系统科学与数学
J. Sys. Sci. & Math. Scis.
9 (1) (1988), 47—52

临界增长拟线性椭圆方程的正则性

朱熹平
(中国科学院武汉数学物理研究所)

杨健夫
(江西大学数学系，南昌)

近年来，非线性临界增长椭圆方程得到了广泛的研究．对于半线性方程，许多正则性结果已经得到[1-3]．本文我们考虑拟线性方程

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}(a_i(x,u,\nabla u))=a(x,u,\nabla u),\ x\in\Omega\subset\mathrm{R}^N \tag{1}$$

的 $W^{1,p}(\Omega)$ 弱解的正则性．

假定 $a_i(x,z,\boldsymbol{q})$, $a(x,z,\boldsymbol{q})$ 是 $\Omega\times\mathrm{R}\times\mathrm{R}^N$ 上的 Carathéodory 函数，且满足如下条件：

$$\left.\begin{array}{l}(\mathrm{a})\ |a_i(x,z,\boldsymbol{q})|\leqslant C|\boldsymbol{q}|^{p-1}+C|z|^{p^*/p'}+C\\(\mathrm{b})\ |a(x,z,\boldsymbol{q})|\leqslant C|\boldsymbol{q}|^{p/(p^*)'}+C|z|^{p^*-1}+C\\(\mathrm{c})\ a_i(x,z,\boldsymbol{q})q_i\geqslant\nu|\boldsymbol{q}|^p-C|z|^{p^*}-C\end{array}\right\}\forall(x,z,\boldsymbol{q})\in\Omega\times\mathrm{R}\times\mathrm{R}^N,$$

其中，$N>p\geqslant1$, $p^*=Np/(N-p)$, $p'=p/(p-1)$, $(p^*)'=p^*/(p^*-1)$, C,ν 是两正数．

由于 $p^*=Np/(N-p)$ 是 Sobolev 嵌入 $W^{1,p}\hookrightarrow L^{p^*}$ 的极限指标，通常的靴带方法不再适用．我们还注意到 Brezis 等人的方法完全依赖于关于 Laplacian Δ 的 Kato 不等式．这样处理半线性方程正则性的方法不适用于讨论拟线性方程 (1) 的正则性．

首先，我们给出方程 (1) 的 $W^{1,p}(\Omega)$ 弱解的局部有界性．

定理 1. 假设 u 是 (1) 的 $W^{1,p}(\Omega)$ 弱解，且 (a), (b), (c) 满足，则 $u\in L^\infty_{\mathrm{loc}}(\Omega)$ (即 $\forall\Omega'\subset\subset\Omega$, $u\in L^\infty(\Omega')$)．

证．由条件 (a), (b), (c)，对于某 $k>0$ 足够大，记 $\bar z=|z|+k$，我们有

$$\left.\begin{array}{l}(\mathrm{a}')\ |a_i(x,z,\boldsymbol{q})|\leqslant C(|\boldsymbol{q}|^{p-1}+|\bar z|^{p^*/p'})\\(\mathrm{b}')\ |a(x,z,\boldsymbol{q})|\leqslant C(|\boldsymbol{q}|^{p/(p^*)'}+|\bar z|^{p^*-1})\\(\mathrm{c}')\ a_i(x,z,\boldsymbol{q})q_i\geqslant\nu|\boldsymbol{q}|^p-C|\bar z|^{p^*}\end{array}\right\}\forall(x,z,\boldsymbol{q})\in\Omega\times\mathrm{R}\times\mathrm{R}^N.$$

这里和以下我们均记 C 为非本质性的正常数．我们只需往证 $u^+=\max\{0,u\}\in L^\infty_{\mathrm{loc}}(\Omega)$ (对于 $u^-=-\min\{0,u\}$ 完全一样证明)．

对任意 $x_0\in\Omega$，取 $R>0$ 足够小(待定)，使 $B_{2R}(x_0)\subset\Omega$．记 $\eta\in C_0^\infty(\Omega)$，使得 $\eta\equiv1$, $x\in B_R(x_0)$ 及 $\eta\equiv0$, $x\bar{\in}B_{R+r}(x_0)$ 且 $|\nabla\eta|\leqslant\frac{2}{r}$, $0<r<R$, $\eta\geqslant0$ 在 Ω 上．

1987 年 11 月 20 日收到．

记

$$\bar{u}_L^+ = \begin{cases} \bar{u}^+, & x \in \{\bar{u}^+ \leqslant L\}, \\ L, & x \in \{\bar{u}^+ > L\}. \end{cases} \quad (L > k > 0)$$

对于(1)的 $W^{1,p}(\Omega)$ 弱解 u, 即 $u \in W^{1,p}(\Omega)$ 且

$$\int_\Omega \left[\sum_{i=1}^N a_i(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} - a(x, u, \nabla u)\varphi\right] dx = 0, \ \forall \varphi \in W_c^{1,p}(\Omega), \tag{2}$$

取

$$\varphi = \eta^p(\bar{u}^+ \cdot \bar{u}_L^{+p(q-1)} - k^{p(q-1)+1}) \quad (q > 1 \text{ 待定}),$$

得

$$\int_\Omega \Big\{\sum_{i=1}^N \Big(a_i(x, u, \nabla u)p(\bar{u}^+ \cdot \bar{u}_L^{+p(q-1)} - k^{p(q-1)+1})\eta^{p-1} \frac{\partial \eta}{\partial x_i}$$
$$+ a_i(x, u, \nabla u) \frac{\partial \bar{u}^+}{\partial x_i} \eta^p \cdot \bar{u}_L^{+p(q-1)} + a_i(x, u, \nabla u)p(q-1)\eta^p$$
$$\cdot \frac{\partial \bar{u}_L^+}{\partial x_i} \bar{u}_L^{+p(q-1)}\Big) - a(x, u, \nabla u)\eta^p(\bar{u}^+ \cdot \bar{u}_L^{+p(q-1)} - k^{p(q-1)+1})\Big\} dx = 0,$$

即

$$\int_\Omega \sum_{i=1}^N \Big(a_i(x, u, \nabla u) \frac{\partial \bar{u}^+}{\partial x_i} \eta^p \cdot \bar{u}_L^{+p(q-1)} + a_i(x, u, \nabla u)p(q-1)\eta^p$$
$$\cdot \frac{\partial \bar{u}_L^+}{\partial x_i} \bar{u}_L^{+p(q-1)}\Big) dx \leqslant -p\int_\Omega \sum_{i=1}^N a_i(x, u, \nabla u)(\bar{u}_L^{+p(q-1)}\bar{u}^+$$
$$- k^{p(q-1)+1})\eta^{p-1} \frac{\partial \eta}{\partial x_i} dx + \int_\Omega a(x, u, \nabla u)\eta^p(\bar{u}^+ \cdot \bar{u}_L^{+p(q-1)} - k^{p(q-1)+1}) dx. \tag{3}$$

由(c′), 有

$$\int_\Omega \sum_{i=1}^N a_i(x, u, \nabla u)\eta^p \frac{\partial \bar{u}^+}{\partial x_i} \cdot \bar{u}_L^{+p(q-1)} dx$$
$$= \int_\Omega \sum_{i=1}^N a_i(x, u^+, \nabla \bar{u}^+)\eta^p \cdot \frac{\partial \bar{u}^+}{\partial x_i} \cdot \bar{u}_L^{+p(q-1)} dx$$
$$\geqslant \int_\Omega \eta^p \bar{u}_L^{+p(q-1)}[\nu|\nabla \bar{u}^+|^p - C|\bar{u}^+|^{p^*}] dx \tag{4}$$

和

$$\int_\Omega \sum_{i=1}^N a_i(x, u, \nabla u)p(q-1)\eta^p \cdot \frac{\partial \bar{u}_L^+}{\partial x_i} \cdot \bar{u}_L^{+p(q-1)} dx$$
$$= \int_{\{0<u<L-k\}} \sum_{i=1}^N a_i(x, u^+, \nabla \bar{u}^+)p(q-1)\eta^p \cdot \frac{\partial \bar{u}^+}{\partial x_i} \cdot \bar{u}_L^{+p(q-1)} dx$$
$$\geqslant p(q-1)\int_{\{0<u<L-k\}} \eta^p \cdot \bar{u}_L^{+p(q-1)}(\nu|\nabla \bar{u}^+|^p - C|\bar{u}^+|^{p^*}) dx$$
$$= p(q-1)\int_{\{0<u<L-k\}} \eta^p \cdot \bar{u}_L^{+p(q-1)}(\nu|\nabla \bar{u}_L^+|^p - C|\bar{u}_L^+|^{p^*}) dx$$
$$\geqslant p(q-1)\int_\Omega \eta^p \cdot \bar{u}_L^{+p(q-1)}(\nu|\nabla \bar{u}_L^+|^p - C|\bar{u}_L^+|^{p^*}) dx. \tag{5}$$

由 (a′) 和 Yaung 不等式，对任意 $\varepsilon>0$，$\exists C_\varepsilon$ 使

$$\begin{aligned}&\left|-p\int_\Omega\sum_{i=1}^N a_i(x,u,\nabla u)(\bar u^+\cdot\bar u_L^{+p(q-1)}-k^{p(q-1)+1})\eta^{p-1}\frac{\partial\eta}{\partial x_i}dx\right|\\&=\left|-p\int_\Omega\sum_{i=1}^N a_i(x,u^+,\nabla u^+)(\bar u^+\cdot\bar u_L^{+p(q-1)}-k^{p(q-1)+1})\eta^{p-1}\frac{\partial\eta}{\partial x_i}dx\right|\\&\leqslant C\cdot p\int_\Omega(|\nabla u^+|^{p-1}+|\bar u^+|^{p^*/p'})|\nabla\eta|\eta^{p-1}\cdot\bar u^+\cdot\bar u_L^{+p(q-1)}dx\\&\leqslant\varepsilon\int_\Omega|\nabla u^+|^p\cdot\eta^p\cdot\bar u_L^{+p(q-1)}dx+C_\varepsilon\int_\Omega(|\nabla\eta|^p\cdot|\bar u^+|^p\\&\quad+|\bar u^+|^{p^*}\cdot\eta^p)\bar u_L^{+p(q-1)}dx.\end{aligned}\tag{6}$$

由 (b′) 和 Yaung 不等式，对任意 $\varepsilon>0$，$\exists C_\varepsilon$ 使

$$\begin{aligned}&\left|\int_\Omega a(x,u,\nabla u)\eta^p(\bar u_L^{+p(q-1)}\cdot\bar u^+-k^{p(q-1)+1})dx\right|\\&=\left|\int_\Omega a(x,u^+,\nabla u^+)\eta^p(\bar u_L^{+p(q-1)}\cdot\bar u^+-k^{p(q-1)+1})dx\right|\\&\leqslant C\int_\Omega(|\nabla u^+|^{p/(p^*)'}+|\bar u^+|^{p^*-1})\eta^p\bar u^+\cdot\bar u_L^{+p(q-1)}dx\\&\leqslant\varepsilon\int_\Omega|\nabla u^+|^p\cdot\eta^p\cdot\bar u_L^{+p(q-1)}dx+C_\varepsilon\int_\Omega|\bar u^+|^{p^*}\cdot\eta^p\cdot\bar u_L^{+p(q-1)}dx.\end{aligned}\tag{7}$$

综合 (3)—(7)，选定 ε 足够小，得

$$\begin{aligned}&\int_\Omega(|\nabla\bar u^+|^p\cdot\eta^p\cdot\bar u_L^{+p(q-1)}+(q-1)|\nabla\bar u_L^+|^p\cdot\eta^p\cdot\bar u_L^{+p(q-1)})dx\\&\leqslant qC\int_\Omega|\bar u^+|^{p^*}\eta^p\cdot\bar u_L^{+p(q-1)}dx+C\int_\Omega|\nabla\eta|^p\cdot|\bar u^+|^p\cdot\bar u_L^{+p(q-1)}dx.\end{aligned}\tag{8}$$

记

$$w_L=\eta\cdot\bar u^+\cdot\bar u_L^{+(q-1)},$$

有

$$\begin{aligned}\int_\Omega|\nabla w_L|^pdx&\leqslant 2^{p-1}\int_\Omega|\nabla\eta|^p\cdot\bar u^{+p}\cdot\bar u_L^{+p(q-1)}dx+4^{p-1}\Big(\int_\Omega\eta^p\cdot\bar u_L^{+p(q-1)}|\nabla\bar u^+|^pdx\\&\quad+\int_\Omega\eta^p\cdot\bar u_L^{+p(q-1)}(q-1)^p|\nabla\bar u_L^+|^p\,dx\Big).\end{aligned}\tag{9}$$

利用 (8),(9) 及 Sobolev 不等式得

$$\begin{aligned}&\left(\int_\Omega(\eta\bar u^+\cdot\bar u_L^{+(q-1)})^{p^*}dx\right)^{p/p^*}\leqslant C\int_\Omega|\nabla(\eta\bar u^+\cdot\bar u_L^{+(q-1)})|^pdx\\&\leqslant Cq^p\left[\int_\Omega|\bar u^+|^{p^*}\cdot\eta^p\cdot\bar u_L^{+p(q-1)}dx+\int_\Omega|\nabla\eta|^p\cdot\bar u^{+p}\cdot\bar u_L^{+p(q-1)}dx\right].\end{aligned}\tag{10}$$

我们断言，存在 $R_0>0$，使得

$$\bar u^+\in L^{(p^*)^2/p}(B_{R_0}(x_0)).\tag{11}$$

事实上，在 (10) 式中令 $q=p^*/p$，有

$$\left(\int_\Omega(\eta\bar u^+\cdot\bar u_L^{+(p^*-p)/p})^{p^*}dx\right)^{p/p^*}\leqslant C\left[\int_\Omega(|\bar u^+|^{p^*}\cdot\eta^p+|\nabla\eta|^p\cdot|\bar u^+|^p)\bar u_L^{+(p^*-p)}dx\right].\tag{12}$$

由于

$$\int_\Omega \eta^p \cdot \bar{u}^{+p^*} \cdot \bar{u}_L^{+p^*-p} dx$$

$$\leqslant \left[\int_\Omega (\eta \bar{u}^+ \cdot \bar{u}_L^{+(p^*-p)/p})^{p^*} dx\right]^{p/p^*} \cdot \left[\int_{B_{R+r}(x_0)} |\bar{u}^+|^{p^*} dx\right]^{(p-p^*)/p^*},$$

选取 $R = R_0 > 0$ 足够小,使

$$C\left[\int_{B_{2R}(x_0)} |\bar{u}^+|^{p^*} dx\right]^{(p^*-p)/p^*} < \frac{1}{2},$$

其中C是(12)式中的常数. 这样我们得

$$\left(\int_\Omega (\eta \bar{u}^+ \cdot \bar{u}_L^{+(p^*-p)/p})^{p^*} dx\right)^{p/p^*} \leqslant 2C\int_\Omega |\nabla\eta|^p \cdot \bar{u}^{+p} \cdot \bar{u}_L^{+p^*-p} dx$$

$$\leqslant 2C\int_\Omega |\nabla\eta|^p \cdot |\bar{u}^+|^{p^*} dx. \qquad (13)$$

所以令 $L \to +\infty$得

$$\left(\int_{B_{R_0}} |\bar{u}^+|^{(p^*)^2/p} dx\right)^{p/p^*} \leqslant \left(\int_\Omega (\eta |\bar{u}^+|^{p^*/p})^{p^*} dx\right)^{p/p^*}$$

$$\leqslant 2C\int_\Omega |\nabla\eta|^p \cdot |\bar{u}^+|^{p^*} dx < +\infty.$$

现在我们往证 $\bar{u}^+ \in L^\infty(B_R(x_0))$, $0 < R < \frac{1}{2} R_0$.

记 $t = (p^*)^2/(p^* - p)p$. 如果 $\bar{u}^+ \in L^{pqt/(t-1)}(B_{R+r}(x_0))$ $(0 < r < R)$, 由(11)式,我们有

$$\int_\Omega \bar{u}^{+p^*} \cdot \eta^p \cdot \bar{u}_L^{+p(q-1)} dx \leqslant \left(\int_{B_{R+r}(x_0)} (\eta^p \cdot \bar{u}^{+pq})^{t/(t-1)} dx\right)^{1-\frac{1}{t}}$$

$$\cdot \left(\int_{B_{R+r}(x_0)} \bar{u}^{+(p^*-p)t} dx\right)^{\frac{1}{t}}$$

$$\leqslant \left[\int_{B_{R+r}(x_0)} (\eta^p \cdot \bar{u}^{+pq})^{t/(t-1)} dx\right]^{1-\frac{1}{t}} \cdot \left[\int_{B_{R_0}(x_0)} \bar{u}^{+(p^*)^2/p} dx\right]^{\frac{1}{t}}$$

$$\leqslant C\left[\int_{B_{R+r}(x_0)} (\eta^p \cdot \bar{u}^{+pq})^{t/(t-1)} dx\right]^{1-\frac{1}{t}},$$

$$\int_\Omega |\nabla\eta|^p \cdot \bar{u}^{+p} \cdot \bar{u}_L^{+p(q-1)} dx \leqslant C r^{-p}\left[\int_{B_{R+r}(x_0)} (\bar{u}^{+pq})^{t/(t-1)} dx\right]^{1-\frac{1}{t}}.$$

则(由(10))

$$\left(\int_\Omega (\eta \bar{u}^+ \cdot \bar{u}_L^{+(q-1)})^{p^*} dx\right)^{p/p^*} \leqslant C q^p \cdot r^{-p}\left[\int_{B_{R+r}(x_0)} (\bar{u}^{+pq})^{t/(t-1)} dx\right]^{1-\frac{1}{t}},$$

即

$$\left(\int_{B_R} \bar{u}^{+p^*q} dx\right)^{1/q} \leqslant C^{1/q} \cdot q^{p^*/q} \cdot r^{-p^*/q}\left[\int_{B_{R+r}(x_0)} \bar{u}^{+pqt/(t-1)} dx\right]^{(t-1)p^*/tpq}, \qquad (14)$$

其中常数C不依赖于 r, q.

记 $\tau = p^*(t-1)/pt$ $(\tau > 1)$, $B_i = B_{R+2^{-i}r}(x_0)$ 及

$$I_i=\left(\int_{B_i}(\bar{u}^{+\,pt/(t-1)})^{\tau^i}dx\right)^{1/\tau^i}.$$

利用(14)式，进行迭代得

$$\begin{aligned}I_{i+1}&=\left(\int_{B_{i+1}}\bar{u}^{+\frac{pt}{t-1}\tau^{i+1}}dx\right)^{1/\tau^{i+1}}=\left(\int_{B_{i+1}}\bar{u}^{+\tau^i p^*}dx\right)^{1/\tau^{i+1}}\\&\leqslant C^{1/\tau^{i+1}}\cdot r^{-p^*/\tau^{i+1}}\cdot 2^{(i+1)p^*/\tau^{i+1}}\cdot(\tau^i p^*)^{1/\tau^{i+1}}\left[\int_{B_i}\bar{u}^{+\frac{pt}{t-1}\tau^i}dx\right]^{1/\tau^i}\\&=C^{1/\tau^{i+1}}\cdot r^{-p^*/\tau^{i+1}}\cdot 2^{(i+1)p^*/\tau^{i+1}}\cdot(\tau^i p^*)^{1/\tau^{i+1}}\cdot I_i\\&\leqslant C^{\sum_{j=0}^{i+1}(\frac{1}{\tau})^j}\cdot(r^{-p^*})^{\sum_{j=0}^{i+1}(\frac{1}{\tau})^j}\cdot(2p^*)^{\sum_{j=0}^{i+1}j(\frac{1}{\tau})^j}\cdot e^{\sum_{j=0}^{i+1}p^*j\cdot(\frac{1}{\tau})^j\ln\tau}\cdot I_0,\end{aligned}$$

而

$$I_0=\int_{B_{R+r}(x_0)}|\bar{u}^+|^{pt/(t-1)}dx<+\infty\quad(pt/(t-1)<p^*),$$

所以得

$$\bar{u}^+\in L^\infty(B_R(x_0)),\quad 0<R<\frac{1}{2}R_0,$$

即

$$u^+\in L^\infty(B_R(x_0)),\quad 0<R<\frac{1}{2}R_0.$$

由于 x_0 是任意的，从而我们完成了定理证明。

现在我们给出方程(1)的 $C^{1,\alpha}$ 正则性。我们附加如下条件：

(d) $a_i\in C^0(\Omega\times\mathrm{R}\times\mathrm{R}^N)\cap C^1(\Omega\times\mathrm{R}\times(\mathrm{R}^N\backslash\{0\}))$ 且

$$a_i(x,z,0)=0,\ (x,z)\in\Omega\times\mathrm{R},\ 1\leqslant i\leqslant N,$$

(e) $\gamma(|z|)|\boldsymbol{q}|^{p-2}\cdot|\xi|^2\leqslant\sum_{i,j=1}^N\frac{\partial a_j}{\partial q_i}(x,z,\boldsymbol{q})\xi_i\xi_j\leqslant\Gamma(|z|)|\boldsymbol{q}|^{p-2}|\xi|^2,$

(f) $\sum_{i,j=1}^N\left(\left|\frac{\partial a_j}{\partial x_i}\right|+\left|\frac{\partial a_j}{\partial z}\right|\right)\leqslant\Gamma(|z|)|\boldsymbol{q}|^{p-1},$

其中 γ，Γ 是 $[0,+\infty)$ 上的正连续函数，$x\in\Omega$，$z\in\mathrm{R}$，$\boldsymbol{q}\in\mathrm{R}^N\backslash\{0\}$ 及 $\xi\in\mathrm{R}^N$。

定理 2. 假设 (a)—(f) 满足，u 是方程(1)的 $W^{1,p}(\Omega)$ 解，$1<p<+\infty$。则对任意 $\Omega'\subset\subset\Omega$，$u\in C^{1,\alpha}(\bar{\Omega}')$，某 $\alpha>0$（依赖于 Ω'）。

证. 由定理 1，对任意 $x_0\in\Omega$，存在 $R>0$，使得 $u\in L^\infty(B_R(x_0))$。这样 u 是方程(1)的 $W^{1,p}(B_R(x_0))\cap L^\infty(B_R(x_0))$ 解。利用[4]中的结果得 $u\in C^{1,\alpha}(B_{R'}(x_0))$，对于固定 $0<R'<R$ 和某 $\alpha>0$。从而我们得所需。

注　本文的正则性结果可应用于[5]中的所有情形([5]中得到了一些非平凡 $W_0^{1,p}(\Omega)$ 解的存在性结果)。

参 考 文 献

[1] Brezis, H. & T. Kato, Remarks on the Schrödinger operator with singular complex potentials, *J. Math. Pures Appl.*, **58**(1979), 137—151.

[2] Brezis, H. & E. H. Lieb, Minimum action solutions of some vector field equations, *Comm.*

Math. Phys., **96**(1984), 97—113

[3] Trüdinger, N., Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, *Ann. Sc. Norm. Sup. Pisa.*, **22**(1968), 265—274.

[4] Tolksdorf, P., Regularity for a more general class of quasilinear elliptic equations, *J. Diff. Eq.*, **51**(1984), 126—150.

[5] 朱熹平,临界增加拟线性椭圆方程的非平凡解,中国科学A辑, 3(1988), 225—237.

REGULARITY FOR QUASILINEAR ELLIPTIC EQUATIONS INVOLVING CRITICAL SOBOLEV EXPONENT

ZHU XI-PING

(*Wuhan Institute of Mathematical Sciences, Academia Sinica*)

YAN JIAN-FU

(*Departement of Mathematics, Jiangxi University*)

ABSTRACT

In this paper, we prove that the $W^{1,p}$ solutions of the following quasilinear elliptic equation possess interior $C^{1,\alpha}$ regularity:

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right)=f(x,u),\ x\in\Omega,$$

where Ω is a domain in $\mathbf{R}^N$, $N>p>1$, and $|f(x,u)|\leqslant C(1+|u|^{\frac{Np}{N-p}-1})$ for any $x\in\Omega$, $u\in\mathbf{R}$.