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THE MIXED FINITE ELEMENT METHOD FOR STOKES EQUATIONS ON UNBOUNDED DOMAINS

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§ 1. INTRODUCTION

The finite element method has recently become of interest in fluid mechanics in both theory and practice[1,2]. The theoretical analysis of the finite element method for Navier-Stokes equations has been made with respect to bounded domains by many authors[3,4]. In this paper, the mixed finite element method for Stokes equations on unbounded domains is considered. Fix and Strang[5] have analyzed the finite element method with respect to an unbounded domain, but their procedure requires the solution of an infinite system of linear algebraic equations. Babuška[6] has proposed an approach for finding a finite element approximate solution of a boundary value problem for an elliptic partial differential equation on an unbounded domain by solving only a finite system of linear algebraic equations, which has been shown on a model equation $-\Delta u+u=f$ on $\mathbf{R}^n$. In fact, this approach corresponds to solving a boundary value problem on a bounded domain Ω_h for each mesh $h>0$, and the bounded domain Ω_h approaches to $\mathbf{R}^n$, when $h\to 0$. It means that a boundary value problem on a quite large domain must be solved by the finite element method. Combining the finite element method and the classical analytical method, we have proposed a local finite element method for finding the numerical solution of a boundary value problem of elliptic partial differential equation on unbounded domains, shown through some exterior boundary value problems of model equation $\Delta u=0$. Its rate of convergence is the same as that of the boundary value problem on bounded domains[7]. This method is closely related with the method of coupling the FEM with canonical reduction proposed by Feng Kang[8,9]. Their difference is in the form of the canonical integral equations. Hence this paper can be understood as an extension of the local finite element or the coupling of the FEM and canonical reduction to Stokes equations. Aside from the above there are other ways of handling problems on unbounded domains, such as the boundary finite element method[10,11], the infinite element method[12], and the infinite elements[13].

§ 2. THE SOLUTION OF STOKES PROBLEM ON AN EXTERIOR CIRCLE

Suppose $\Omega_\infty=\{x^2+y^2\geqslant r_0^2\}$, and let Γ_0 denote the boundary of Ω_∞. We consider the boundary value problem of Stokes equations on an unbounded domain Ω_∞:

$$\Delta u_1-\frac{\partial p}{\partial x_1}=0,\ \Omega_\infty, \tag{2.1}$$

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$$\Delta u_2 - \frac{\partial p}{\partial x_2} = 0, \ \Omega_\infty, \tag{2.2}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \ \Omega_\infty, \tag{2.3}$$

$$u_1|_{\Gamma_0} = f_1(\theta), \ u_2|_{\Gamma_0} = f_2(\theta), \tag{2.4}$$

$$u_1, u_2 \text{ are bounded when } r \to +\infty, \tag{2.5}$$

$$p \to 0 \text{ when } r \to +\infty, \tag{2.6}$$

where $f_i(\theta)$ $(i = 1, 2)$ are given, and (r, θ) are polar coordinates. In this section we shall solve problem (2.1)—(2.6) by series expansion. From (2.1)—(2.3), we have

$$\Delta p = \Delta\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) = 0, \ \Omega_\infty, \tag{2.7}$$

$$\Delta^2 u_1 = \frac{\partial}{\partial x_1}(\Delta p) = 0, \ \Omega_\infty, \tag{2.8}$$

$$\Delta^2 u_2 = \frac{\partial}{\partial x_2}(\Delta p) = 0, \ \Omega_\infty. \tag{2.9}$$

Hence, we know that p is a harmonic function on Ω_∞, and u_1, u_2 are biharmonic functions on Ω_∞. Therefore, there are four harmonic functions W_1, G_1, W_2, G_2 such that

$$u_1 = (r^2 - r_0^2)W_1 + G_1, \tag{2.10}$$

$$u_2 = (r^2 - r_0^2)W_2 + G_2. \tag{2.11}$$

By boundary conditions (2.4), G_1 and G_2 can be uniquely determined by the following Fourier series

$$G_1(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^{-n}, \tag{2.12}$$

$$G_2(r, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} (c_n \cos n\theta + d_n \sin n\theta) r^{-n}, \tag{2.13}$$

where

$$a_n = \frac{r_0^n}{\pi} \int_0^{2\pi} f_1(\theta) \cos n\theta d\theta, \ n = 0, 1, 2, \cdots, \tag{2.14}$$

$$b_n = \frac{r_0^n}{\pi} \int_0^{2\pi} f_1(\theta) \sin n\theta d\theta, \ n = 1, 2, \cdots, \tag{2.15}$$

$$c_n = \frac{r_0^n}{\pi} \int_0^{2\pi} f_2(\theta) \cos n\theta d\theta, \ n = 0, 1, 2, \cdots, \tag{2.16}$$

$$d_n = \frac{r_0^n}{\pi} \int_0^{2\pi} f_2(\theta) \sin n\theta d\theta, \ n = 1, 2, \cdots. \tag{2.17}$$

By equation (2.7) and boundary condition (2.6), we have

$$p(r, \theta) = \sum_{n=1}^{\infty} \{p_n^1 \cos n\theta + p_n^2 \sin n\theta\} r^{-n}, \tag{2.18}$$

where constants $\{p_n^1, p_n^2\}$ are yet to be determined.

Now we determine functions W_1 and W_2. From equations (2.1) and (2.2), we obtain

$$\Delta\{(r^2 - r_0^2)W_1\} = \frac{\partial p}{\partial x_1},$$

$$\Delta\{(r^2 - r_0^2)W_2\} = \frac{\partial p}{\partial x_2}.$$

A simple computation shows

$$\frac{\partial}{\partial r}(rW_1) = \frac{1}{4}\frac{\partial p}{\partial x_1}, \tag{2.19}$$

$$\frac{\partial}{\partial r}(rW_2) = \frac{1}{4}\frac{\partial p}{\partial x_2}. \tag{2.20}$$

On the other hand, we have

$$\frac{\partial p}{\partial x_1} = \frac{\partial p}{\partial r}\cos\theta - \frac{\partial p}{\partial \theta}\frac{\sin\theta}{r}$$
$$= \sum_{n=1}^{\infty}(-n)\{p_n^1\cos(n+1)\theta + p_n^2\sin(n+1)\theta\}r^{-n-1},$$
$$\frac{\partial p}{\partial x_2} = \frac{\partial p}{\partial r}\sin\theta + \frac{\partial p}{\partial \theta}\frac{\cos\theta}{r}$$
$$= \sum_{n=1}^{\infty}(-n)\{p_n^1\sin(n+1)\theta - p_n^2\cos(n+1)\theta\}r^{-n-1}.$$

Therefore, we obtain

$$W_1 = \frac{1}{4}\sum_{n=1}^{\infty}\{p_n^1\cos(n+1)\theta + p_n^2\sin(n+1)\theta\}r^{-n-1}, \tag{2.21}$$

$$W_2 = \frac{1}{4}\sum_{n=1}^{\infty}\{p_n^1\sin(n+1)\theta - p_n^2\cos(n+1)\theta\}r^{-n-1}. \tag{2.22}$$

If we know constants $\{p_n^1, p_n^2\}$, then p, W_1, W_2 are determined uniquely by (2.18),(2.21) and (2.22). Unfortunately, $\{p_n^1, p_n^2\}$ are unknown. The remainder is to determine $\{p_n^1, p_n^2\}$. By equation (2.3), we obtain

$$(x_1W_1 + x_2W_2)|_{\Gamma_0} = -\frac{1}{2}\left(\frac{\partial G_1}{\partial x_1} + \frac{\partial G_2}{\partial x_2}\right)\Big|_{\Gamma_0}. \tag{2.23}$$

On the other hand, we have

$$x_1W_1 + x_2W_2 = r(W_1\cos\theta + W_2\sin\theta) = \frac{1}{4}p(r,\theta). \tag{2.24}$$

Combining (2.23) and (2.24), we obtain

$$p|_{\Gamma_0} = -2\left(\frac{\partial G_1}{\partial x_1} + \frac{\partial G_2}{\partial x_2}\right)\Big|_{\Gamma_0}. \tag{2.25}$$

Furthermore, we have

$$\frac{\partial G_1}{\partial x_1} = \sum_{n=1}^{\infty}(-n)\{a_n\cos(n+1)\theta + b_n\sin(n+1)\theta\}r^{-n-1},$$

$$\frac{\partial G_2}{\partial x_2} = \sum_{n=1}^{\infty}(-n)\{c_n\sin(n+1)\theta - d_n\cos(n+1)\theta\}r^{-n-1},$$

and

$$\begin{cases} p_n^1 = 2(n-1)(a_{n-1} - d_{n-1}), \\ p_n^2 = 2(n-1)(b_{n-1} + c_{n-1}), \end{cases} \quad n = 2, 3, \cdots, \tag{2.26}$$

$$p_1^1 = p_1^2 = 0. \tag{2.27}$$

Hence problem (2.1)—(2.6) has been solved. Finally, we present two integral boundary value conditions on Γ_0. By computation, we obtain

$$\left(-\frac{\partial u_1}{\partial r} + p\cos\theta\right)\Big|_{\Gamma_0} = -2\frac{\partial G_1}{\partial r}\Big|_{r=r_0} - \frac{1}{r_0}\frac{\partial G_2}{\partial \theta}\Big|_{r=r_0},$$

$$\left(-\frac{\partial u_2}{\partial r} + p\sin\theta\right)\Big|_{\Gamma_0} = -2\frac{\partial G_2}{\partial r}\Big|_{r=r_0} + \frac{1}{r_0}\frac{\partial G_1}{\partial \theta}\Big|_{r=r_0},$$

and

$$\frac{\partial G_1}{\partial r}\Big|_{r=r_0} = -\frac{1}{2\pi r_0}\int_0^{2\pi} \ln[1 - \cos(\varphi - \theta)]\frac{\partial^2 u_1(r_0, \varphi)}{\partial \varphi^2}\,d\varphi,$$

$$\frac{\partial G_2}{\partial r}\Big|_{r=r_0} = -\frac{1}{2\pi r_0}\int_0^{2\pi} \ln[1 - \cos(\varphi - \theta)]\frac{\partial^2 u_2(r_0, \varphi)}{\partial \varphi^2}\,d\varphi.$$

Namely, we have

$$\left(-\frac{\partial u_1}{\partial r} + p\cos\theta\right)\Big|_{\Gamma_0} = \frac{1}{\pi r_0}\int_0^{2\pi} \ln[1 - \cos(\varphi - \theta)]\frac{\partial^2 u_1(r_0, \varphi)}{\partial \varphi^2}\,d\varphi - \frac{1}{r_0}\frac{\partial u_2(r_0, \theta)}{\partial \theta}, \tag{2.28}$$

$$\left(-\frac{\partial u_2}{\partial r} + p\sin\theta\right)\Big|_{\Gamma_0} = \frac{1}{\pi r_0}\int_0^{2\pi} \ln[1 - \cos(\varphi - \theta)]\frac{\partial^2 u_2(r_0, \varphi)}{\partial \varphi^2}\,d\varphi + \frac{1}{r_0}\frac{\partial u_1(r_0, \theta)}{\partial \theta}. \tag{2.29}$$

§ 3. An Equivalent Variational Problam on a Bounded Domain

Let Γ be a simple closed curve satisfying the Lipchitz condition, and let Ω_e and Ω_i be the unbounded and bounded domain with boundary Γ respectively. Consider the following boundary value problem of Stokes equations:

$$\Delta u_1 - \frac{\partial p}{\partial x_1} = 0, \ \Omega_e, \tag{3.1}$$

$$\Delta u_2 - \frac{\partial p}{\partial x_2} = 0, \ \Omega_e, \tag{3.2}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \ \Omega_e, \tag{3.3}$$

$$u_1|_\Gamma = f_1, \tag{3.4}$$

$$u_2|_\Gamma = f_2, \tag{3.5}$$

$$u_1, u_2 \text{ are bounded when } r \to +\infty, \tag{3.6}$$

$$p \to 0 \text{ when } r \to +\infty. \tag{3.7}$$

In Ω_e, we draw a circumference Γ_0 with radius r_0. Then domain Ω_e is divided to two parts:

the bounded part is denoted by Ω_0 and $\Omega_\infty = \Omega_e \setminus \Omega_0$. Suppose the center of this circumference is the origin of the coordinates (as shown in Figure 1). On the bounded domain Ω_0, we consider the following boundary value problem:

$$\Delta u_1 - \frac{\partial p}{\partial x_1} = 0, \ \Omega_0, \tag{3.8}$$

$$\Delta u_2 - \frac{\partial p}{\partial x_2} = 0, \ \Omega_0, \tag{3.9}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0, \ \Omega_0, \tag{3.10}$$

$$u_1|_\Gamma = f_1, \tag{3.11}$$

$$u_2|_\Gamma = f_2, \tag{3.12}$$

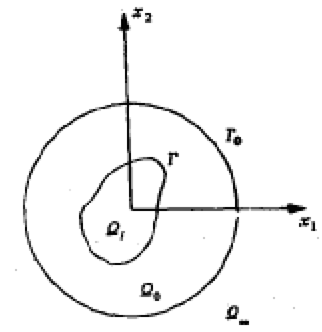

Figure 1

$$\left(-\frac{\partial u_1}{\partial r} + p\cos\theta\right)\Big|_{r_0} = \frac{1}{\pi r_0}\int_0^{2\pi} \ln[1-\cos(\varphi-\theta)]\,\frac{\partial^2 u_1(r_0,\varphi)}{\partial \varphi^2}\,d\varphi - \frac{1}{r_0}\,\frac{\partial u_2(r_0,\theta)}{\partial\theta}, \tag{3.13}$$

$$\left(-\frac{\partial u_2}{\partial r} + p\sin\theta\right)\Big|_{r_0} = \frac{1}{\pi r_0}\int_0^{2\pi} \ln[1-\cos(\varphi-\theta)]\,\frac{\partial^2 u_2(r_0,\varphi)}{\partial \varphi^2}d\varphi + \frac{1}{r_0}\,\frac{\partial u_1(r_0,\theta)}{\partial\theta}. \tag{3.14}$$

Obviously, (3.8)—(3.14) is a boundary value problem of Stokes equations on the bounded domain Ω_0. Let $\hat{u}_1, \hat{u}_2, \hat{p}$ denote the restrictions of u_1, u_2, p, the solution of problem (3.1)—(3.7), on Ω_0. Then we know that $\hat{u}_1, \hat{u}_2, \hat{p}$ is the solution of problem (3.8)—(3.14) by (2.28) and (2.29).

Suppose that $f_i(x_1,x_2)$ $(i=1,2)$ denotes the extension of f_i on Ω_0, and

$$f_i(x_1,x_2)|_{r_0} = 0, \quad \frac{\partial f_i(x_1,x_2)}{\partial r}\Big|_{r_0} = 0.$$

Let $v_1 = u_1 - f_1(x_1,x_2)$, $v_2 = u_2 - f_2(x_1,x_2)$. Then the boundary problem(3.8)—(3.14) can be rewritten as follows:

$$-\Delta v_1 + \frac{\partial p}{\partial x_1} = F_1, \ \Omega_0, \tag{3.15}$$

$$-\Delta v_2 + \frac{\partial p}{\partial x_2} = F_2, \ \Omega_0, \tag{3.16}$$

$$-\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\right) = g, \ \Omega_0, \tag{3.17}$$

$$v_1|_\Gamma = 0, \tag{3.18}$$

$$v_2|_\Gamma = 0, \tag{3.19}$$

$$\left(-\frac{\partial v_1}{\partial r} + p\cos\theta\right)\Big|_{r_0} = \frac{1}{\pi r_0}\int_0^{2\pi} \ln[1-\cos(\varphi-\theta)]\,\frac{\partial^2 v_1(r_0,\varphi)}{\partial \varphi^2}\,d\varphi$$

$$-\frac{1}{r_0}\frac{\partial v_2(r_0,\theta)}{\partial\theta}, \tag{3.20}$$

$$\left(-\frac{\partial v_2}{\partial r}+p\sin\theta\right)\Big|_{\Gamma_0}=\frac{1}{\pi r_0}\int_0^{2\pi}\ln[1-\cos(\varphi-\theta)]\frac{\partial^2 v_2(r_0,\varphi)}{\partial\varphi^2}d\varphi$$
$$+\frac{1}{r_0}\frac{\partial v_1(r_0,\theta)}{\partial\theta}, \tag{3.21}$$

where $F_1=\Delta f_1$, $F_2=\Delta f_2$, and $g=\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}$ are given.

Let $W^{m,p}(\Omega_0)$, $H^m(\Omega_0)$, and $H^s(\Gamma_0)$ denote the usual Sobolev spaces with norms $\|\cdot\|_{m,p,\Omega_0}$, $\|\cdot\|_{m,\Omega_0}$, $\|\cdot\|_{s,\Gamma_0}$. Suppose $\mathring{H}^1(\Omega_0)=\{v\in H^1(\Omega_0), v|_\Gamma=0\}$, and $X=\mathring{H}^1(\Omega_0)\times\mathring{H}^1(\Omega_0)$, $M=H^0(\Omega_0)$ with norms $\|\cdot\|_X$, $\|\cdot\|_M$. Then the boundary value problem (3.15)—(3.21) is equivalent to the following variational problem:

Find $(\boldsymbol{v},p)\in X\times M$, such that

$$a(\boldsymbol{v},\boldsymbol{w})+a_0(\boldsymbol{v},\boldsymbol{w})+b(\boldsymbol{w},p)=\langle\boldsymbol{F},\boldsymbol{w}\rangle,\ \forall\boldsymbol{w}\in X, \tag{3.22}$$
$$b(\boldsymbol{v},q)=\langle g,q\rangle,\ \forall q\in M,$$

where

$$b(\boldsymbol{w},q)=-\int_{\Omega_0}(\mathrm{div}\boldsymbol{w})q\,dx,$$

$$a(\boldsymbol{v},\boldsymbol{w})=\int_{\Omega_0}\nabla v_1\cdot\nabla w_1dx+\int_{\Omega_0}\nabla v_2\cdot\nabla w_2dx,$$

$$a_0(\boldsymbol{v},\boldsymbol{w})=\int_0^{2\pi}\left[-\frac{\partial v_2(r_0,\theta)}{\partial\theta}w_1(r_0,\theta)+\frac{\partial v_1(r_0,\theta)}{\partial\theta}w_2(r_0,\theta)\right]d\theta$$
$$-\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi}\ln[1-\cos(\varphi-\theta)]\frac{\partial v_1(r_0,\theta)}{\partial\varphi}\frac{\partial w_1(r_0,\theta)}{\partial\theta}d\varphi d\theta$$
$$-\frac{1}{\pi}\int_0^{2\pi}\int_0^{2\pi}\ln[1-\cos(\varphi-\theta))\frac{\partial v_2(r_0,\theta)}{\partial\varphi}\frac{\partial w_2(r_0,\theta)}{\partial\theta}d\varphi d\theta$$
$$=\int_0^{2\pi}\left[-\frac{\partial v_2(r_0,\theta)}{\partial\theta}w_1(r_0,\theta)+\frac{\partial v_1(r_0,\theta)}{\partial\theta}w_2(r_0,\theta)\right]d\theta$$
$$+\frac{2}{\pi}\sum_{i=1}^{2}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\int_0^{2\pi}\cos n\varphi\frac{\partial v_i(r_0,\varphi)}{\partial\varphi}d\varphi\cdot\int_0^{2\pi}\cos n\varphi\frac{\partial w_i(r_0,\theta)}{\partial\theta}d\theta\right.$$
$$\left.+\int_0^{2\pi}\sin n\varphi\frac{\partial v_i(r_0,\varphi)}{\partial\varphi}d\varphi\cdot\int_0^{2\pi}\sin n\theta\frac{\partial w_i(r_0,\theta)}{\partial\theta}d\theta\right\}, \tag{3.24}$$

$$\langle\boldsymbol{F},\boldsymbol{w}\rangle=\sum_{i=1}^{2}\int_{\Omega_0}F_iw_idx,$$
$$\langle g,q\rangle=\int_{\Omega_0}gq\,dx.$$

Obviously, $a(\boldsymbol{v},\boldsymbol{w})$ is a bilinear, continuous and V-elliptic form on $X\times X$, and $b(\boldsymbol{w},q)$ is a bilinear, continuous form on $X\times M$. For $a_0(\boldsymbol{v},\boldsymbol{w})$, we have

Lemma 3.1. *$a_0(\boldsymbol{v},\boldsymbol{w})$ is a bilinear form on $X\times X$, and the following inequality hold*

$$|a_0(\boldsymbol{v},\boldsymbol{w})|\leqslant C\|\boldsymbol{v}\|_X\|\boldsymbol{w}\|_X,\ \forall\boldsymbol{v},\boldsymbol{w}\in X, \tag{3.25}$$

where C is a positive constant.

Proof. For any $\boldsymbol{v}$, $\boldsymbol{w} \in X$, we know that v_i, w_i $(i = 1, 2) \in H^1(\Omega_0)$. By the trace theorem, we obtain $v_i|_{\Gamma_0}$ and $w_i|_{\Gamma_0} \in H^{1/2}(\Gamma_0)$; moreover,

$$\|v_i|_{\Gamma_0}\|_{1/2,\Gamma_0} \leqslant c\|v_i\|_{1,\Omega_0}, \tag{3.26}$$

$$\|w_i|_{\Gamma_0}\|_{1/2,\Gamma_0} \leqslant c\|w_i\|_{1,\Omega_0}. \tag{3.27}$$

On the other hand, for any function $h(\theta) \in H^{1/2}(\Gamma_0)$, let

$$\begin{cases} \alpha_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} h(\theta)\cos n\theta d\theta, \\ \beta_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} h(\theta)\sin n\theta d\theta, \end{cases} \quad n = 0, 1, 2, \cdots.$$

Then

$$\left\|\frac{dh}{d\theta}\right\|_{-1/2,\Gamma_0} \leqslant \sqrt{\sum_{n=1}^{\infty} n(\alpha_n^2 + \beta_n^2)} \leqslant \|h\|_{1/2,\Gamma_0}. \tag{3.28}$$

The conclusion follows from (3.24), (3.26)—(3.28). Furthermore, it is not difficult to show

$$a_0(\boldsymbol{w},\boldsymbol{w}) \geqslant 0, \quad \forall \boldsymbol{w} \in X. \tag{3.29}$$

Lemma 3.2. *For any $p \in M = H^0(\Omega_0)$, there is an element $\boldsymbol{v} \in X$, such that*

$$\operatorname{div} \boldsymbol{v} = -p \tag{3.30}$$

and

$$\|\boldsymbol{v}\|_X \leqslant c_0\|p\|_M. \tag{3.31}$$

Proof. Let $V = \{\boldsymbol{v} \in X,\ \operatorname{div}\boldsymbol{v} = 0\}$, and we denote by $V^{\perp}$ the orthogonal complement of V in X for the scalar product $\langle \operatorname{grad}\boldsymbol{w}, \operatorname{grad}\boldsymbol{v}\rangle$. Thus $\operatorname{div} \in \mathscr{L}(X; H^0(\Omega_0))$. Let us show that div is a one-to-one mapping from $V^{\perp}$ onto $H^0(\Omega_0)$. For any $p \in H^0(\Omega_0)$, we seek $\boldsymbol{v} \in X$ such that $\operatorname{div}\boldsymbol{v} = -p$. As Ω_0 is bounded, there exists some function $\theta \in H^2(\Omega_0)$, and

$$\Delta\theta = -p \qquad \text{in } \Omega_0.$$

We set $\boldsymbol{v}_1 = \operatorname{grad}\theta \in (H^1(\Omega_0))^2$; then

$$\operatorname{div} \boldsymbol{v}_1 = -p,$$

and $\boldsymbol{v}_1|_{\Gamma} \in H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$. Hence there exists $\boldsymbol{w}_1 \in (H^1(\Omega))^2$ such that $\operatorname{div}\boldsymbol{w}_1 = 0$, and $\boldsymbol{w}_1|_{\Gamma} = \boldsymbol{v}_1|_{\Gamma}$. Let $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{w}_1$; then $\boldsymbol{v} \in X$ and $\operatorname{div}\boldsymbol{v} = -p$. So $\boldsymbol{v}$ is required. By the Banach inverse operator theorem, we know that the inverse of the div operator is a bounded linear operator. Inequality (3.1) then follows immediately.

Lemma 3.3. *There exists a constant $\beta > 0$, such that*

$$\sup_{\boldsymbol{w} \in X \setminus \{0\}} \frac{b(\boldsymbol{w}, q)}{\|\boldsymbol{w}\|_X} \geqslant \beta\|q\|_M, \quad \forall q \in M. \tag{3.32}$$

Proof. For any $q \in M$, there is an element $\boldsymbol{v} \in X$, satisfying

$$\operatorname{div}\boldsymbol{v} = -q, \quad \|\boldsymbol{v}\|_X \leqslant c_0\|q\|_M.$$

Therefore

$$\sup_{\boldsymbol{w} \in X \setminus \{0\}} \frac{b(\boldsymbol{w}, q)}{\|\boldsymbol{w}\|_X} \geqslant \frac{b(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_X} = \frac{\|q\|_M^2}{\|\boldsymbol{v}\|_X} \geqslant \frac{1}{c_0}\|q\|_M.$$

Then inequality (3.32) is proved with $\beta = \frac{1}{c_0}$.

By Lemmas 3.1, 3.3 and Theorem 4.1 in Chapter I of [4], we obtain

Theorem 3.1. *Suppose $\boldsymbol{F} \in X'$, $g \in M'$. Then the variational problem* (3.22)—(3.23) *has a unique solution* $(\boldsymbol{v}, p) \in X \times M$.

§ 4. A Finite Element Approximation of Problem (3.22)—(3.23)

In this section, we suppose Γ is a polygonal line for the sake of simplicity. Domain Ω_0 is subdivided into a finite number of triangles K and curved triangles $\widetilde{K}$. Let $\mathscr{T}_h$ denote this triangulation satisfying

1° $\Omega_0 = \left(\bigcup_{K \in \mathscr{T}_h} K\right) \cup \left(\bigcup_{\widetilde{K} \in \mathscr{T}_h} \widetilde{K}\right)$.

2° For each $\widetilde{K} \in \mathscr{T}_h$, $\widetilde{K}$ is a curved triangle with a curved side(as shown in Figure 2).

3° If $\widetilde{K} \in \mathscr{T}_h$ is a curved triangle with vertices a_1, a_2, a_3, then triangle K with vertices a_1, a_2, a_3 is contained in $\widetilde{K}$. Let $\hat{S}$ denote the circumscribed circle of K. Then $\widetilde{K}$ is contained in $\hat{S}$. Hence

$$\Omega_h = \left(\bigcup_{K \in \mathscr{T}_h} K\right) \cup \left(\bigcup_{\widetilde{K} \in \mathscr{T}_h} \widetilde{K}\right) \subset \Omega_0.$$

4° $\mathscr{T}_h$ is a regular triangulation in the following sense:

There is a constant $\sigma > 0$, such that

$$\rho_K / h_K \leqslant \sigma, \quad \forall K, \ \widetilde{K} \in \mathscr{T}_h, \tag{4.1}$$

where

$$h_K = \begin{cases} \operatorname{diam}\{K\}, \\ \operatorname{diam}\{\widetilde{K}\}, \end{cases}$$

$\rho_K =$ the diameter of the circumscribed circle of K.

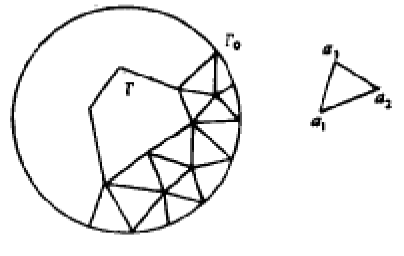

Figure 2

Let $h = \max_{K \in \mathscr{T}_h} \{h_K\}$. Now we construct a finite dimensional subspace of X by quadratic triangular elements. For any element K (or $\widetilde{K}$) $\in \mathscr{T}_h$, the space P_K (or $P_{\widetilde{K}}$) is $P_2(K)$ (or $P_2(\widetilde{K})$) and Σ_K consists of the values at the vertices and middle points of the straight sides as shown in Figure 3.

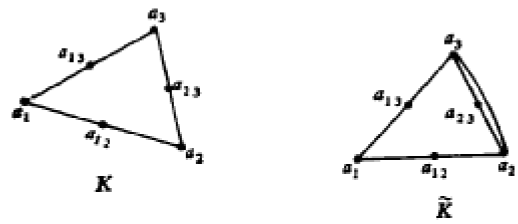

Figure 3

Let $S^h = \{v \in H^1(\Omega_0),\ v|_K (v|_{\widetilde{K}})$ is a quadratic polynomial, $\forall K(\widetilde{K}) \in \mathscr{T}_h$, and $v|_\Gamma = 0\}$. Obviously, $X_h = S^h \times S^h$ is a finite dimensional subspace of X. Suppose

$$M_h = \{q \in M,\ q|_K(q|_{\tilde{K}}) \text{ is a constant}, \ \forall K(\tilde{K}) \in \mathscr{T}_h\}.$$

Then M_h is a finite dimensional subspace of M. Consider the following finite element approximation of problem (3.22)—(3.23):

Find $(\boldsymbol{v}_h, p_h) \in X_h \times M_h$, such that

$$a(\boldsymbol{v}_h, \boldsymbol{w}) + a_0(\boldsymbol{v}_h, \boldsymbol{w}) + b(\boldsymbol{w}, p_h) = \langle \boldsymbol{F}, \boldsymbol{w}\rangle, \ \forall \boldsymbol{w} \in X_h, \tag{4.2}$$

$$b(\boldsymbol{v}_h, q) = \langle g, q\rangle, \ \forall q \in M. \tag{4.3}$$

The bilinear form $b(\boldsymbol{v}_h, q)$ can be rewritten

$$\begin{aligned} b(\boldsymbol{v}_h, q) &= -\int_{\Omega_h} (\operatorname{div} \boldsymbol{v}_h) q dx - \int_{\Omega_0 \setminus \Omega_h} (\operatorname{div} \boldsymbol{v}_h) q dx \\ &\equiv b_1(\boldsymbol{v}_h, q) + b_2(\boldsymbol{v}_h, q). \end{aligned} \tag{4.4}$$

We have

Lemma 4.1. *There is an operator Π_h: $X \to X_h$, such that*

$$b_1(\boldsymbol{v} - \Pi_h \boldsymbol{v}, q_h) = 0, \ \forall q_h \in M_h, \ \boldsymbol{v} \in X, \tag{4.5}$$

and

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_X \leqslant c_1 h^{-\varepsilon} \|\boldsymbol{v}\|_X, \tag{4.6}$$

$$\|\Pi_h \boldsymbol{v}\|_X \leqslant c_1 h^{-\varepsilon} \|\boldsymbol{v}\|_X, \tag{4.6$'$}$$

where $\varepsilon > 0$ is a constant given in the proof.

A similar lemma for a convex polygonal domain is given in [4] (chapter II, Lemma 2.5). The domain Ω_0 in our problem is not convex. Hence this lemma can be understood as an extension of Lemma 2.5 in [4] to the non-convex domain.

Proof. For any $\boldsymbol{v} = (v_1, v_2) \in X$, we take the orthogonal projection $\boldsymbol{w}_h$ of $\boldsymbol{v}$ on X_h for the scalar product of $(\mathring{H}^1(\Omega))^2$:

$$A(\boldsymbol{w}_h - \boldsymbol{v}, \boldsymbol{z}_h) \equiv \langle \nabla(\boldsymbol{w}_h - \boldsymbol{v}), \nabla \boldsymbol{z}_h \rangle = \boldsymbol{0}, \ \forall \boldsymbol{z}_h \in X_h.$$

Then on each element $K(\tilde{K})$, we define $\boldsymbol{v}_h$ of X_h by

$$\begin{cases} \boldsymbol{v}_h(a_i) = \boldsymbol{w}_h(a_i), \text{ for } 1 \leqslant i \leqslant 3, \\ \displaystyle\int_{[a_i, a_j]} (\boldsymbol{v}_h - \boldsymbol{v}) dl = \boldsymbol{0}, \text{ for } 1 \leqslant i < j \leqslant 3. \end{cases}$$

Let $\Pi_h \boldsymbol{v} = \boldsymbol{v}_h$, $\Pi_h \in \mathscr{L}(X, X_h)$. Obviously, $\Pi_h \boldsymbol{v}$ satisfies (4.5). Now we establish estimate (4.6). Let

$$\boldsymbol{e}_h = \boldsymbol{v}_h - \boldsymbol{w}_h \in X_h, \ \boldsymbol{e} = \boldsymbol{v} - \boldsymbol{w}_h. \tag{4.7}$$

Therefore

$$|\boldsymbol{v}_h|_{1,\Omega_0} \leqslant |\boldsymbol{w}_h|_{1,\Omega_0} + |\boldsymbol{e}_h|_{1,\Omega_0} \leqslant |\boldsymbol{v}|_{1,\Omega_0} + |\boldsymbol{e}_h|_{1,\Omega_0}.$$

Hence, we only need to estimate $|\boldsymbol{e}_h|_{1,\Omega_0}$. A computation shows

$$|\boldsymbol{e}_h|^2_{1,\Omega_0} \leqslant C(h^{-2}\|\boldsymbol{e}\|^2_{0,\Omega_0} + |\boldsymbol{e}|^2_{1,\Omega_0}) \tag{4.8}$$

(for the detail the reader can refer to the proof of Lemma 2.5 in chapter II of [4]). By equality (4.7), we have

$$|\boldsymbol{e}|_{1,\Omega_0} \leqslant |\boldsymbol{v}|_{1,\Omega_0}.$$

It remains to evaluate $h^{-2}\|\boldsymbol{e}\|^2_{0,\Omega_0}$. We have

$$\|e_i\|^2_{0,\Omega_0} = \langle e_i, e_i\rangle = \langle e_i, \Delta\varphi_i\rangle,\ i = 1, 2,$$

where φ_i is the solution of the auxiliary problem:

$$\begin{cases} -\Delta\varphi_i = e_i,\ \Omega_0, \\ \varphi_i|_\Gamma = 0, \\ \left.\dfrac{\partial\varphi_i}{\partial n}\right|_{\Gamma_0} = 0. \end{cases} \tag{4.9}$$

Let $\Theta_j, 1 \leqslant j \leqslant N$, denote the vertices of Γ with θ_j being the interior angle of Ω_0. Furthermore, suppose

$$\gamma = \min_{\substack{\theta_j > \pi \\ 1 \leqslant j \leqslant N}} \pi/\theta_j.$$

Then we have $1/2 < \gamma < 1$. By a regularity result of φ_i from Kondatév [14], see also [15], we obtain

$$\varphi_i \in \overset{*}{H}{}^1(\Omega_0) \cup W^{2,p}(\Omega_0),$$

and

$$\|\varphi_i\|_{2,p,\Omega_0} \leqslant c\|e_i\|_{0,\Omega_0}, \tag{4.10}$$

where $4/3 < \rho = \dfrac{1}{\dfrac{2-\gamma}{2} + \delta} < 2$, and $\delta > 0$ is a sufficiently small constant. Let $\varepsilon = 2\left(\dfrac{1}{\rho} - \dfrac{1}{2}\right)$. Then $0 < \varepsilon < \dfrac{1}{2}$. By

$$\|e_i\|^2_{0,\Omega_0} = \langle e_i, -\Delta\varphi_i\rangle = \langle \nabla e_i, \nabla\varphi_i\rangle = \langle \nabla e_i, \nabla\varphi_i - \nabla z_h\rangle,\ \forall\ z_h \in S^h,$$

we have

$$\|e_i\|^2_{0,\Omega_0} \leqslant |e_i|_{1,\Omega_0}\ |\varphi_i - z_h|_{1,\Omega_0},\ \forall\ z_h \in S^h.$$

By Theorem 3.1.6 in [16], we obtain

$$\inf_{z_h \in S^h} |\varphi_i - z_h|_{1,\Omega_0} \leqslant ch^{1-\varepsilon}|\varphi_i|_{2,p,\Omega_0} \leqslant ch^{1-\varepsilon}\|e_i\|_{0,\Omega_0}.$$

It yields

$$\|e_i\|_{0,\Omega_0} \leqslant ch^{1-\varepsilon}|e_i|_{1,\Omega_0}. \tag{4.11}$$

Combining inequalities (4.8) and (4.11), we have

$$|\boldsymbol{e}_h|_{1,\Omega_0} \leqslant ch^{-\varepsilon}|\boldsymbol{e}|_{1,\Omega_0}. \tag{4.12}$$

Finally, we find

$$\begin{aligned} \|\boldsymbol{v} - \Pi_h\boldsymbol{v}\|_X = \|\boldsymbol{v} - \boldsymbol{v}_h\|_X &\leqslant c|\boldsymbol{v} - \boldsymbol{v}_h|_{1,\Omega_0} \\ &\leqslant c\{|\boldsymbol{e}_h|_{1,\Omega_0} + |\boldsymbol{e}|_{1,\Omega_0}\} \\ &\leqslant c_1h^{-\varepsilon}\|\boldsymbol{v}\|_X, \end{aligned}$$

and

$$\|\Pi_h\boldsymbol{v}\|_X \leqslant c_1h^{-\varepsilon}\|\boldsymbol{v}\|_X.$$

Thus the proof is completed.

Lemma 4.2. *There exists a constant c_2 independent of h such that*

$$|b(\boldsymbol{v}-\Pi_h\boldsymbol{v},q_h)|\leqslant c_2h^{\frac{1}{2}-\varepsilon}\|\boldsymbol{v}\|_X\|q_h\|_M,$$
$$\forall \boldsymbol{v}\in X,\ q_h\in M_h. \tag{4.13}$$

Proof. By Lemma 4.1, we have

$$\begin{aligned}|b(\boldsymbol{v}-\Pi_h\boldsymbol{v},q_h)| &= |b_2(\boldsymbol{v}-\Pi_h\boldsymbol{v},q_h)| = \left|\int_{\Omega_0\setminus\Omega_h}[\operatorname{div}(\boldsymbol{v}-\Pi_h\boldsymbol{v})]q_h dx\right|\\
&\leqslant \sum_{\tilde K\in\mathcal{T}_h}\left|\int_{\tilde K\setminus K}[\operatorname{div}(\boldsymbol{v}-\Pi_h\boldsymbol{v})q_{h,\tilde K}dx\right|\\
&\overset{1)}{\leqslant} \sum_{\tilde K\in\mathcal{T}_h}|q_{h,\tilde K}|\int_{\tilde K\setminus K}|\operatorname{div}(\boldsymbol{v}-\Pi_h\boldsymbol{v})dx|\\
&\leqslant c\sum_{\tilde K\in\mathcal{T}_h}|q_{h,\tilde K}|\,|\boldsymbol{v}-\Pi_h\boldsymbol{v}|_{1,\tilde K}[\operatorname{meas}(\tilde K\setminus K)]^{\frac{1}{2}}\\
&\overset{2)}{\leqslant} c\sum_{\tilde K\in\mathcal{T}_h}|q_{h,\tilde K}|\,|\boldsymbol{v}-\Pi_h\boldsymbol{v}|_{1,\tilde K}h^{\frac{3}{2}}\\
&\leqslant c\sum_{K\in\mathcal{T}_h}h^{1/2}|\boldsymbol{v}-\Pi_h\boldsymbol{v}|_{1,\tilde K}\|q_h\|_{0,\tilde K}\\
&\leqslant ch^{1/2}\|\boldsymbol{v}-\Pi_h\boldsymbol{v}\|_X\|q_h\|_M\\
&\leqslant c_2h^{\frac{1}{2}-\varepsilon}\|\boldsymbol{v}\|_X\|q_h\|_M.\end{aligned}$$

Lemma 4.3. *There exist two constants $h_0>0$, $\beta^*>0$, independent of h, such that for $0<h\leqslant h_0$,*

$$\sup_{\boldsymbol{w}_h\in X_h\setminus\{0\}}\frac{b(\boldsymbol{w}_h,q_h)}{\|\boldsymbol{w}_h\|_X}\geqslant\beta^*h^{\varepsilon}\|q_h\|_M,\quad\forall q_h\in M_h.$$

Proof. For an arbitrary $q_h\in M_h\subset M$, there is $\boldsymbol{v}\in X$, such that

$$\operatorname{div}\,\boldsymbol{v}=-q_h,\ \|\boldsymbol{v}\|_X\leqslant c_0\|q_h\|_M.$$

Hence

$$\begin{aligned}\sup_{\boldsymbol{w}_h\in X_h\setminus\{0\}}\frac{b(\boldsymbol{w}_h,q_h)}{\|\boldsymbol{w}_h\|_X}&\geqslant\frac{b(\Pi_h\boldsymbol{v},q_h)}{\|\Pi_h\boldsymbol{v}\|_X}\\
&=\frac{b(\boldsymbol{v},q_h)-b(\boldsymbol{v}-\Pi_h\boldsymbol{v},q_h)}{\|\Pi_h\boldsymbol{v}\|_X}\\
&\geqslant\frac{\|q_h\|_M^2-c_0c_2h^{\frac{1}{2}-\varepsilon}\|q_h\|_M^2}{c_1h^{-\varepsilon}\|q_h\|_M}\\
&\geqslant\beta^*h^{\varepsilon}\|q_h\|_M,\end{aligned}$$

with

$$\beta^*=\frac{1}{2c_1},\quad h_0=\left(\frac{1}{2c_0c_2}\right)^{\varepsilon-1/2}.$$

Finally, by Theorem 1.1 in Chapter II of [4], we obtain the following error estimate:

Theorem 4.1. *There exists a constant $h_0>0$, such that when $0<h\leqslant h_0$, problem*

1) $q_{h,\tilde K}$ is a constant, and $q_h|_{\tilde K}=q_{h,\tilde K}$.

2) By inequality $\operatorname{meas}(\tilde K\setminus K)\leqslant ch^3$, $\forall\tilde K\in\mathcal{T}_K$.

(4.2)—(4.3) *has a unique solution* $(\boldsymbol{v}_h, p_h) \in X_h \times M_h$, *and*

$$\|\boldsymbol{v}-\boldsymbol{v}_h\|_X \leqslant ch^{-s}\{\inf_{\boldsymbol{w}_h\in X_h}\|\boldsymbol{v}-\boldsymbol{w}_h\|_X+\inf_{q_h\in M_h}\|p-q_h\|_M\},$$

$$\|p-p_h\|_M \leqslant ch^{-2s}\{\inf_{\boldsymbol{w}_h\in X_h}\|\boldsymbol{v}-\boldsymbol{w}_h\|_X+\inf_{q_h\in M_h}\|p-q_h\|_M\}.$$

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无界区域上 Stokes 方程组的混合有限元方法

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摘　要

本文讨论无界区域上 Stokes 方程组边值问题的有限元近似解．为了克服区域的无界性所造成的困难，本文采用“局部化”技巧，首先将问题化为一个等价的有界区域上的边值问题，然后求解这个等价问题的混合有限元近似解，最后给出了有限元近似解的误差分析．