitional NTU (non-transferable utility) game given below:

$$G_{CF} = \{N, V(\cdot)\},$$

where for each S, $V(S) \neq \emptyset \subset \mathbf{R}^S$ is its set of payoffs for S that satisfies:[3]

(a) it is closed and bounded from above;

(b) $y \in V(S)$ and $u \leq y$ imply $u \in V(S)$; and

(c) $\exists\, y \in V(S)$ such that $y_i \geq Max\ \{x_i \mid x_i \in V(i)\}$, all $i \in S$.

Given a set of coalitions $\mathcal{B} = \{T_1, \ldots, T_k\}$, let $\mathcal{B}(i) = \{T \in \mathcal{B} \mid i \in T\}$ denote the subset of coalitions that include i as a member. $\mathcal{B}$ is balanced if $\exists\, w_T > 0$ for each $T \in \mathcal{B}$ such that $\Sigma_{T \in \mathcal{B}(i)} w_T = 1$ holds for all i. A game is balanced if for any balanced $\mathcal{B}$, $u \in V(N)$ must hold if $u_S \in V(S)$ for all $S \in \mathcal{B}$. Let

3 We simplify $V(\{i\})$ as $V(i)$.

$$\partial V(S) = \{y \in V(S) \mid \exists \text{ no } x \in V(S) \text{ with } x>>y\} \neq \emptyset$$

be the weakly efficient set of $V(S)$. A vector u is blocked by S if there is $y \in v(S)$ with $y >> u_S$ (i.e., $u_S \in v(S)/\partial v(S)$), or in other words if S can obtain a higher payoff for each of its members than that given by u.

A payoff vector $u \in \partial v(N)$ is in the core if it is unblocked by all $S \neq N$ (i.e., if it is *S-efficient* or if $u_S \notin v(S)/\partial v(S)$ for all $S \neq N$). Let

$$C(G_{CF}) = \{u \in \partial V(N) \mid u_S \notin V(S) \backslash \partial V(S), \text{ all } S \neq N\}$$

be the core.

(iii) (Scarf, [7]) If G_{CF} is balanced, then $C(G_{CF}) \neq \emptyset$.

Finally, consider the hybrid equilibrium for $G = \{N, X_i, u_i\}$. Let the set of weakly efficient solutions be denoted as

$$X^*_{we} = \{x \in X \mid \exists \text{ no } y \in X \text{ with } u(y)>>u(x)\},$$

where $u(x) = \{u_i(x) \mid i \in N\}$, and let the coalitional game derived from G in the α-fashion,

$$G_\alpha = \{N, V_\alpha(\cdot)\},$$

be given by:

$$V_\alpha(S) = \bigcup_{x_S} \{ \bigcap_{y_{-S}} \{w_S \in \mathbf{R}^S \mid w_S \leq u_S(x_S, y_{-S})\}\} \text{ for } S \neq N, \text{ and}$$

$$V(N) = V_\alpha(N) = \{ w \in \mathbf{R}^n \mid w \leq u(x) \text{ for some } x \in X^*_{we}\}.$$

Then, $x \in X^*_{we}$ is an α-core solution if $u(x) \in C(G_\alpha)$. A partition $D = \{S_1, ..., S_k\}$ (i.e., $\cup S_i = N$, $S_i \cap S_j = \emptyset$, all $i \neq j$) defines k parametric games in normal form:

$$G_S(y_{-S}) = \{S, X_i, u_i(x_S, y_{-S})\},\ S \in D.$$

A choice $x = \{x_S | S \in D\} \in X$ is a hybrid equilibrium for D if each x_S is an α-core solution of $G_S(x_{-S})$.

(iv) (Zhao, [10]) Given $D = \{S_1, \ldots, S_k\}$ *in G. For each* $S \in D$*, assume: (a)* X_i*,* $i \in S$*, all are compact and convex; (b)* $u_i(x) = u_i(x_S, x_{-S})$*,* $i \in S$*, all are continuous in x and quasi-concave in* x_S*. Then there exists at least one hybrid equilibrium for D.*

As shown in Figure 1, Nash equilibrium assumes an institution with no binding contract, core assumes another institution with binding contracts for N, while hybrid equilibrium assumes a general institution with binding contracts for each $S \in D$. Hence, hybrid equilibria include core (when $D = D_m = \{N\}$)[4] and Nash equilibrium (when $D = D_0 = \{\{1\}, \ldots, \{n\}\}$) as two limiting cases.

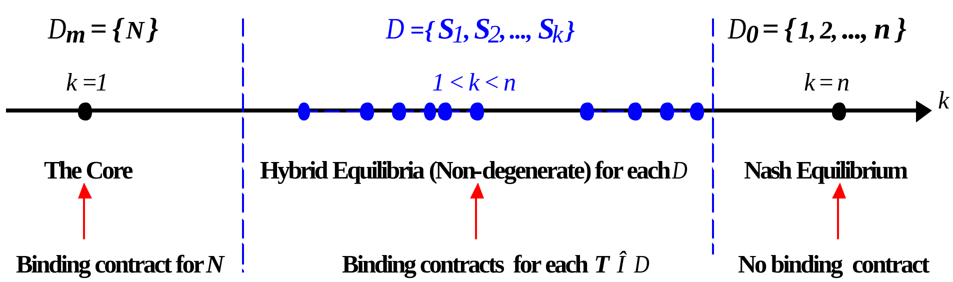

Figure 1. The spectrum of hybrid equilibria, where $k\ (1 \le k \le n)$ is the number of coalitions in D.

4 Note that *(iv)* (Theorem 2 in [10]) becomes the α-core existence in [8] when $D = D_m = \{N\}$. The general version of *(iv)* (Theorem 3 in [10]) implies *(iii)* directly.

3. The Equivalence Theorem

Theorem: *The statements (i) -(iv) are equivalent.*

Proof: Our proof uses three other equivalent claims *(v)*-*(vii)* as given below. Statement *(v)* is Brouwer's fixed point theorem, *(vi)* is equivalent to Theorem 2 in [7], and *(vii)* is a new claim that reveals the geometry of *(iii)* and *(vi)*.

(v) (Brouwer, [3]) Let $X \neq \emptyset \subset \mathbf{R}^n$ *be compact and convex, and* $f: X \to X$ *be a continuous function. Then there exists at least one* $x \in X$ *such that* $f(x) = x$.

Given a finite game $G_{CF} = \{N, V\}$, where each

$$V(S) = \{y \in \mathbf{R}^S | y \leq u_{jS}, \text{ all } j\}$$

is defined by $k(S)$ corners u_{jS}, $j = 1, \ldots, k(S)$. Let $K(S) = \{1, \ldots, k(S)\}$,

$$Q > Max\{u_{ijS} \mid i \in S, j \in K(S), S \neq N\}$$

be a large number;

$$m = \sum_{S \neq N} k(S),\ C = C_{n \times m} = \{c_k\}$$

be given by:

$$c_k = c_{jS} = (u_{jS}, Qe_{-S})$$

be the column vector for each S and j $\in K(S)$; and

$$A = A_{n \times m} = \{a_k\} = \{e_{jS}\}$$

be the incidence matrix for all S in C (i.e., $e_{jS} \equiv (e_S, 0_{-S})$, all j $\in K(S)$).[5] The following *(vi)*, is equivalent to Theorem 2 in [7].

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5 $c_k = c_{jS}$ and a_k, for unique j $\in K(S)$, are given by: $c_{ik} = u_{ijS}$, $a_{ik} \equiv 1$ if $i \in S$; $c_{ik} \equiv Q$, $a_{ik} \equiv 0$ if $i \notin S$.

(vi) (Scarf, [7]) For the above A and C, $\exists\, x = (x_B, 0_{-B}),\ x_B >> 0$ *such that* $Ax = e$, *and that if we define* $u_i = Min\{c_{ij} \mid j \in B\}$, *then for each* $k \in M = \{1,\ldots,m\}$, $\exists\, i$ *with* $u_i \geq c_{ik}$.

The following *(vii)* reveals the geometry of the core theorem *(iii)* and the above claim *(vi)*. For each coalition $S \neq N$, let $\tilde{v}(S) = V(S) \times \boldsymbol{R}^{-S} \subset \boldsymbol{R}^n$ denote the n-dimensional cylinder associated with $V(S)$. For each balanced collection $\mathcal{B}$, let

$$(1) \qquad GP(\mathcal{B}) = \bigcap_{S \in \mathcal{B}} \tilde{v}(S) \subset \boldsymbol{R}^n$$

denote the n-dimensional payoffs generated by $\mathcal{B}$, and

$$(2) \qquad GP = GP(G_{CF}) = \bigcup_{Balanced\ \mathcal{B};\ N \notin \mathcal{B}} GP(\mathcal{B})$$

be the set of all generated payoffs.

It is straightforward to see that G_{CF} is balanced $\Leftrightarrow GP \subset V(N)$.

Observe that u is unblocked by $S \Leftrightarrow u \in [V(S) \backslash \partial V(S)]^C \times \boldsymbol{R}^{-S} \subset \boldsymbol{R}^n$, where the superscript C denotes the complement set. Hence, the set of payoffs unblocked by all $S \neq N$ is given by

$$(3) \qquad UBP = UBP(G_{CF}) = \bigcap_{S \neq N} \{[V(S) \backslash \partial V(S)]^C \times \boldsymbol{R}^{-S}\},$$

and the core can alternatively be given by $C(G_{CF}) = \partial V(N) \cap UBP$.

Now, one sees that statement *(vi)* is equivalent to the following three claims:

a) the base B *for* x_B *forms a balanced set;*[6]

b) $u \in GP$*; and*

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6 Let $\mathcal{B} = \{S \mid x_{jS} > 0$ for some $j \in K(S)\}$, $d_S = \sum_{k \in S} x_k$ for each $S \in \mathcal{B}$. One sees that $\mathcal{B}$ is balanced.

c) $u \in UBP$ *(by* $u_i \geq c_{ik}$*, all k).*

Hence, *(vi)* ⇔ $GP \cap UBP \neq \emptyset$ for finite games. Since any G_{CF} can be approximated by a sequence of finite games (§6 in [7]), one obtains *(vii)* below:

(vii) $GP(G_{CF}) \cap UBP(G_{CF}) \neq \emptyset$ *holds for all* G_{CF}.

Now, come back to the proof for our theorem. By *(i)* ⇒ *(iv)* ⇒ *(ii)* [10], *(ii)* ⇒ *(iii)* [6], *(vii)* ⇔ *(vi)*, *(vi)* ⇒ *(v)* [7], and *(v)* ⇔ *(i)*[4], our proof completes by showing *(iii)* ⇒ *(vii)*.

Since $V(S)\backslash\partial V(S)$ is open in $\mathbf{R}^S$, each $\{[V(S)\backslash\partial V(S)]^C \times \mathbf{R}^{-S}\}$ is closed in $\mathbf{R}^n$. Hence,

(4) $$UBP = \bigcap_{S \neq N} \{[V(S)\backslash\partial V(S)]^C \times \mathbf{R}^{-S}\} = \bigcap_{\mathcal{B} \text{ with } N \notin \mathcal{B}} \{\bigcap_{S \in \mathcal{B}} \{[V(S)\backslash\partial V(S)]^C \times \mathbf{R}^{-S}\}\}$$

is a closed subset in $\mathbf{R}^n$. Let ∂GP be the upper surface of GP, $\underline{GP^C}$ be the enclosure of GP^C, one has $\underline{GP^C} = \partial GP \cup GP^C$. By (1) and (2), one has

(5) $$GP^C = \{\bigcup_{\mathcal{B} \text{ with } N \notin \mathcal{B}} \{\bigcap_{S \in \mathcal{B}} [V(S) \times \mathbf{R}^{-S}]\}\}^C = \bigcap_{\mathcal{B} \text{ with } N \notin \mathcal{B}} \{\bigcup_{S \in \mathcal{B}} [V(S)^C \times \emptyset^{-S}]\}.$$

Let $\underline{V(S)^C}$ be the enclosure of $V(S)^C$. By $[V(S)\backslash\partial V(S)]^C = \underline{V(S)^C}$, one has

$$\bigcap_{S \in \mathcal{B}} \{[V(S)\backslash\partial V(S)]^C \times \mathbf{R}^{-S}\} \subset \bigcup_{S \in \mathcal{B}} [\underline{V(S)^C} \times \emptyset^{-S}].$$

By (4), (5), and the above expression, one has

(6) $$UBP = UBP(G_{CF}) \subset \underline{GP^C}.$$

Given *(iii)*, assume by way of contradiction that $GP \cap UBP = \emptyset$. By (6), UBP is included in the interior of $\underline{GP^C}$. Since ∂GP is the lower surface of $\underline{GP^C}$, one must have

$$(7) \qquad d^* = d(\partial GP, UBP) = Min\{\, ||x-y|| \mid x \in \partial GP,\ y \in UBP\} > 0.$$

Define a new game $G' = \{N, V(\cdot)'\}$ from G_{CF} by: $V(S)' = V(S)$, all $S \neq N$, and

$$(8) \qquad V(N)' = GP(G_{CF}) \cup \{ \bigcup_{x \in \partial GP(G_{CF})} \{y \mid y \geq x \text{ and } d(x,y) < d^*/2\}\},$$

one has: $GP(G') = GP(G_{CF}) \subseteq V(N)'$, so G' is balanced. However, by (7)-(8), and by $UBP(G') = UBP(G_{CF})$ and $d(\partial V(N)', UBP(G')) \geq d^*/2 > 0$, one has

$$C(G') = \partial V(N)' \cap UBP(G') = \emptyset,$$

which contradicts to *(iii)*. Therefore, *(vii)* must hold. ***Q.E.D.***

4. Conclusion and Open Problems

We have established the equivalence between Kakutani's fixed point theorem and the existence theorems for Nash equilibrium, core, and hybrid equilibrium. We completed the equivalence by showing that Scarf's core theorem implies Brouwer's fixed point theorem.

The equivalence leaves with us three classes of problems for future research. The first class of open problems are to prove directly those relations in the equivalence that had been indirectly established, these are marked as the dotted arrows in Figure 2. Due to Nash's popularity, it will be interesting to directly derive Brouwer's or Kakutani's fixed point theorem using Nash's theorem. Although an indirect proof follows from [6] and this paper, it remains to be seen when direct proofs for *(ii)* $\Rightarrow$ *(i)* can be discovered.

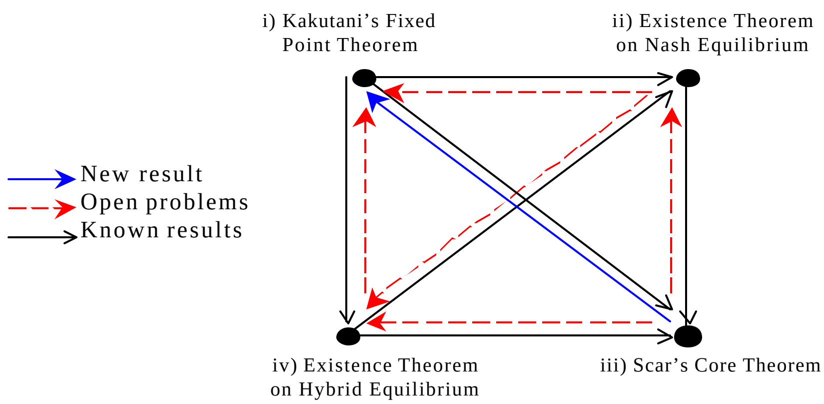

Figure 2. The equivalence between Kakutani' fixed point theorem and the existence theorems of Nash equilibrium, core, and hybrid equilibrium.

Because the concepts of Nash equilibrium, core, and hybrid equilibrium are the outcomes in three different social institutions, the equivalence theorem implies that the outcome in each institution can be theoretically implemented as an outcome in each of the two other institutions. This suggests a new approach to implementation and hence another class of open problems: It will be useful to apply the equivalence theorem to the growing literature on implementation and mechanism design.

Finally, it will be useful to find the equivalent versions of Nash equilibrium, core and hybrid equilibrium associated with each of the more advanced fixed point theorems. Given the success of the finite-dimensional fixed point theorems in game theory, this line of research will provide a rich source of future applications in economics and game theory for the fixed point literature.

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