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三合板梁彎曲問題*

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一. 引　言

三合板的彎曲問題在航空工業上有廣泛的應用. 近十年來出現了不少關於這方面的理論，但是這些理論都是以線性問題觀點來處理的，在實驗結果與理論出入太大時，一般都認爲是材料的品質和結構的缺點的問題. 本文的理論分析了非線性的因素，無論是線性及由線性擴展到非線性範圍；而且根據本文的理論結果得到了很好的實驗證明. 本文處理了在平面應變情形—維空間的三合板彎曲的問題，以一三合板肱梁末端受集中載荷情況求解. 同樣方法可以推廣於三合板梁彎曲的其他問題.

二. 標　號

1. 附號:

　　c　夾心

　　f　表鈑

　　+　上表鈑　$z = +h/2$

　　−　下表鈑　$z = -h/2$

　　1　線性項次，或 “ϵ” 之係數

　　2,3　非線性項次. “ϵ^2” “ϵ^3” 之係數，依此類推.

2. 數學記號:

　　$' = \dfrac{d}{d\xi}$　　$'' = \dfrac{d^2}{d\xi^2}$……依此類推.

　　$\bar{\ }$　複數之共軛數

　　Re　函數之實數部分

*1954 年 12 月 19 日收到.

Im　函數之虛數部分

[………]$_p$ 微分方程之特解

[………]$_c$ 微分方程之補解

3. 坐標： x, z. （圖 1）

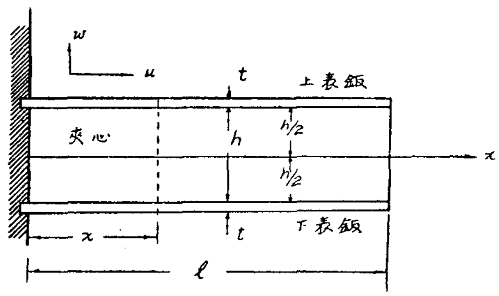

圖 1. 三合板肱梁

4. 位移： u, w.

5. 力及應力：

Q　載荷

S　表鈑横截面剪力

M　表鈑撓矩

p　x–方向表鈑內力

τ, σ_{z+}, σ_{z-}　剪應力及正應力（圖 2）

$\sigma_{1,2}$　線性及非線性表鈑彎曲正應力

σ_p　p 力引起之表鈑正應力

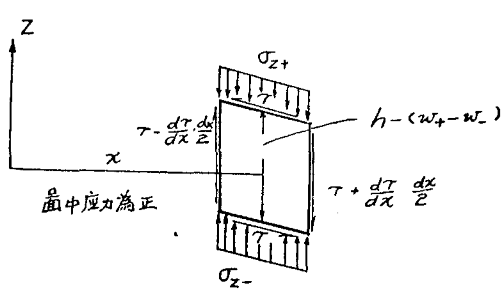

圖 2. 夾心應力

6. 彈性係數：

E_c　夾心彈性係數

G_c　夾心剪彈性係數

E_f　表鈑彈性係數

μ_f　表鈑泊松比

7. 三合板參數:

1　無量綱彈性係數

$$E = E_c / E_f \qquad G = G_c / E_f$$

2　無量綱比值 (圖 1)

$a = l / h \qquad b = h / t$

l　三合板梁長度

h　夾心厚度

t　表鈑厚度

8. 無量綱變數:

$$\xi = x/l,\ W = w/h,\ \epsilon = Q/hG_c,\ P = P/lG_c$$

9. 無量綱常數:

$$k^4 = 6\,(1-\mu_f^2)\,E\,a^4\,b^3$$

$$K_1 = 12\,a^2\,E\,/\,G$$

$$K_2 = 72\,(1-\mu_f^2)\,a^4\,(b+1)^2\,b\,E\left[1 + \frac{1}{3(b+1)^2}\right]$$

$$K_3 = 72\,(1-\mu_f^2)\,a^4\,(b+1)\,b^2\,E$$

$$K_4 = \frac{b}{b+1}\cdot\frac{1}{\left[1 + \frac{1}{3(b+1)^2}\right]}$$

$$K_5 = \frac{1}{12\,(1-\mu_f^2)\,a^3\,(b+1)\,b^2\,G}$$

$$K_6 = \frac{b}{1+b} - K_4$$

$$K_7 = -\,144\,(1-\mu_f^2)^2\,a^6\,b^5\,(b+1)\,G^2$$

$$K_8 = -\,144\,(1-\mu_f^2)^2\,a^6\,b^6\,G^2$$

$$K_9 = K_8' - 2\,K_4\,K_7 = 144\,(1-\mu_f^2)^2\,a^6\,b^6\,G^2\,\frac{3\,(b+1)^2-1}{3\,(b+1)^2+1}$$

$$K_{10} = \frac{K_4}{4\,k^4}\,(K_8 - K_4\,K_1) = -6\,(1-\mu_f^2)\,a^2\,b^2\,(1+b)\,\frac{K_4\,K_6\,E}{(E/G)^2}$$

10. $K_1^2 > 4\,K_2$ 情況下之無量綱常數:

$$\alpha_1 = \pm\sqrt{\frac{K_1}{2} + \sqrt{\left(\frac{K_1}{2}\right)^2 - K_2}}$$

$$\alpha_2 = \pm\sqrt{\frac{K_1}{2} - \sqrt{\left(\frac{K_1}{2}\right)^2 - K_2}}$$

11. $K_1^2 < 4\,K_2$ 情況下之無量綱常數:

$$\alpha_1 = \sqrt{\sqrt{\frac{K_2}{4}} + \frac{K_1}{4}}$$

$$\beta_1 = \sqrt{\sqrt{\frac{K_2}{4}} + \frac{K_1}{4}}$$

$$\gamma_1 = \alpha_1 + i\,\beta_1, \quad \text{複數}$$

$$\mathrm{Re}\,\gamma_1^2 = \alpha_1^2 - \beta_1^2$$

$$\mathrm{Im}\,\gamma_1^2 = 2\,\alpha_1\,\beta_1$$

$$\gamma_1\,\bar{\gamma}_1 = \alpha_1^2 + \beta_1^2$$

$$m = \mathrm{Re}\,\gamma_1^3 = \alpha_1^3 - 3\,\alpha_1\,\beta_1^2$$

$$n = \mathrm{Im}\,\gamma_1^3 = 3\,\alpha_1^2\,\beta_1 - \beta_1^2$$

$$f = \mathrm{Re}\,\gamma_1^4 = \alpha_1^4 + \beta_1^4 - 6\,\alpha_1^2\,\beta_1^2$$

$$g = \mathrm{Im}\,\gamma_1^4 = 4\,\alpha_1\,\beta_1\,(\alpha_1^2 - \beta_1^2)$$

$$X_1 = \mathrm{Re}\,\frac{\gamma_1^3}{(4\,k^4+\gamma_1^4)^2} = \frac{m\,(4\,k^4+f)^2 + 2\,ng\,(4\,k^4+f) - mg^2}{[(4\,k^4+f)^2 + g^2]^2}$$

$$Y_1 = \mathrm{Im}\,\frac{\gamma_1^3}{(4\,k^4+\gamma_1^4)^2} = \frac{n\,(4\,k^4+f)^2 + 2\,mg\,(4\,k^4+f) - ng^2}{[(4\,k^4+f)^2 + g^2]^2}$$

$$X_2 = \mathrm{Re}\,\frac{\gamma_1^3}{4\,k^2+\gamma_1^4} = \frac{4\,k^4 + f}{(4\,k^4+f) + g^2}$$

$$Y_2 = \mathrm{Im}\,\frac{1}{4\,k + \gamma_1^4} = \frac{-g}{(4\,k^4+f)^2 + g^2}$$

$$X_3 = \mathrm{Re}\,\frac{1}{16\,\gamma_1^4+4\,k^4} = \frac{4\,k^4 + 16\,f}{(4\,k^4+16\,f)^2 + 256\,g^2}$$

$$Y_3 = \text{Im}\,\frac{1}{16\,\gamma_1^4+4\,k^4} = \frac{-16\,g}{(4\,k^4+16\,f)^2+256\,g^2}$$

$$X_4 = \text{Re}\,\frac{\gamma_1^4}{(4\,k^4+\gamma_1^4)^2} = \alpha_1\,X_1 - \beta_1\,Y_1$$

$$Y_4 = \text{Im}\,\frac{\gamma^4}{(4\,k^4+\gamma_1^4)} = \alpha_1\,Y_1 + \beta_1\,X_1$$

$$X_5 = \text{Re}\,\frac{\gamma_1}{4\,k^4+\gamma_1^4} = \alpha_1\,X_2 - \beta_1\,Y_2$$

$$Y_5 = \text{Im}\,\frac{\gamma_1}{4\,k^4+\gamma_1^4} = \alpha_1\,Y_2 + \beta_1\,X_2$$

$$X_6 = \text{Re}\,\frac{\gamma_1}{4\,k^4+16\,\gamma_1^4} = \alpha_1\,X_3 - \beta_1\,Y_3$$

$$Y_6 = \text{Im}\,\frac{\gamma_1}{4\,k^4+16\,\gamma_1^4} = \alpha_1\,Y_3 + \beta_1\,X_3$$

三．應力應變關係、平衡方程及連續條件

本文的理論將根據下列假設進行：

1. 三合板梁之寬度甚大(與厚度比較)，因而作爲平面應變處理.
2. 夾心應力應變關係根據 [1] 加上平面應變條件.
3. 表鈑爲受彎曲之薄鈑.
4. 根據第三假設表面中間層之位移將等於夾心與表鈑接觸層之位移.

I. 夾心方程

在平面應變條件下 [1] 提供之夾心方程簡化爲：

$$\frac{d\tau}{dx} = -\frac{1}{h}(\sigma_{z+}-\sigma_{z-}),\qquad (1)$$

$$2(u_+-u_-) + h\frac{d}{dx}(w_++w_-) = -2h\frac{\tau}{G_C} - \frac{h^2}{bEc}\frac{d}{dx}(\sigma_{z+}-\sigma_{z-}),\qquad (2)$$

$$\frac{2Ec}{h}(w_+-w_-) = -(\sigma_{z+}+\sigma_{z-}).\qquad (3)$$

II. 表鈑方程

表鈑爲受撓矩及由夾心剪應力導致之鈑向力共同作用之薄鈑. 下列諸式適用於肱梁或簡支梁跨度中間受集中載荷之情况. 在梁上受分佈載荷 q_+ 時僅在 (5) 式右側加一已知項 q_+，其他一般程序不受影響. 根據圖 3 之標號－微分單元之平衡引出：

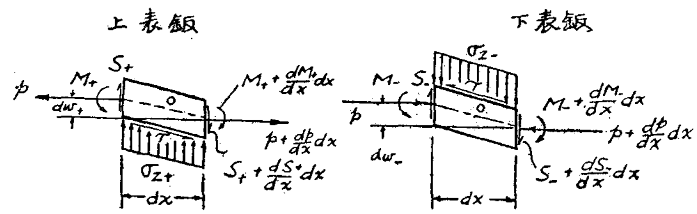

圖 3. 表鈑平衡

$$\Sigma F_x = 0 \qquad \tau\, dx + \frac{dp}{dx}\, dx = 0\,, \tag{4}$$

$$\Sigma F_z = 0 \qquad \frac{dS_\pm}{dx}\, dx \mp \sigma_z\, dx \pm \tau\, dx \left(\frac{dw_\pm}{dx}\right) = 0\,, \tag{5}$$

$$\Sigma M_0 = 0 \qquad \frac{dM_\pm}{dx}\, dx \mp p\, dw_\pm - \frac{t}{2}\tau\, dx + S_\pm\, dx = 0\,. \tag{6}$$

考慮表鈑的彎曲如圖 3 所示,得

$$M_\pm = -D\frac{d^2 w_\pm}{dx^2}\,;$$

其中

$$D = \frac{E_f t^3}{12\,(1-\mu_f^2)}\,. \tag{7}$$

(7) 式代入 (6) 式用 (5) 式消去 $S_\pm$, 得:

$$D\frac{d^4 w_\pm}{dx^4} \pm p\frac{d^2 w_+}{dx^2} \mp \sigma_{z\pm} + \frac{t}{2}\frac{d\tau}{dx} = 0\,. \tag{8}$$

(8) 式用 (4) 式消去 τ 可以寫成另一形式:

$$D\frac{d^4 w_\pm}{dx^4} \pm p\frac{d^2 w_\pm}{dx^2} \mp \sigma_{z\pm} - \frac{t}{2}\frac{d^2 p}{dx^2} = 0\,. \tag{9}$$

III. 連續條件

根據第四假設夾心與上下表鈑之連續要求:

$$\frac{du_\pm}{dx} = \pm\frac{t}{2}\frac{d^2 w_+}{dx^2} \pm \frac{(1-\mu_f^2)}{E_f}\frac{p}{t}\,. \tag{10}$$

四. 三合板梁之微分方程及邊界條件

I. 微分方程

方程 (1), (2), (3), (4), (8) 及 (10) 代表八個方程包含八個變數 ($w_{\pm}$, $u_{\pm}$, $\sigma z_{\pm}$, p 及 τ). 爲了簡化計加以運算得到以 $(w_{+}+w_{-})$, $(w_{+}-w_{-})$ 及 p 元三個方程:

$$D\frac{d^4}{dx^4}(w_{+}+w_{-})+p\frac{d^2}{dx^2}(w_{+}-w_{-})-(h+t)\frac{d^2p}{dx^2}=0, \tag{11}$$

$$D\frac{d^4}{dx^4}(w_{+}-w_{-})+p\frac{d^2}{dx^2}(w_{+}+w_{-})+\frac{2E_c}{h}(w_{+}-w_{-})=0, \tag{12}$$

$$(h+t)\frac{d^2}{dx^2}(w_{+}+w_{-})+\frac{4(1-\mu_f^2)}{tE_f}p- -\frac{2h}{G_c}\frac{d^2p}{dx^2}+\frac{2h}{G_c}\frac{d^2p}{dx^2}-\frac{h^3}{6E_c}\frac{d^4p^2}{dx^4}=0. \tag{13}$$

(11), (12) 及 (13) 式可逕由 [1] 簡化而獲得. 至於 $u_{\pm}$, $\sigma z_{\pm}$, τ, 可再用 (4), (10) 及 (1), (3) 諸式求出.

II. 邊界條件

三合板肱梁的端點條件爲:

在固定端　$x=0$,

$$w_{\pm}=\frac{dw_{\pm}}{dx}=u_{\pm}=0; \tag{14}$$

在載荷端　$x=l$,

$$\frac{d^2w_{\pm}}{dx^2}=p=0; \tag{15}$$

而由 (10) 式

$$\frac{du_{\pm}}{dx}=0. \tag{15a}$$

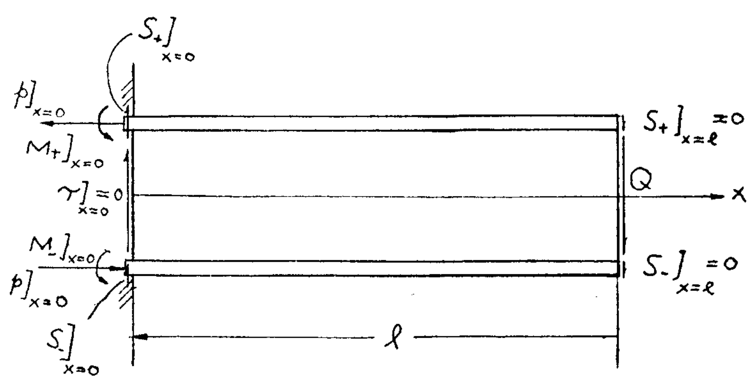

圖 4. 邊界力及力偶

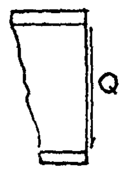

圖 4a.

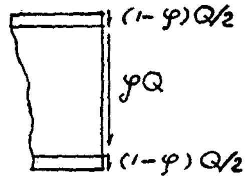

圖 4b.

設在載荷端,載荷 Q 由夾心承受[1] (參看圖 4),則在載荷端: $x = l$,

$$\tau]_{x=l} = \frac{Q}{h}; \qquad S_{\pm}]_{x=l} = 0; \tag{16}$$

在固定端: $x = 0$,

$$\tau]_{x=0} = 0; \qquad D\frac{d^3}{dx^3}(w_+ + w_-)]_{x=0} = Q; \tag{17}$$

$$D\frac{d^2}{dx^2}(w_+ + w_-)]_{x=0} = P(h+t) - Ql.$$

用方程 (15),(16),(6),(7) 可以得到

$$\frac{d^3}{dx^3}(w_+ - w_-)]_{x=l} = 0, \tag{18}$$

$$\frac{d^3}{dx^3}(w_+ + w_-)\Big]_{x=l} = -\frac{t\tau}{D}\Big]_{x=l} = -\frac{Qt}{hD}; \tag{19}$$

而 (4) 式及 (16) 式指出

$$\frac{dp}{dx}\Big]_{x=l} = -\frac{Q}{h}, \tag{20}$$

(1),(2),(14) 及 (17) 式指出:

$$\frac{d^3p}{dx^3}\Big]_{x=0} = 0. \tag{21}$$

III. p 及綜合變數 $(w_+ + w_-)$,$(w_+ - w_-)$ 之微分方程及邊界條件

1) 載荷端不同載荷分佈對於固定端應力的影響: 載荷在末端的分佈一般而言可以不像圖 $4a$ 所示而爲圖 $4b$ 之情況. 但變更 φ 祇影響方程 (51) (52) 或 (66) (67) 之右側. 數字計算指出 φ 並不影響與應力有關之 A_2, B_2, N_3, N_4. 由此可見,倘若僅注意固定端應力中之非線性因素載荷在末端之分佈不同並不重要(這樣的結論與梵南原理並不矛盾).

(11) 式可以加以積分,但需運用 z 方向力之平衡條件:

$$(\sigma_{z+}-\sigma_{z-})\,dx+\frac{d}{dx}\{\tau[h-(w_+-w_-)]\}\,dx-\tau\frac{d}{dx}(w_+-w_-)=0, \quad (22)$$

因而:

$$\left[(\sigma_{z+}-\sigma_{z-})+h\frac{d\tau}{dx}\right]-\left[\frac{d\tau}{dx}(w_+-w_-)+2\tau\frac{d}{dx}(w_+-w_-)\right]=0. \quad (23)$$

由 (4) 式及 (23) 式得

$$\frac{d^2p}{dx^2}(w_+-w_-)+2\frac{dp}{dx}\cdot\frac{d}{dx}(w_+-w_-)=0, \quad (24)$$

(24) 式加 (11) 式積分二度運用端點條件 (17) 得:

$$D\frac{d^2}{dx^2}(w_++w_-)+p(w_+-w_-)-(h+t)p=Q(x-l). \quad (25)^{1)}$$

(12) 式重列於此,

$$D\frac{d^4}{dx^4}(w_+-w_-)+p\frac{d^2}{dx^2}(w_++w_-)+\frac{2E_c}{h}(w_+-w_-)=0. \quad (26)$$

而將 (25) 式代入 (13) 式得:

$$\frac{Dh^3}{6(h+t)E_c}\frac{d^4p}{dx^4}-\frac{2hD}{(h+t)G_c}\frac{d^2p}{dx^2}+$$
$$+\left[(h+t)+\frac{4(1-\mu_f^2)D}{(h+t)tE_f}\right]p=p(w_+-w_-)+Q(l-x). \quad (27)$$

(14—21) 式提供 (w_++w_-), (w_+-w_-), 及 p 之端點條件如次:

$$(w_+-w_-)\Big]_{x=0}=\frac{d}{dx}(w_++w_-)=0; \qquad \frac{d^2}{dx^2}(w_++w_-)\Big]_{x=l}=0,$$
$$\frac{d^3}{dx^3}(w_++w_-)\Big]_{x=l}=-\frac{Qt}{hD}, \quad (28)$$
$$(w_+-w_-)\Big]_{x=0}=\frac{d}{dx}(w_+-w_-)\Big]_{x=0}=0;$$

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1) (25) 式可以從 x 截面以右的力矩平衡得到.

$$\frac{d^2}{dx^2}(w_+ - w_-)\Big]_{x=l} = 0, \qquad \frac{d^3}{dx^3}(w_+ - w_-)\Big]_{x=l} = 0, \tag{29}$$

$$\frac{dp}{dx}\Big]_{x=0} = \frac{d^3}{dx^3}\Big]_{x=0} = 0; \qquad p\Big]_{x=l} = 0, \qquad \frac{dp}{dx}\Big]_{x=l} = -\frac{Q}{h}. \tag{30}$$

五. 非線性微分方程之近似解

方程 (25), (26) 及 (27)俱包含非線性項次. 因之本文採取的載荷參數"ϵ"冪級數方法求出線性範圍以外的解. 同時決定線性理論範疇. 令

$$p = \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + \cdots\cdots, \tag{31}$$

$$w_\pm = \epsilon w_{1\pm} + \epsilon^2 w_{2\pm} + \epsilon^3 w_{3\pm} + \cdots\cdots; \tag{32}$$

其中 $p_1, p_2, p_3, \cdots\cdots$; $w_{1\pm}$, $w_{2\pm}$, $w_{3\pm}$ $\cdots\cdots$ 爲 x 之函數與載荷 Q 無關. 而 $\epsilon = \frac{Q}{hG_c}$ 爲無量綱之載荷參數. 由 (31), (32) 式可見當 $Q \to 0$, p, $w_\pm$ 亦爲零. 但並非載荷之線性函數, 而在 ϵ 甚小時, 問題將近接線性. 由於 Q 易號, 或 ϵ 易號時 p 必易號, 故 (31) 式僅有 ϵ 之奇次項. 而 $w_+(-\epsilon) = -w_-(\epsilon)$, $w_-(-\epsilon) = -w_+(\epsilon)$, 因之 $(w_+ + w_-)$ 爲 ϵ 之奇次級數, $(w_+ - w_-)$ 爲 ϵ 之偶次級數. 亦卽,

$$p_2 = p_4 = p_6 = \cdots = 0, \tag{33}$$

$$(w_{2+} + w_{2-}) = (w_{4+} + w_{4-}) = \cdots = 0, \tag{34}$$

$$(w_{1+} - w_{1-}) = (w_{3+} - w_{3-}) = \cdots = 0. \tag{35}$$

I. 無量綱之微分方程

爲了處理方便計 (25), (26), 及 (27) 三式可以引用無量綱常數 a, b, E, G, μ_f, G 及無量綱變數 ξ, $W_\pm$, P (參看標號) 表示:

$$(W_+ + W_-)'' + 12(1-\mu_f^2)\, G a^3 b^3 P (W_+ - W_-) - \\ - 12(1-\mu_f^2)\, G a^3 b^2 (1+b) P = 12(1-\mu_f^2)\, G a^3 b^3 \epsilon (\zeta - 1), \tag{36}$$

$$(W_+ - W_-)'''' + 12(1-\mu_f^2)\, G a^3 b^3 P (W_+ + W_-) + \\ + 24(1-\mu_f^2)\, E a^4 b^3 (W_+ - W_-) = 0, \tag{37}$$

$$P'''' - 12 a^2 \frac{E}{G} P'' + 72(1-\mu_f^2)\, E a^4 b (b+1)^2 \left(1 + \frac{1}{3(b+1)^2}\right) P = \\ = 72(1-\mu_f^2)\, E a^4 (1+b) b^2 P (W_+ - W_-) + 72(1-\mu_f^2)\, E a^4 b^2 (1-\zeta)\epsilon. \tag{38}$$

無量綱形式之邊界條件爲:

$$(W_+ + W_-)\Big]_{\xi=0} = (W_+ + W_-)'\Big]_{\xi=0} = 0; \quad (W_+ + W_-)''\Big]_{\xi=1} = 0,$$

$$(W_+ + W_-)'''\Big]_{\xi=1} = -12(1-\mu_f^2)\,G\,a^3\,b^2\,\epsilon, \tag{39}$$

$$(W_+ - W_-)\Big]_{\xi=0} = (W_+ - W_-)'\Big]_{\xi=0} = 0; \quad (W_+ - W_-)''\Big]_{\xi=1} = 0,$$

$$(W_+ - W_-)'''\Big]_{\xi=1} = 0. \tag{40}$$

$$P'\Big]_{\xi=0} = P'''\Big]_{\xi=0} = 0; \quad P\Big]_{\xi=1} = 0; \quad P'\Big]_{\xi=1} = -\epsilon. \tag{41}$$

(31) 及 (32) 式無量綱形式爲:

$$P = \epsilon P_1 + \epsilon^3 P_3 + \cdots, \tag{42}$$

$$W_\pm = \epsilon W_{1\pm} + \epsilon^2 W_{2\pm} + \epsilon^3 W_{3\pm} + \cdots. \tag{43}$$

由於引入 (33) 式,故 (42) 式無偶次 ϵ 項.

II. 線性項解

將 (42) 及 (43) 式代入 (36), (37) 及 (38) 式, 令 ϵ 之係數爲零時得到二個微分方程:

$$K_5(W_{1+} + W_{1-})'' = P_1 + \frac{b}{(b+1)}(\xi - 1), \tag{44}$$

$$P_1'''' - K_1 P_1'' + K_2 P_1 = K_3(1-\xi); \tag{45}$$

其中 K_1, K_2, K_3 及 K_5 如標號中之規定.

解 (44) 及 (45) 式必須用 (42) 及 (43) 式代入 (39) 及 (41) 式所得之末端條件

$$P_1'\Big]_{\xi=0} = P'''\Big]_{\xi=0} = 0; \quad P_1\Big]_{\xi=1} = 0, \quad P_1'\Big]_{\xi=1} = -1, \tag{46}$$

$$(W_{1+} + W_{1-})\Big]_{\xi=0} = (W_{1+} + W_{1-})\Big]_{\xi=0} = 0; \quad (W_{1+} + W_{1-})''\Big]_{\xi=1} = 0,$$

$$(W_{1+} + W_{1-})'''\Big]_{\xi=1} = -12(1-\mu_f^2)\,G\,a^3\,b^2. \tag{47}$$

(45) 式之解可以表達如下列形式:

$$P_1 = c_1 e^{\gamma_1 \xi} + \bar{c}_1 e^{\bar{\gamma}_1 \xi} + c_2 e^{\gamma_2 \xi} + \bar{c}_2 e^{\bar{\gamma}_2 \xi} + K_4 (1-\xi), \qquad (48)$$

其中 $c_1 = A_1 + i B_1, \quad c_2 = A_2 + i B_2,$

$$\gamma_1 = \alpha_1 + i \beta_1, \quad \gamma_2 = \alpha_2 + i \beta_2, \qquad (48a)$$

及 $\gamma_1 = -\bar{\gamma}_2,$

符號上之橫線表示爲共軛數.

(a) 當 $K_1^2 < 4K_2$, γ 爲複數

$$\alpha_1 = -\alpha_2 = \sqrt{\left(\frac{K_2}{4}\right)^{1/2} + \left(\frac{K_1}{4}\right)},$$

$$\beta_1 = \beta_2 = \sqrt{\left(\frac{K_2}{4}\right)^{1/2} - \left(\frac{K_1}{4}\right)}. \qquad (48b)$$

(b) 當 $K^2 > 4K_2$, γ 爲實數

$$\alpha_1 = \pm \sqrt{\frac{K_1}{2} + \sqrt{\left(\frac{K_1}{2}\right)^2 - K_2}},$$

$$\alpha_2 = \pm \sqrt{\frac{K_1}{2} - \sqrt{\left(\frac{K_1}{2}\right)^2 - K_2}}.$$

其分界情况 $K_1^2 = 4K_2$ 四常數中有二數消失, 四冪數亦消失其中之二, 此時微分方程之解應爲

$$P_1 = (N_1 + N_2 \xi) e^{\alpha_1 \xi} + (N_3 + N_4 \xi) e^{-\alpha_1 \xi} + K_4 (1-\xi),$$

式中 N_1, N_2, N_3 及 N_4 爲任意常數, $\alpha_1 = \sqrt{\frac{K_1}{2}} = \sqrt{K_2}$.

在實際問題中, E, G, γ 等並不會在數學上造成"恰如"上述條件, 而且數學上也沒有特異, 故此一情况將不加詳細討論.

IIa. $K_1^2 < 4K_2$ 情况下的線性解

爲決定 C_1 及 C_2 二複數. 亦卽四個實數常數 A_1, B_1, A_2 及 B_2, 引用末端條件可得下列四式:

$$\alpha_1 (A_1 - A_2) \qquad - \beta_1 (B_1 + B_2) = \frac{K_4}{2}, \qquad (49)$$

$$(\alpha_1^3-3\alpha_1\beta_1^2)(A_1-A_2)\qquad -(3\alpha_1^2\beta_1-\beta_1^3)(B_1+B_2)=0, \tag{50}$$

$$e^{\alpha_1}\cos\beta_1 A_1-e^{\alpha_1}\sin\beta_1 B_1\qquad +e^{\alpha_1}\cos\beta_1 A_2-e^{-\alpha_1}\sin\beta_1 B_2=0, \tag{51}$$

$$\begin{aligned}&e^{\alpha_1}(\alpha_1\cos\beta_1-\beta_1\sin\beta)A_1-e^{\alpha_1}(\alpha_1\sin\beta_1+\beta_1\cos\beta)B_1-\\&\quad -e^{\alpha_1}(\alpha_1\cos\beta_1+\beta_1\sin\beta_1)A_2+e^{\alpha_1}(\alpha_1\sin\beta-\beta\cos\beta)B_2=\frac{(K_4-1)}{2}.\end{aligned} \tag{52}$$

解 (49),(50) 式中之 A_1-A_2, B_1+B_2, 得

$$(A_1-A_2)=\frac{K_4}{4\alpha_1}\cdot\frac{3\alpha_1^2-\beta_1^2}{\alpha_1^2+\beta_1^2}, \tag{53}$$

$$(B_1+B_2)=\frac{K_4}{4\beta_1}\cdot\frac{\alpha_1^2-3\beta_1^2}{\alpha_1^2+\beta_1^2}. \tag{54}$$

由於

$$\alpha_1=\sqrt{\left(\frac{K_2}{4}\right)^{1/2}+\frac{K_1}{4}}\geqslant\sqrt{\frac{K_1}{2}}=\sqrt{\frac{6E_c}{G_c}}\,a,$$

在一般實用於三合板的 $\alpha_1>10$, 因之包括 $e^{-\alpha_1}$ 項次將可略去[1]. 由此將得到下列結果:

$$A_1=-\frac{\sin\beta_1}{\beta_1}e^{-\alpha}\left(\frac{K_4-1}{2}\right), \tag{55}$$

$$B_1=-\frac{\cos\beta_1}{\beta_1}e^{-\alpha}\left(\frac{K_4-1}{2}\right), \tag{56}$$

$$A_2=\frac{K_4}{4\alpha_1}\frac{\beta_1^2-3\alpha_1^2}{\alpha_1^2+\beta_1^2}, \tag{57}$$

$$B_2=\frac{K_4}{4\beta_1}\frac{\alpha_1^2-3\beta_1^2}{\alpha_1^2+\beta_1^2}, \tag{58}$$

(55),(56),(57),(58) 為近似之解. 若求精確解或 $\beta_1\to 0$ 時,應逕由解(49),(50),(51) 及 (52) 得到 $A_1, A_2, \beta_1 B_1$ 及 $\beta_2 B_2$. 由於 A_1, B_1 包括因子 $e^{-\alpha_1}$, 因而為小量,在以後所述之 (82), (86), (95) 及 (96) 式俱略去. 至於 (94) 為精確解,故相應的該用 A_1, A_2, B_1 及 B_2 之精確值.

解 (44) 式時可用上述 P 的解,並由末端條件

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1) [4] 中曾詳細討論.

$$(W_{1+}+W_{1-})'\Big]_{\xi=0}=0,\qquad (W_{1+}+W_{1-})\Big]_{\xi=0}=0$$

得:

$$K_5(W_{1+}+W_{1-})=2\,\mathrm{Re}\left[\frac{c_1}{\gamma_1^2}(e^{\gamma_1\xi}-1)+\right.$$

$$\left.+\frac{c_2}{\gamma_2^2}(e^{\gamma_2\xi}-1)-\left(\frac{c_1}{\gamma_1}+\frac{c_2}{\gamma_2}\right)\xi\right]+K_6\left(\frac{\xi^3}{6}-\frac{\xi^2}{2}\right);\qquad(59)$$

其中 $K_6=\dfrac{b}{1+b}-K_4$, γ_1, γ_2, c_1 及 c_2 如前述.

由於計算中僅注意 (59) 式之末端值,故此時不將 (59) 改成實數形式.

其餘線性之末端值如次:

$$P_1\Big]_{\xi=0}=2(A_1+A_2)+K_4\doteq K_4\left(1+\frac{\beta_1^2-3\alpha_1^2}{2\alpha_1(\alpha_1^2+\beta_1^2)}\right),\qquad(60)$$

$$(W_{1+}+W_{1-})\Big]_{\xi=1}=-\frac{1}{K_5}\left[\frac{K_6}{3}+2K_4\frac{\alpha_1^2-\beta_1^2}{(\alpha_1^2+\beta_1^2)}+\right.$$

$$\left.+2\frac{A_2(\alpha_1^2-\beta_1^2)B_2(2\alpha_1\beta_1)}{(\alpha_1^2+\beta_1^2)^2}+2\left(\frac{K_4-1}{2\beta_1}\frac{2\alpha_1\beta_1}{(\alpha_1^2+\beta_1^2)^2}\right),\qquad(61)$$

$$(W_{1+}+W_{1-})\Big]_{\xi=0}=-\frac{1}{K_5}[K_6-2(A_1+A_2)]\doteq-\frac{1}{K_5}[K_6-2A_2].\qquad(62)$$

IIb. $K_1^2>4K_2$ 情况下線性解

在此情况下 (45) 式之解將爲:

$$P_1=N_1e^{\alpha_1\xi}+N_2e^{-\alpha_1\xi}+N_3e^{\alpha_2\xi}+N_4e^{-\alpha_2\xi}+K_4(1-\xi),\qquad(63)$$

其中 N_1, N_2, N_3 及 N_4 爲任意實數常數:

應用末端條件得到四個聯立式:

$$\alpha_1N_1-\alpha_1N_2+\alpha_2N_3-\alpha_2N_4=K_4,\qquad(64)$$

$$\alpha_1^3N_1-\alpha_1^3N_2+\alpha_2^3N_3-\alpha_2^3N_4=0,\qquad(65)$$

$$e^{\alpha_1}N_1+e^{\alpha_1}N_2+e^{\alpha_2}N_3+e^{-\alpha_2}N_4=0,\qquad(66)$$

$$\alpha_1e^{\alpha_1}N_1-\alpha_1e^{-\alpha_1}N_2+\alpha_2e^{\alpha_2}N_3+\alpha_2e^{-\alpha_2}N_4=(K_4-1);\qquad(67)$$

解之得:

$$N_1=\frac{[\alpha_1^3\alpha_2-\alpha_2^3e^{-\alpha_1}(\alpha_1\cosh\alpha_2+\alpha_2\sinh\alpha_2)]K_4-\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2)\cosh\alpha_2(K_4-1)}{2\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2)(\alpha_2\sinh\alpha_2\cosh\alpha_1-\alpha_1\sinh\alpha_1\cosh\alpha_2)}, \quad (68)$$

$$N_2=\frac{[\alpha_2^3\alpha_1-\alpha_1^3e^{-\alpha_1}(\alpha_2\sinh\alpha_2-\alpha_1\cosh\alpha_2)]K_4-\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2)\cosh\alpha_2(K_4-1)}{2\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2)(\alpha_2\sinh\alpha_2\cosh\alpha_1-\alpha_1\sinh\alpha_1\cosh\alpha_2)}, \quad (69)$$

$$N_3=\frac{[\alpha_2^3\alpha_1-\alpha_1^3e^{-\alpha_2}(\alpha_1\sinh\alpha_1+\alpha_2\cosh\alpha_1)]K_4-\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2)\cosh\alpha_1(K_4-1)}{2\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2)(\alpha_2\sinh\alpha_2\cosh\alpha_1-\alpha_1\sinh\alpha_1\cosh\alpha_2)}, \quad (70)$$

$$N_4=\frac{[\alpha_2^3\alpha_1+\alpha_1^3e^{+\alpha_2}(\alpha_1\sinh\alpha_1-\alpha_2\cosh\alpha_1)]K_4+\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2)\cosh\alpha_1(K_4-1)}{2\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2)(\alpha_2\sinh\alpha_2\cosh\alpha_1-\alpha_1\sinh\alpha_1\cosh\alpha_2)}. \quad (71)$$

而在 $K_1^2>4K_2$ 情況下可得:

$$K_5(W_{1+}+W_{1-})=\frac{N_1}{\alpha_1^2}(e^{\alpha_1\xi}-1)+\frac{N_2}{\alpha_1^2}(e^{-\alpha\xi}-1)+\frac{N_3}{\alpha_2^2}(e^{\alpha_2\xi}-1)+$$
$$+\frac{N_4}{\alpha_2^2}(e^{-\alpha_2\xi}-1)-\left(\frac{N_1-N_2}{\alpha_1}+\frac{N_3-N_4}{\alpha_2}\right)\xi+K_6\left(\frac{\xi^3}{6}-\frac{\xi^2}{2}\right), \quad (72)$$

$$P_1\Big]_{\xi=0}=N_1+N_2+N_3+N_4+K_4, \quad (73)$$

$$(W_{1+}+W_{1-})\Big]_{\xi=1}=-\frac{1}{K_5}\left\{\frac{K_6}{3}-\frac{N_1e^{\alpha_1}+N_2e^{-\alpha_1}}{\alpha_1^2}-\frac{N_3e^{\alpha_2}+N_4e^{-\alpha_2}}{\alpha_2^2}+\right.$$
$$\left.+\frac{N_1+N_2}{\alpha_1^2}+\frac{N_3+N_4}{\alpha_2^2}+\frac{K_4}{\alpha_1^2-\alpha_2^2}\left[\left(\frac{\alpha_1}{\alpha_2}\right)^2-\left(\frac{\alpha_2}{\alpha_1}\right)^2\right]\right\}, \quad (74)$$

$$(W_{1+}+W_{1-})''\Big]_{\xi=0}=\frac{1}{K_5}(N_1+N_2+N_3+N_4-K_6). \quad (75)$$

IIIa. $K_1^2<4K_2$ 情況下的非線性解

將 (43) 式代入 (36) 式並將 ϵ^2 項之係數令爲零,得:

$$(W_{2+}-W_{2-})''''+4k^4(W_{2+}-W_{2-})=-12(1-\upsilon_f^2)\cdot Ga^3b^3P_1(W_{1+}+W_{1-})''; \quad (76)$$

其中 $4k^4$ 如標號中所規定:

末端條件可由 (43) 式及 (40) 式求得:

$$(W_{2+}-W_{2-})\Big]_{\xi=0}=(W_{2+}-W_{2-})'\Big]_{\xi=0}=0;$$

$$(W_{2+}-W_{2-})\Big]_{\xi=1}=(W_{2+}-W_{2-})'''\Big]_{\xi=1}=0. \quad (77)$$

將 (48) 式及 (44) 式代入 (76) 式,得

$$(W_{2+}-W_{2-})'''' + 4k^2 (W_{2+}-W_{2-}) = K_7 P_1^2 + K_8 (\xi - 1) P_1; \tag{78}$$

其中
$$K_7 = -144 (1 - v_f^2)^2 a^6 b^5 (b + 1) G^2,$$

$$K_8 = 144 (1 - v_f^2)^2 a^6 b^6 G^2.$$

(78) 式之解可以分爲二部,一爲一般解,一爲特殊解:

$$(W_{2+} - W_{2-}) = (W_{2+} - W_{2-})_c + (W_{2+} - W_{2-})_p,$$
$$(W_{2+} - W_{2-})_c = M_1 \cosh k\xi \cos k\xi + M_2 \cosh k\xi \sin k\xi + $$
$$+ M_3 \sinh k\xi \cos k\xi + M_4 \sinh k\xi \sin k\xi, \tag{79}$$

$$\begin{aligned}(W_{2+}-W_{2-})_p = 2K_9 \,\mathrm{Re}\Big\{&C_1 e^{\gamma_1\xi} \frac{1}{4k^4+\gamma_1^4}\Big(\xi - \frac{4\gamma_1^3}{4k^4+\gamma_1^4}\Big) - C_1 \frac{e^{\gamma_1\xi}}{4k^4+\gamma_1^4} + \\ &+ C_2 e^{\gamma_2\xi} \frac{1}{4k^4+\gamma^4}\Big(\xi - \frac{4\gamma_2^3}{4k^4+\gamma_2^4}\Big) - C_2 \frac{e^{\gamma_1\xi}}{4k^4+\gamma_2^4}\Big\} + \\ &+ K_7 \Big\{2\,\mathrm{Re}\Big[\frac{C_1^2 e^{2\gamma_1\xi}}{16\gamma_1^4+4k^4} - \frac{C_2^2 e^{2\gamma_2\xi}}{16\gamma_2^4+4k^4}\Big] + \\ &+ \frac{1}{4\alpha_1^4+k^4}\Big[\frac{A_1^2+B_1^2}{2} e^{2\alpha_1\xi} + \frac{A_2^2+B_2^2}{2} e^{-2\alpha_1\xi}\Big] + \\ &+ \frac{1}{4\beta^4+k^4}\Big[(A_1A_2-B_1B_2)\cos 2\beta_1\xi - (A_1B_2+B_1A_2)\sin 2\beta\xi\Big] + \\ &+ \frac{A_1A_2+B_1B_2}{k^4}\Big\} - K_{10}\Big\{1 - 2\xi + \xi^2\Big\}. \end{aligned} \tag{80}$$

引入標號中所示實數 $X_1, Y_1, \cdots, X_6, Y_6$ 代表複數常數之實數及虛數部分. 應用末端條件

$$(W_{2+} - W_{2-})]_{\xi=0} = 0,$$

由此得出
$$M_1 = \delta_1. \tag{81}$$

而 $\delta_1 = -[W_{2+} - W_{2-})]_{\xi=0}$. 如 IIa 中略去小量諸項

$$\begin{aligned} = K_{10} - 2K_9 &\Big[A_2 (4X_1 - X_2) + B_2 (4Y_1 - Y_2)\Big] \\ &- 2K_7 \Big[(A_2^2 - B_2^2) X_3 + 2A_2 B_2 Y_3 + \frac{A_2^2+B_2^2}{4}\Big(\frac{1}{4\alpha_1^4+k^4}\Big)\Big]. \end{aligned} \tag{82}$$

同理應用末端條件

$$(W_{2+}-W_{2-})'\Big]_{\xi=0}=0,\ (W_{2+}-W_{2-})''\Big]=0\ 及\ (W_{2+}-W_{2-})'''\Big]=0,$$

得
$$M_2+M_3=\delta_2, \tag{83}$$

$$M_4\cosh k\cos k+M_2\sinh k\cos k-M_3\cosh k\sin k-M_1\sinh k\sin k=\delta_3, \tag{84}$$

$$(M_2-M_3)\cosh k\cos k+(M_4-M_1)\sinh k\cos k-(M_1+M_4)\cosh k\sin k+ \\ +(M_2+M_3)\sinh k\sin k=\delta_4, \tag{85}$$

其中
$$\delta_2=-(W_{2+}-W_{2-})'_p\Big]_{\xi=0}= \\ =\frac{2}{k}\Big\{K_9\Big[A_2(4X_4-X_5-X_2)+B_2(4Y_4-Y_5-Y_2)\Big]+ \\ +2K_7\Big[(A_2^2-B_2^2)X_6+2A_2B_2Y_6+\frac{\alpha_1(A_2^2+B_2^2)}{4(4\alpha_1^4+k^4)}\Big]- \\ -K_{10}\Big\}, \tag{86}$$

$$\delta_3=-(W_{2+}-W_{2-})''_p\Big]_{\xi=1},$$

$$k^2\cdot\delta_3=2K_9\left(\frac{K_4-1}{2\beta}\right)\Big\{2\Big[(\alpha_1^2-\beta_1^2)X_1-2\alpha_1\beta_1Y_1\Big]-(\alpha_1Y_2+\beta_1X_2)\Big\} \\ +2K_7\left(\frac{K_4-1}{2\beta}\right)^2\Big\{2\Big[(\alpha_1^2-\beta_1^2)X_3-2\alpha_1\beta_1Y_3\Big]-\frac{\alpha_1^4}{4\alpha_1^4+k^4}, \tag{87}$$

$$\delta_4=-(w_{2+}-w_{2-})'''_p\Big]_{\xi=1},$$

$$k^3\cdot\delta_4=K_9\left(\frac{K_4-1}{2\beta_1}\right)\Big\{4(mX_1-nY_1)-3\Big[(\alpha_1^3-\beta_1^3)Y_2+2\alpha_1\beta_1X_2\Big]\Big\} \\ +4K_7\left(\frac{K_4-1}{2\beta_1}\right)^2\Big\{2(mX_3-nY_3)-\frac{\alpha_1^3}{4\alpha^4+k^4}. \tag{88}$$

解 (81),(83),(84),(85) 四聯立式中之 M_1, M_2, M_3, M_4, 得

$$M_1=\delta_1, \tag{89}$$

$$M_2=\frac{\delta_2\cosh^2 k+\delta_1[\cosh k\sinh k+\cos k\sin k]-\delta_3[\sinh k\cos k-\cosh k\sin k]+\delta_4\cos k\cosh k}{(\cosh^2 k+\cos^2 k)} \tag{90}$$

$$M_3=\frac{\delta_2\cos^2 k-\delta_1[\cosh k\sinh k+\cos k\sin k]+\delta_3[\sinh k\cos k-\cosh k\sin k]-\delta_4\cosh k\cos k}{(\cosh^2 k+\cos^2 k)} \tag{91}$$

$$M_4=\frac{2\delta_3\cosh k\cos k-\delta_1[\cosh^2 k\sin^2 k+\sinh^2 k\sin^2 k]+\delta_2[\sin k\cos k-\sin k\cosh k]-\delta_4(\cosh k\sin k+\sinh k\cos k)}{(\cosh^2 k+\cos^2 k)}; \quad (92)$$

但由於 k 之數值一般大於 5，則 (89),(90),(91),(92) 可以近似取爲:

$$M_1=\delta_1\,,\ M_2=\delta_1+\delta_2\,,\ M_3=-\delta_3\,,\ M_4=-(\delta_1+\delta_2)\,. \quad (93)$$

整理常數由 (79) 及 (80) 式可得精確表達式:

$$\begin{aligned}(W_{2+}-W_{2-})=&M_1\cosh k\xi\cos k\xi+M_2\cosh k\xi\sin k\xi+M_3\sinh k\xi\cos k\xi+\\&+M_4\sinh k\xi\sin k\xi+\\&+2K_9\{e^{\alpha_1\xi}[(A_1\cos\beta_1\xi-B_1\sin\beta_1\xi)(X_2\xi-4X_1-X_2)-\\&-(A_1\sin\beta_1\xi+B\cos\beta\xi)(Y_2\xi-4Y_1-Y_2)+\\&+e^{-\alpha_1\xi}[(A_2\cos\beta_1\xi-B_2\sin\beta_1\xi)(X_2\xi-4X_1-X_2)+\\&+(A_2\sin\beta_1\xi+B_2\cos\beta\xi)(Y_2\xi+4Y_1-Y_2)]\}+\\&+K_7\Big\{2e^{2\alpha_1\xi}\Big[[(A_1^2-B_1^2)\cos 2\beta_1\xi-2A_1B_1\sin 2\beta_1\xi]X_3-\\&-[(A_2^2+B_2^2)\sin\beta_1\xi+2A_1B_1\cos\beta\xi]Y_3\Big]+\\&+2e^{-2\alpha_1\xi}\Big[[(A_2^2-B_2^2)\cos 2\beta_1\xi-2A_2B_2\sin 2\beta_1\xi]X_3+\\&+[(A_2^2-B_2^2)\sin 2\beta_1\xi+2A_2B_2\sin\beta_1\xi]Y_3\Big]+\\&+\frac{1}{4\alpha_1^4+k^4}\Big[\frac{1}{2}(A_1^2+B_1^2)e^{2\alpha_1\xi}+\frac{1}{2}(A_2^2+B_2^2)e^{-2\alpha_1\xi}\Big]+\\&+\frac{1}{4\beta_1^4+k^4}(A_1A_2-B_1B_2)\cos 2\beta_1\xi-(A_1B_2+B_1A_2)\sin 2\beta_1\xi]+\\&+\frac{A_1A_2+B_1B_2}{k^4}\Big\}+\\&+K_{10}\{1+2\xi+\xi^2\}.\end{aligned} \quad (94)$$

微分 (94) 式二次，注意 $A_1\cos\beta_1=B_1\sin\beta$，而 $\xi\to 1$，$C_2e^{\gamma\xi}$ 甚小，同理 $C_2^2e^{2\gamma_2\xi}$ 甚小，由此可得:

$$\begin{aligned}(W_{2+}-W_{2-})\Big]_{\xi=1}=&-8K_9\Big(\frac{K_4-1}{2\beta_1}\Big)Y_1-K_7\Big(\frac{K_4-1}{2\beta_1}\Big)^2\Big[2X_3-\frac{1}{2(4\alpha_1^4+k^4)}\Big]+\\&+\cosh k[(M_1+M_3)\cos k+(M_2+M_4)\sin k]\,,\end{aligned} \quad (95)$$

$$\begin{aligned}(W_{2+}-W_{2-})''\Big]_{\xi=0}=&2k^2M_4-2K_{10}+\\&+2K_9\{(4X_1-X_2)[(\alpha_1^2-\beta_1^2)A_2+2\alpha_1\beta_1B_2]-\end{aligned}$$

$$-2X_2(A_2\alpha_1+\beta_1 B_2)-2Y(B_2\alpha_1-A_2\beta_1)-$$

$$-(4Y_1-Y_2)[(\beta_1^2-\alpha_1^2)B_2+2\alpha_1\beta_1A_2]\}+$$

$$+2K_7\left\{8X_3[(\alpha_1^2-\beta_1^2)(A_2^2-B_2^2)+4A_2B_2\alpha_1\beta_1]+\frac{2\alpha_1^2(A_2^2+B_2^2)}{A\alpha_1^4+k^4}-\right.$$

$$\left.-8Y_3[(\beta_1^2-\alpha_1^2)\alpha A_2B_2+2\alpha_1\beta_1(A_2^2-B_2^2)]\right\}. \tag{96}$$

IIIb. $K_1^2>4K_2$ 情况下非線性解

按照 IIIa 辦法唯一與 IIIa 不同處本節引用 IIb 之結果此時:

$$(W_{2+}-W_{2-})p$$

$$=K_9\left\{(\xi-1)\left[\frac{N_1e^{\alpha_1\xi}+N_2e^{-\alpha_1\xi}}{\alpha_1^4+4k^4}+\frac{N_3e^{\alpha_2\xi}+N_4e^{-\alpha_2\xi}}{\alpha_2^4+4k^4}\right]-\right.$$

$$\left.-\frac{4\alpha_1^3(N_1e^{\alpha_1\xi}-N_2e^{-\alpha_1\xi})}{(4k^4+\alpha_1^4)^2}-\frac{4\alpha_2^3(N_3e^{\alpha_2\xi}-N_4e^{-\alpha_1\xi})}{(4k^4+\alpha_2^4)^2}\right\}+$$

$$+K_7\left\{2\frac{N_1N_3e^{(\alpha_1+\alpha_2)\xi}+N_2N_4e^{-(\alpha_1+\alpha_2)\xi}}{(\alpha_1+\alpha_2)^4+4k^4}-\right.$$

$$-2\frac{N_1N_4e^{(\alpha_1-\alpha_2)\xi}+N_2N_3e^{-(\alpha_1-\alpha_2)\xi}}{(\alpha_1-\alpha_2)^4+4k^4}+\frac{N_1^2e^{2\alpha_1\xi}+N_2^2e^{-2\alpha\xi}}{16\alpha_1^4+4k^4}+$$

$$\left.+\frac{N_3^2e^{2\alpha_2\xi}+N_4e^{-2\alpha_2\xi}}{16\alpha_2^4+4k^4}+2\frac{N_1N_2+N_3N_4}{4k^4}\right\}-$$

$$-K_{10}\{1-2\xi+\xi^2\}, \tag{97}$$

其中 K_9, K_7, K_{10} 如標號所示. 而 $(w_{2+}-w_{2-})\,c$ 之形式與 (79), (81), (83), (84), 及 (85) 相同,但:

$$\delta_1=K_9\left\{\frac{4\alpha_1^3(N_1-N_2)+(\alpha_3^2+4k^4)(N_1+N_2)}{(\alpha_1^4+4k^4)^2}+\right.$$

$$\left.+\frac{4\alpha_2^3(N_3-N_4)+(\alpha_2^4+4k^4)(N_2+N_4)}{(\alpha_2^4+4k^4)^2}\right\}-$$

$$-K_7\left\{\frac{N_1^2+N_2^2}{16\alpha_1^4+4k^4}+\frac{N_3^2+N_4^2}{16\alpha_2^4+4k^4}+\frac{2(N_1N_3+N_2N_4)}{(\alpha_1+\alpha_2)^4+4k^4}+\right.$$

$$\left.+\frac{2(N_1N_4+N_2N_3)}{(\alpha_1-\alpha_2)^4+4k^4}+\frac{2(N_1N_2+N_3N_4)}{4k^4}\right\}+K_{10}, \tag{98}$$

$$k\delta_2=K_9\left\{\frac{\alpha_1(N_1-N_2)}{\alpha_1^4+4k^4}+\frac{\alpha_2(N_3-N_4)}{\alpha_2^4+4k^4}+\right.$$

$$\left.+\frac{(3\alpha_1^4-4k^4)(N_1+N_2)}{(\alpha_1^4-4k^4)^2}+\frac{(3\alpha_2^4-4k^4)(N_3+N_4)}{(\alpha_2^4+4k^4)^2}\right\}-$$

$$-K_7\left\{\frac{2(\alpha_1+\alpha_2)(N_1N_3-N_2N_4)}{(\alpha_1+\alpha_2)^4+4k^4}-\frac{2(\alpha_1-\alpha_2)(N_1N_4-N_2N_3)}{(\alpha_1-\alpha_2)^4+4k^4}+\right.$$

$$\left.+\frac{2\alpha_1(N_1^2+N_2^2)}{16\alpha_1^4+4k^4}+\frac{2\alpha_2(N_3^2-N_4^2)}{16\alpha_2^4+4k^4}\right\}-2K_{10}, \tag{99}$$

$$k^2\delta_3=K_9\left\{\frac{\alpha_1(\alpha_1^4-4k^4)(N_1e^{\alpha_1}N_2e^{-\alpha_1})}{(\alpha_1^4+4k^4)^2}+\frac{\alpha_2(\alpha_2^4-4k^4)(N_3e^{\alpha_2}-N_4{}^{-\alpha_2})}{(\alpha_2^4+4k^4)^2}\right\}-$$

$$-K_7\left\{\frac{(\alpha_1+\alpha_2)(N_1N_3e^{(\alpha_1+\alpha_2)}+N_2N_4e^{-(\alpha_1+\alpha_2)}}{(\alpha_1+\alpha_2)^4+4k^4}+\right.$$

$$+\frac{(\alpha_1-\alpha_2)^2(N_1N_4e^{(\alpha_1+\alpha_2)}+N_2N_3e^{-(\alpha_1+\alpha_2)})}{(\alpha_1-\alpha_2)^4+4k^4}+$$

$$\left.+\frac{2\alpha_1^2(N_1^2e^{2\alpha_1}+N_2^2e^{-2\alpha_1})}{16\alpha_1^4+4k^4}-\frac{2\alpha_2^2(N_3^2e^{2\alpha_2}+N_4^2e^{-2\alpha_2})}{16\alpha_2^4+4k^4}\right\}+K_{10}, \tag{100}$$

$$2k^3\delta_4=K_9\left\{\frac{\alpha_1^2(\alpha_1^4-12k^4)(N_1e^{\alpha_1}+N_2e^{-\alpha_1})}{(\alpha_1^4+4k^4)^2}+\frac{\alpha_2^2(\alpha_2^4-12k^4)(N_3e^{\alpha_2}+N_4e^{-\alpha_2})}{(\alpha_2^4+4k^4)^2}\right\}-$$

$$-K_7\left\{\frac{2(\alpha_1+\alpha_2)^3(N_1N_3e^{(\alpha_1+\alpha_2)}-N_2N_4e^{-(\alpha_1+\alpha_2)})}{(\alpha_1+\alpha_2)^4+4k^4}+\right.$$

$$+\frac{2(\alpha_1-\alpha_2)^3(N_1N_4e^{(\alpha_1-\alpha_2)}-N_2N_3e^{-(\alpha_1-\alpha_2)})}{(\alpha_1-\alpha_2)^4+4k^4}+$$

$$\left.+\frac{8\alpha_1^3(N_1^2e^{2\alpha_1}-N_2^2e^{-2\alpha_1})}{16\alpha_1^4-4k^4}+\frac{8\alpha_2^3(N_3^2e^{2\alpha_2}-N_4^2e^{-2\alpha_2})}{16\alpha_2^4+4k^4}\right\}. \tag{101}$$

$(W_{2+}-W_{2-})$ 之精確解可以示爲:

$$(W_{2+}-W_{2-})=(W_{2+}-W_{2-})_c+(W_{2+}-W_{2-})_p$$
$$=M_1\cosh k\xi\cos k\xi+M_2\cosh k\xi\sin k\xi+M_3\sin k\xi\cos k\xi+$$
$$+M_4\sinh k\xi\sin k\xi+(91)\text{ 式之右側}. \tag{102}$$

相當於 (95) 及 (96) 式之末端值爲:

$$(W_{2+}-W_{2-})\Big]_{\xi=1}=-K_9\left\{\frac{4\alpha_1^3(N_1e^{\alpha_1}-N_2e^{-\alpha_1})}{(4k^4+\alpha_1^4)^2}+\frac{4\alpha_2^3(N_3e^{\alpha_2}-N_4e^{-\alpha_2})}{(4k^4+\alpha_2^4)^2}\right\}+$$

$$+K_7\left\{\frac{2(N_1N_3e^{(\alpha_1+\alpha_2)}+N_2N_4e^{-(\alpha_1+\alpha_2)})}{(\alpha_1-\alpha_2)^4+k^4}+\right.$$

$$+\frac{2(N_1N_4e^{(\alpha_1-\alpha_2)}+N_2N_3e^{-(\alpha_1-\alpha_2)})}{(\alpha_1-\alpha_2)^4+k^4}+$$

$$+\frac{N_1^2 e^{2\alpha_1}+N_2^2 e^{-2\alpha_1}}{16\alpha_1^4+4k^4}+\frac{N_3^2 e^{2\alpha_2}+N_4^2 e^{-2\alpha_2}}{16\alpha_2^4+4k^4}+\frac{2(N_1N_2+N_3N_4)}{4k^4}\Bigg\}+$$
$$+\cosh k\,[(M_1+M_3)\cos k+(M_2+M_4)\sin k], \tag{103}$$

$$(W_{2+}-W_{2-})''\Big]_{\xi=0}=2k^2M_4$$
$$+K_9\left\{\frac{2\alpha_1(4k^4-\alpha_1^4)(N_1-N_2)}{(\alpha_1^4+4k^4)^2}+\frac{2\alpha_2(4k^4-\alpha_2^4)(N_3-N_4)}{(\alpha_2^4+4k^4)^2}-\right.$$
$$\left.-\frac{\alpha_1^2(N_1+N_2)}{\alpha_1^4+4k^4}-\frac{\alpha_2^2(N_3+N_4)}{\alpha_2^4+4k^4}\right\}+$$
$$+K_7\left\{\frac{2(\alpha_1+\alpha_2)^2(N_1N_3+N_2N_4)}{(\alpha_1+\alpha_2)^4+4k^4}+\frac{2(\alpha_1-\alpha_2)^2(N_1N_4+(N_2N_3)}{(\alpha_1-\alpha_2)^4+4k^4}+\right.$$
$$\left.+\frac{4\alpha_1^2(N_1^2+N_2^2)}{16\alpha_1^4+4k^4}+\frac{4\alpha_2^2(N_3^2+N_4^2)}{16\alpha_2^4+4k^4}\right\}-2K_{10}. \tag{104}$$

IV. 理論結果及計算公式

從以上所得結果,已經求出:

$$\begin{aligned}P&=\epsilon P_1\\(W_+-W_-)&=\epsilon^2(W_{2+}-W_{2-})\\(W_++W_-)&=\epsilon(W_{1+}+W_{1-})\\W_+&=\epsilon\left(\frac{W_{1+}+W_{1-}}{2}\right)+\epsilon\left(\frac{W_{2+}-W_{2-}}{2}\right)\\W_-&=\epsilon\left(\frac{W_{1+}+W_{1-}}{2}\right)-\epsilon^2\left(\frac{W_{2+}-W_{2-}}{2}\right).\end{aligned} \tag{105}$$

爲了計算便利起見,線性部分結果可以寫成下列計算公式:

1. 固定端的表鈑平均應力:

$$\sigma_p=\frac{p}{t}=\psi_p\frac{Ql}{(h+t)}, \tag{106}$$

其中 $\psi_p=\frac{b+1}{b}P_1$, P_1 可用 (60) 或 (73) 式計算.

2. 固定端的表鈑最大彎曲正應力:

$$\sigma_1=\frac{M_{1+}]_{x=0}}{t^2/6}=\psi_1\frac{3Ql}{2t^2}, \tag{107}$$

其中 $\psi_1=K_5\frac{2(b+1)}{b}[W_{1+}+W_{1-}]''_{\xi=0}$ 可以根據 (62) 或 (75) 式計算.

3. 截荷端之撓度：

$$(w_+ + w_-)/2\Big]_{x=l} = \psi_w \frac{2(1-\mu_f^2)\,Ql^3}{3Ef\,(h+t)^2\,t}, \tag{108}$$

其中 $\psi_w = \frac{9(b+1)^3}{b} K_5[W_{1+} + W_{1-}]_{\xi=1}$ 可以根據（61）或（74）式計算.

至於非線性解的結果可以表達如下：

1. 截荷端之撓度：

$$(w_+ + w_-)/2\Big]_{x=l} = (1+\Omega)\,\psi_w \frac{2(1-\mu_f^2)\,Ql^3}{3Ef\,(h+t)^2\,t}, \tag{109}$$

其中[1] $\Omega = \left.\frac{\epsilon^2\left(\frac{W_{2+}+W_{2-}}{2}\right)}{\epsilon\left(\frac{W_{1+}+W_{1-}}{2}\right)}\right]_{\xi=1}$ 可以根據（95），（61）或（103），（74）式計算.

Ω 表出與線性結果之差別.

2. 固定端之最大正應力：

$$\sigma = \sigma_p + \sigma_1(1+\lambda),$$

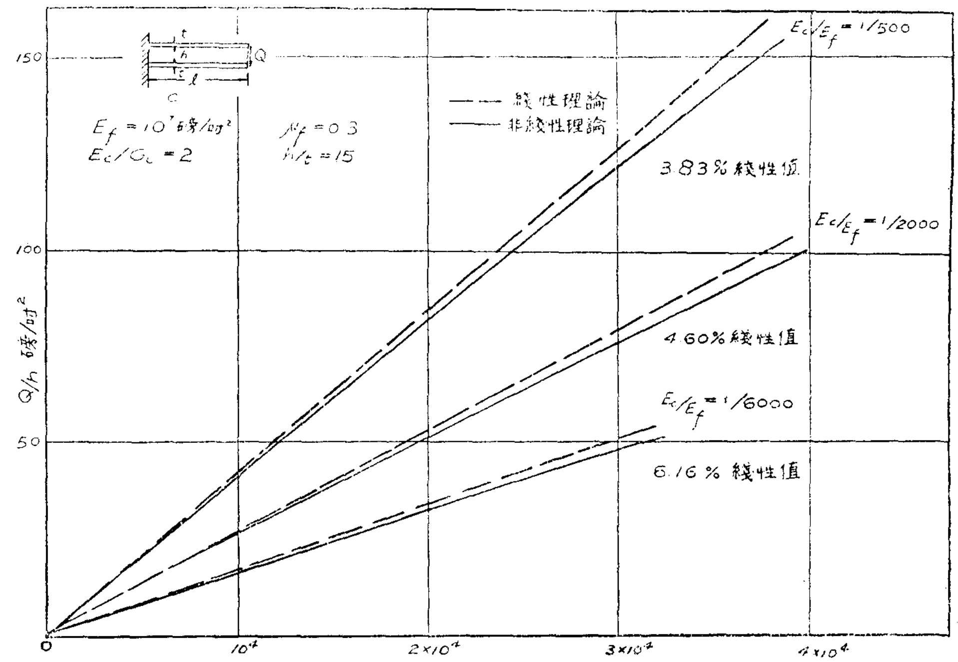
其中 $\lambda = \left.\frac{\epsilon\left(\frac{W_{2+}-W_{2-}}{2}\right)''}{\epsilon\left(\frac{W_{1+}+W_{1-}}{2}\right)''}\right]_{\xi=0}$ 可以根據（96），（62）或（104），（75）式計算.

$\lambda\,\sigma_1$ 表出非線性彎曲正應力

六. 數字計算及實驗驗證

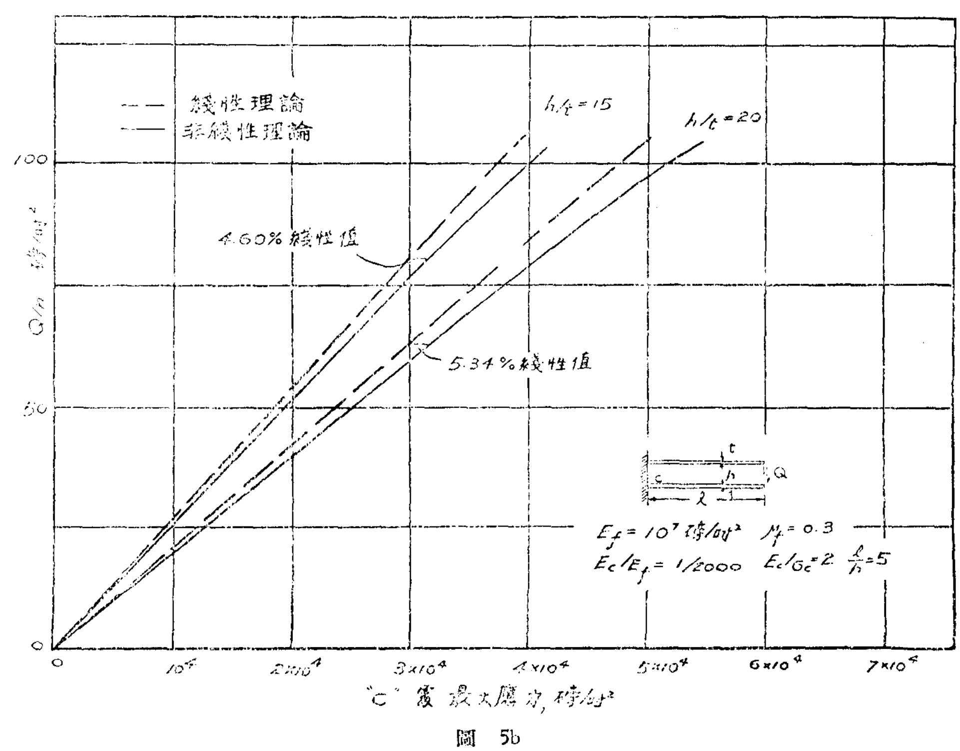
根據 $E_c/G_c = 2$ 之情況（接近於各向同性材料中泊松比爲零之情況. 有關三合板的著作中不少是根據這樣情況分析的）進行計算可以得到圖 5a,5b,5c, 圖表示線性理論及非線性理論. 圖 5a 示出非線性的表鈑應力在總應力中的百分比隨夾心彈性係數之降低而增加. 圖 5b,c 示出非線性因素隨厚度比 h/t 增加. 由此可見在 $h/t > 40$ 時應包括至少 5% 至 10% 之非線性部分，也就是線性理論將不復精確.

1) [4]中曾將現用三合板之實用數字分析一般較長之三合板得出結論 Ω 小於 2.3%，但此項結論並不導致三合板梁之撓度將爲線性性質. 例如圖 8 中由於 B 處的斜度包含有非線性性質因而 A 處之撓度將有顯著的非線性性質.



150
100
50
0
Q/h 磅/吋²
$E_f = 10^7$ 磅/吋²　$\mu_f = 0.3$
$E_c/G_c = 2$　$h/t = 15$
—— 線性理論
—— 非線性理論
$E_c/E_f = 1/500$
3.83% 線性值
$E_c/E_f = 1/2000$
4.60% 線性值
$E_c/E_f = 1/6000$
6.16% 線性值
10^4
2×10^4
3×10^4
4×10^4
"c" 處最大應力 磅/吋²

圖 5a



--- 線性理論
—— 非線性理論
$h/t = 15$
$h/t = 20$
100
50
0
Q/h 磅/吋²
4.60% 線性值
5.34% 線性值
$E_f = 10^7$ 磅/吋²　$\mu_f = 0.3$
$E_c/E_f = 1/2000$　$E_c/G_c = 2$　$l/h = 5$
10^4
2×10^4
3×10^4
4×10^4
5×10^4
6×10^4
7×10^4
"c" 處最大應力, 磅/吋²

圖 5b

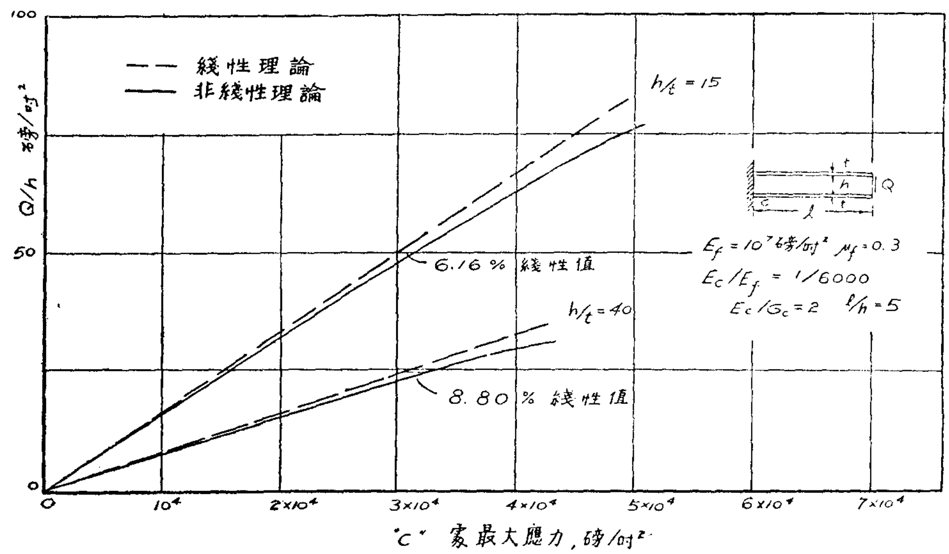


圖 5c

以下列數據進行計算所得結果為:

$\dfrac{h}{t} = 400$, $E_c/E_f = \dfrac{1}{6000}$, $E_c/G_c = 2$, 最大應力 $= 30{,}000$ 磅/吋^2, 非線性應力/線性應力 $= 1.49$.

$\dfrac{h}{t} = 60$, $E_c/E_f = \dfrac{1}{500}$, $E_c/G_c = 2$, 最大應力 $= 30{,}000$ 磅/吋^2, 非線性應力/線性應力 $= 0.81$.

由於以上計算非線性成分甚大. 本文分析方法並不易肯定精確程度. 但顯然由此說明線性理論的應用在以上的情況下是並不合宜的.

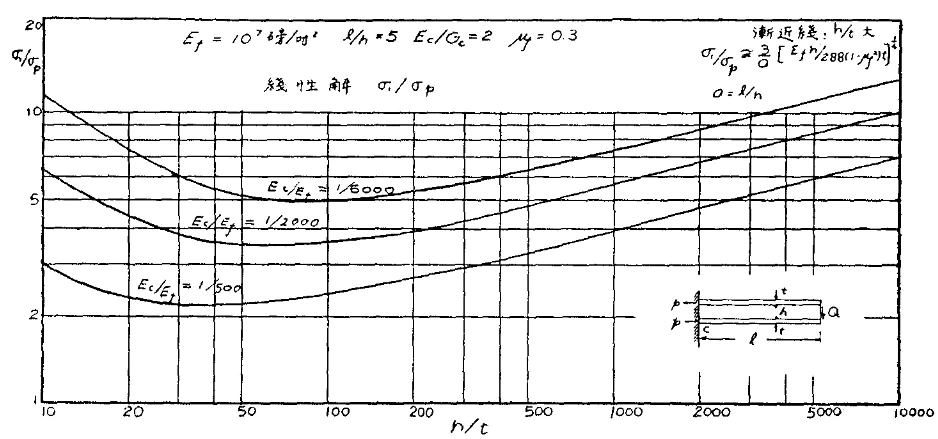


圖 6a

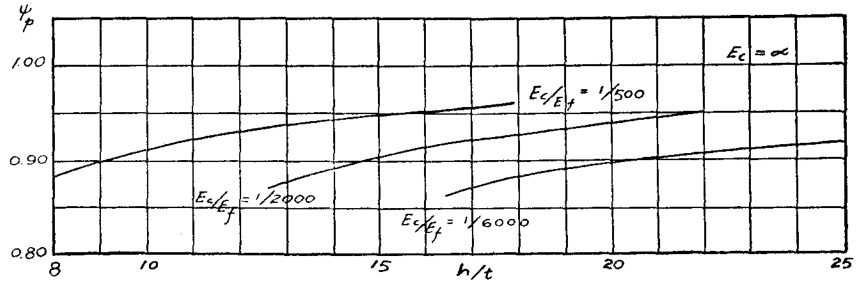


圖 6b　　線 性 解 $\sigma_p = \psi_p \frac{Ql}{t(h+t)}$

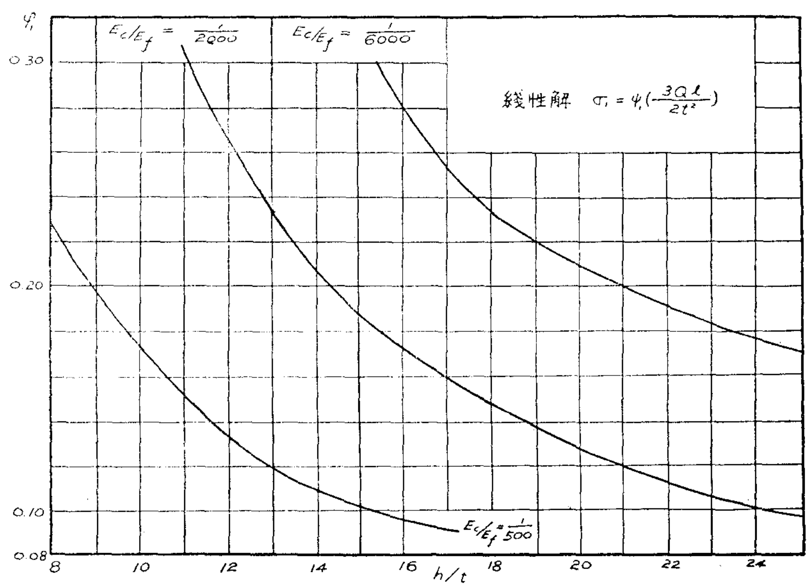


圖 6c

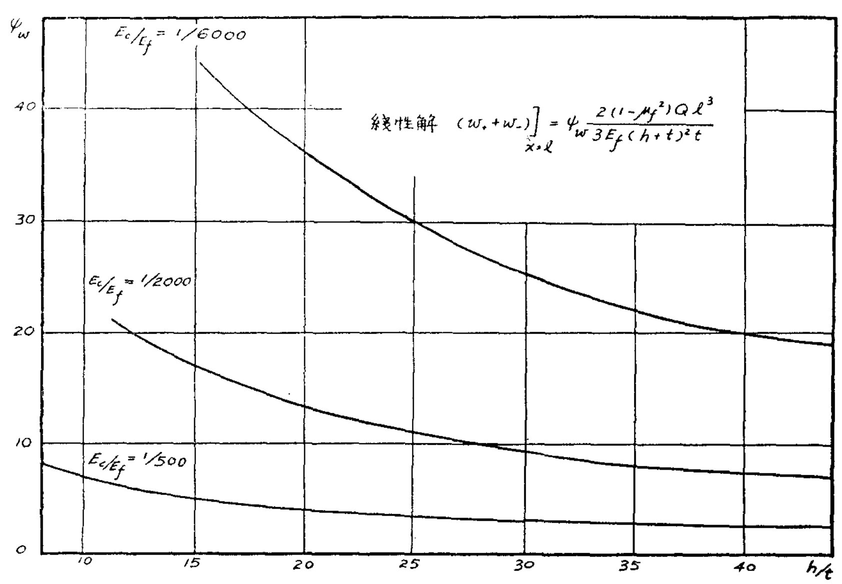


圖 6d

圖 6a, 6b, 6c 爲線性理論的計算. 圖 6a 指出在不同厚度比及彈性常數比的情況下表鈑彎曲應力與薄膜平均應力二者之中以前者爲主. 圖 6b, c, d 指出夾心受壓縮的"彈性墊子"作用. 圖 6b 表示夾心壓縮與夾心不可壓縮 ($E_c=\infty$) 中之差別. 圖 6c 表示倘若夾心不起作用而上下表鈑各負担一半載荷所產生之彎曲應力與夾心存在之差別. 圖 6d 示出載荷端之撓度倘若根據薄膜表鈑及 $E_c=G_c=\infty$ 情況計算應力之校正.

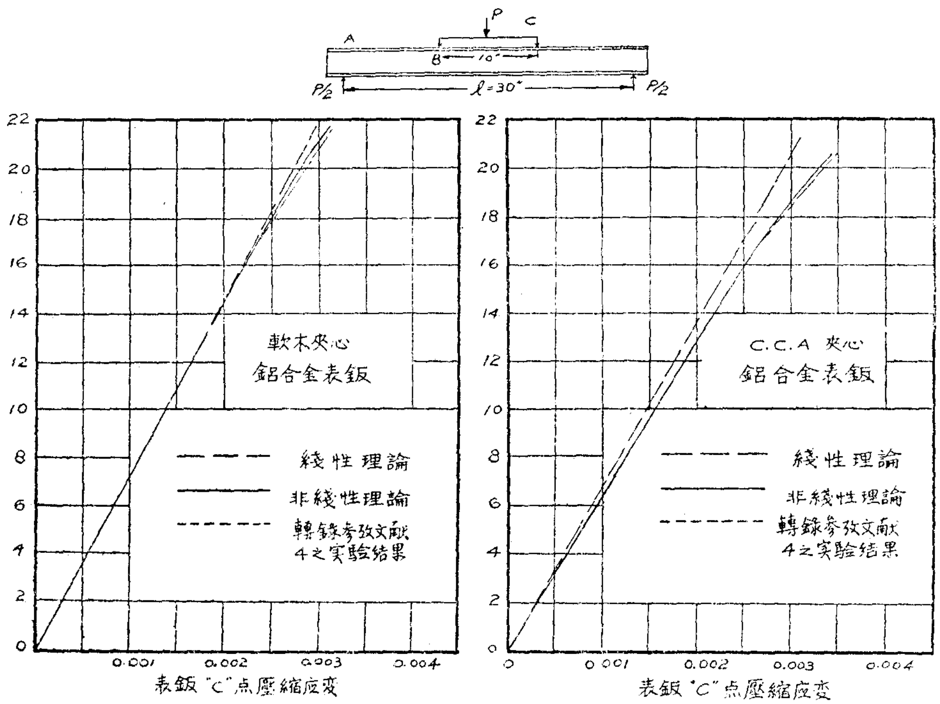


圖 7a 圖 7b

根據[4]所舉實驗結果對下列二種三合板的理論結果與實驗結果進行比較, 可以示出如圖 7. 7a 爲輭木夾心, 鋁合金表鈑: 梁寬——上表鈑 0.996 吋, 下表鈑 0.995 吋, $(h+t)=0.286$ 吋, $t=0.012$ 吋, $E_f=10^7$ 磅/吋^2, $E_c=500,000$ 磅/吋^2, $G_c=15,000$ 磅/吋^2; 7b 爲 C. C. A 夾心, 鋁合金表鈑: 梁寬——上表鈑 0.980 吋, 下表鈑 0.979 吋, $(h+t)=0.270$ 吋, $t=0.012$ 吋, $E_f=10^7$ 磅/吋^2, $E_c=40,000$ 磅/吋^2, $G_c=3500$ 磅/吋^2, 彈性常數俱由[5]中提供爲同一來源之資料.

由於試驗之情况並非肱梁, 因此計算圖 7 中 C 處之應力必須應用二次肱梁問題的叠加. 將圖 7 之問題作爲長爲 $\frac{l}{2}$ 之肱梁在 B 處及 A 處受到方向相反之

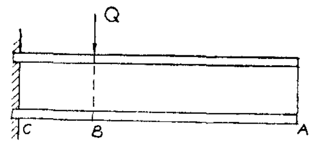


圖 8

$P/2$ 之載荷由於 B 處接近 C, 所以 B 處載荷引起之非線性應力較小[1]. (參閱 280 頁附註及圖 8) 這樣疊加中所引起的誤差是較小的,同時也說明我們所計算的非線性應力主要是由 A 處 $P/2$ 力引起.

圖 7 中的結果指本文的理論從實驗結果中得到良好的驗證.

七. 結 論

從理論的分析和實驗的驗證可以看出, 三合板梁問題包含有非線性的因素, 其主要原因為表鈑內存在有沿幅度分佈的壓力(由夾心傳來). 本文所用的方法僅只擴大了線性理論可以適用的範圍, 但從理論與實驗結果的符合上看,顯然非線性的應力因素是不能忽略的, 正如上節中及附圖所示,三合板肱梁在固定端表鈑上倘若應力到達比例極限,幾乎不可避免含有顯著的非線性應力 σ_2, 這種非線性應力的存在清晰地指出現存的理論 (例如 [2] 以及根據此文獻作出的一系列 N. A. C. A. 研究結果)、其應用是可疑的. 因而 [2] 提供的"綜合三合板彈性常數" 也可以說隨着線性理論的限制而成為狹窄的經驗公式. 本文指出表鈑上的彎曲應力與表鈑薄膜應力較前者為主要應力, 這樣即使在線性範圍內一些薄膜表鈑理論(例如 [3])的使用是必然會引起極大誤差的.

本文的圖及計算主要為了說明原理的本質同時可以利用於實際計算. 最後作者認為對於尋求非線性方程的較為便利的解法, 實為三合板彎曲問題的一重要努力方向.

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BENDING OF SANDWICH BEAMS

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ABSTRACT

Bending of sandwich beams has been treated as a non-linear problem. The non-linearity is called forth by the distributed thrust in the compression face. The non-linear differential equations for cantilevered sandwich beams with concentrated load at end are solved by using load parameter power series. On account of the labour involved in this method, the solution has been calculated up only to terms containing the square of the load. This will be an approximation going a little beyond the usual small load range which can be covered by a linear theory. Its value is not so much that it gives a correction to linear theory, but rather that it shows the latter's range of validity. The results of the present theory show that if the fixed faces are stressed to their elastic limit, they almost always have appreciable non-linear stresses. The theoretical expressions give good experimental checks. The existence of such non-linearities clearly demonstrate that the proposed use of the experimentally determined "composite plate constants" from the linear theory will be of doubtful validity. In the face the linear bending stress of the outmost fibre at the fixed end is found always greater than the linear mean normal stress. Thus treatment of the faces as mere membrances, to simplify the bending problem of the sandwich plate, is likely to lead to large errors.