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两个高维 loop 代数及应用

张玉峰

辽宁师范大学数学学院 大连 116029
E-mail: zhang_yfshandong@163.com

张鸿庆

大连理工大学应用数学系 大连 116024

摘　要 借助于循环数, 构造了维数分别是 $5(s+1)$ 和 $4(s+1)$ 的两个高维 loop 代数. 为了计算方便, 本文只考虑 $s=1$ 时的应用. 利用第一个 loop 代数 $\tilde{A}_1^*$ 得到了具有 4-Hamilton 结构的一个广义 AKNS 族, 该方程族可约化为著名的 AKNS 族. 利用第二个 loop 代数 $\tilde{A}_2^*$, 得到了具有 4 个分量位势函数的 4-Hamilton 结构方程族, 该族可约化为一个非线性耦合 Burgers 方程和一个耦合的 KdV 方程.

关键词 循环数; loop 代数; Hamilton 结构
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Two Higher-Dimensional loop Algebras and Their Applications

Yu Feng ZHANG

Mathematical School, Liaoning Normal University, Dalian 116029*, P. R. China*
E-mail: zhang_yfshandong@163.com

Hong Qing ZHANG

Department of Applied Mathematics, Dalian University of Technology,
Dalian 116024*, P. R. China*

Abstract With the help of the cycled numbers, two higher-dimensional loop algebras are constructed, whose dimension numbers are 5(s+1) and 4(s+1), respectively. For the sake of simple calculation, we only take $s=1$ in the paper for illustrating their applications. By employing the first loop algebra $\tilde{A}_1^*$, a generalized AKNS hierarchy is obtained, possessing 4-Hamiltonian structure, which is also reduced to the well-known AKNS hierarchy. By making use of the second loop algebra $\tilde{A}_2^*$, a new integrable hierarchy with 4-potential functions is generated, also possessing 4-Hamiltonian structure, which is redued to a nonlinear couled Burgers equation and a coupled KdV equation, respectively.

Keywords cycled number; loop algebra; Hamiltonian structure
MR(2000) Subject Classification 35Q51
Chinese Library Classification O175.29

人们利用屠格式已得到许多可积演化方程族, 如文献 [1–11] 中的结果. 马文秀和胡星标等进一步发展了屠格式并得到了一些有趣结果 [2–7]. 郭福奎利用 loop 代数 $\tilde{A}_1$ 的一些子代数和屠格

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式获得了几类具有多位势函数的可积 Hamilton 系统 [10, 11]. 我们怎样才能把一个已知的可积系统扩展成为更大的可积系统? 一个办法是可积耦合法 [12], 利用这种方法, 我们已得到了已知可积系的扩展可积模型 [13−15]. 可是很难求出这些扩展可积模型的 Hamilton 结构. 为了克服这个难题, 本文借助于循环数及 loop 代数 $\tilde{A}_1$ 或 loop 代数 $\tilde{A}_2$ 构造了两类维数分别是 $5(s+1)$ 和 $4(s+1)$ 的 loop 代数, 这里的 s 表示一个任意非负整数. 为应用方便起见, 只考虑 $s=1$ 的情形. 这样我们就给出了两个 loop 代数 $\tilde{A}_1^*$ 和 $\tilde{A}_2^*$. 由 $\tilde{A}_1^*$ 得到了具有 10 个位势函数分量的广义 AKNS 方程族, 而著名的 AKNS 方程族仅是其一个特例. 由 loop 代数 $\tilde{A}_2^*$ 得到了具有 4 位势函数分量的可积方程族, 由此可分别约化成非线性 Burgers 方程和耦合的 KdV 方程.

1 Loop 代数 $\tilde{A}_1^*$ 及其应用

首先, 给出循环数的定义.

定义 1 数集 $\{\epsilon_0,\epsilon_1,\dots,\epsilon_s\}$ 称为循环的, 如果下面关系成立

$$\epsilon_i\epsilon_j=\begin{cases}\epsilon_{i+j}, & i+j\le s,\\ \epsilon_{i+j-s-1}, & i+j\ge s+1,\end{cases} \tag{1}$$

其中 $\epsilon_i\neq 0$, $0\le i\le s$, $\epsilon_k\neq\epsilon_j$, $k\neq j$.

定义 2 对于任意实数或复数 a,b, 若 $i=j$, $a=b$, 定义

$$a\epsilon_i=b\epsilon_j \tag{2}$$

的充分必要条件是

$$i=j,\quad a=b. \tag{3}$$

定义 3 如果 $\{e_1,e_2,\dots,e_n\}$ 是 Lie 代数 A_{n-1} 的一个子代数, 定义一个新的 Lie 代数 A_{n-1}^*, 其元素为

$$\epsilon_k e_i,\quad k=0,1,2,\dots,s;\quad i=1,2,\dots,n, \tag{4}$$

换位运算定义为

$$[\epsilon_k e_i,\epsilon_l e_j]=\begin{cases}\epsilon_{k+l}[e_i,e_j], & k+l\le s,\\ \epsilon_{k+l-s-1}[e_i,e_j], & k+l\ge s+1.\end{cases} \tag{5}$$

这里 $i\neq j$. 相应的 loop 代数 $\tilde{A}_{n-1}^*$ 定义为

$$\epsilon_k e_i(m),\quad k=0,1,2,\dots,s;\quad i=1,2,\dots,n;\quad m=0,\pm1,\pm2,\dots, \tag{6}$$

其中的换位运算为

$$[\epsilon_k e_i(m),\epsilon_l e_j(n)]=[\epsilon_k e_i,\epsilon_l e_j]\lambda^{m+n}, \tag{7}$$

于是等式

$$a(x,t)\epsilon_k e_i(m)=b(x,t)\epsilon_l e_i(n) \tag{8}$$

成立仅当

$$k=l,\quad m=n,\quad a(x,t)=b(x,t). \tag{9}$$

这里 $a(x,t),b(x,t)$ 为任意函数.

考虑一个特殊的 loop 代数 $\tilde{A}_1^*$:

$$\begin{aligned}&\epsilon_k h(0,m),\ \epsilon_k h(1,m),\ \epsilon_k e(0,m),\ \epsilon_k e(1,m),\ \epsilon_k f(0,m),\ \epsilon_k f(1,m),\\ &k=0,1,2,\dots,s;\quad m=0,\pm1,\pm2,\dots,\end{aligned} \tag{10}$$

其中

$$h(0,m)=\begin{pmatrix}\lambda^{2m} & 0\\ 0 & -\lambda^{2m}\end{pmatrix},\ h(1,m)=\begin{pmatrix}\lambda^{2m+1} & 0\\ 0 & -\lambda^{2m+1}\end{pmatrix},\ e(0,m)=\begin{pmatrix}0 & \lambda^{2m}\\ 0 & 0\end{pmatrix},$$

$$e(1,m)=\begin{pmatrix}0 & \lambda^{2m+1}\\ 0 & 0\end{pmatrix},\ f(0,m)=\begin{pmatrix}0 & 0\\ \lambda^{2m} & 0\end{pmatrix},\ f(1,m)=\begin{pmatrix}0 & 0\\ \lambda^{2m+1} & 0\end{pmatrix}.$$

按照上面的 loop 代数 (取 $s=1$), 设计下面的等谱问题

$$\begin{cases}\varphi_x=U\varphi,\quad \lambda_t=0,\\ U=\epsilon_0 h(1,0)+(\epsilon_0 q_0+\epsilon_1 q_1)e(0,0)+(\epsilon_0 r_0+\epsilon_1 r_1)f(0,0)\\ \qquad +(\epsilon_0 v_0+\epsilon_1 v_1)e(1,-1)+(\epsilon_0 w_0+\epsilon_1 w_1),\\ f(1,-1)+(\epsilon_0 s_0+\epsilon_1 s_1)h(1,-1).\end{cases}\tag{11}$$

设 $V=\sum_{m\geq 0}[(\epsilon_0 a(0,m)+\epsilon_1 a(1,m))h(0,-m)+(\epsilon_0 b(0,m)+\epsilon_1 b(1,m))h(1,-m)+(\epsilon_0 g(0,m)+\epsilon_1 c(1,m))e(0,-m)+(\epsilon_0 d(0,m)+\epsilon_1 d(1,m))e(1,-m)+(\epsilon_0 g(0,m)+\epsilon_1 g(1,m))f(0,-m)+(\epsilon_0 p(0,m)+\epsilon_1 p(1,m))f(1,-m)]$, 求下面静态零曲率方程

$$V_x=[U,V]\tag{12}$$

得解

$$a_x(0,m)=q_0 g(0,m)+q_1 g(1,m)-r_0 c(0,m)-r_1 c(1,m)+v_0 p(0,m)+v_1 p(1,m)-w_0 d(0,m)-w_1 d(1,m),$$
$$a_x(1,m)=q_0 g(1,m)+q_1 g(0,m)-r_0 c(1,m)-r_1 c(0,m)+v_0 p(1,m)+v_1 p(0,m)+w_0 d(1,m)+w_1 d(0,m),$$
$$b_x(0,m+1)=q_0 p(0,m+1)+q_1 p(1,m+1)-r_0 d(0,m+1)-r_1 d(1,m+1)+v_0 g(0,m)$$
$$\qquad +v_1 g(1,m)-w_0 c(0,m)-w_1 c(1,m),$$
$$b_x(1,m+1)=q_0 p(1,m+1)+q_1 p(0,m+1)-r_0 d(1,m+1)-r_1 d(0,m+1)+v_0 g(1,m)$$
$$\qquad +v_1 g(0,m)-w_0 c(1,m)-w_1 c(0,m),$$
$$c_x(0,m)=2d(0,m+1)-2q_0 a(0,m)-2q_1 a(1,m)-2v_0 b(0,m)-2v_1 b(1,m)+2s_0 d(0,m)+2s_1 d(1,m),$$
$$2d(1,m+1)=c_x(1,m)+2q_0 a(1,m)+2q_1 a(0,m)+2v_0 b(1,m)+2v_1 b(0,m)-2s_0 d(1,m)-2s_1 d(0,m),$$
$$d_x(0,m+1)=2c(0,m+1)-2q_0 b(0,m+1)-2q_1 b(1,m+1)-2v_0 a(0,m)-2v_1 a(1,m)$$
$$\qquad +2s_0 c(0,m)+2s_1 c(1,m),$$
$$d_x(1,m+1)=2c(1,m+1)-2q_0 b(1,m+1)-2q_1 b(0,m+1)-2v_0 a(1,m)-2v_1 a(0,m)$$
$$\qquad +2s_0 c(1,m)+2s_1 c(0,m),$$
$$2p(0,m+1)=-g_x(0,m)+2r_0 a(0,m)+2r_1 a(1,m)+2w_0 b(0,m)+2w_1 b(1,m)-2s_0 p(0,m)-2s_1 p(1,m),$$
$$2p(1,m+1)=-g_x(1,m)+2r_0 a(1,m)+2r_1 a(0,m)+2w_0 b(1,m)+2w_1 b(0,m)-2s_0 p(1,m)-2s_1 p(0,m),$$
$$2g(0,m+1)=-p_x(0,m+1)+2r_0 b(0,m+1)+2r_1 b(1,m+1)+2w_0 a(0,m)+2w_1 a(1,m)$$
$$\qquad -2s_0 g(0,m)-2s_1 g(1,m),$$
$$2g(1,m+1)=-p_x(1,m+1)+2r_0 b(1,m+1)+2r_1 b(0,m+1)+2w_0 a(1,m)+2w_1 a(0,m)$$
$$\qquad -2s_0 g(1,m)-2s_1 g(0,m),$$
$$a(0,0)=\alpha=\text{const.},\ a(1,0)=\beta=\text{const.},\ c(0,0)=c(1,0)=d(0,0)=d(1,0)=p(1,0)=p(0,0)$$
$$\qquad =g(0,0)=g(1,0)=b(0,0)=b(1,0)=0,$$
$$d(0,1)=\alpha q_0+\beta q_1,\quad d(1,1)=\alpha q_1+\beta q_0, p(0,1)=\alpha r_0+\beta r_1,\quad p(1,1)=\alpha r_1+\beta r_0,$$
$$b(0,1)=b(1,1)=0,\quad g(0,1)=-\frac{\alpha}{2}r_{0x}-\frac{\beta}{2}r_{1x}+\alpha w_0+\beta w_1,$$
$$g(1,1)=-\frac{\alpha}{2}r_{1x}-\frac{\beta}{2}r_{0x}+\alpha w_1+\beta w_0,\quad c(0,1)=\frac{1}{2}(\alpha q_{0x}+\beta q_{1x})+\alpha v_0+\beta v_1,$$
$$c(1,1)=\frac{1}{2}(\alpha q_{1x}+\beta q_{0x})+\alpha v_1+\beta v_0,\quad a(0,1)=-\frac{\alpha}{2}(q_0 r_0+q_1 r_1)-\frac{\beta}{2}(q_0 r_1+q_1 r_0),$$

$$a(1,1)=-\frac{\alpha}{2}(q_0r_1+r_0q_1)-\frac{\beta}{2}(q_0r_0+q_1r_1). \tag{13}$$

记

$$\begin{cases} V_+^{(n)}=\sum_{m=0}^{n}[(\epsilon_0a(0,m)+\epsilon_1a(1,m))h(0,n-m)+(\epsilon_0b(0,m)+\epsilon_1b(1,m))h(1,n-m) \\ \qquad +(\epsilon_0c(0,m)+\epsilon_1c(1,m))e(0,n-m)+(\epsilon_0d(0,m)+\epsilon_1d(1,m))e(1,n-m)+(\epsilon_0g(0,m) \\ \qquad +\epsilon_1g(1,m))f(0,n-m)+(\epsilon_0p(0,m)+\epsilon_1p(1,m))f(1,n-m)], \\ V_-^{(n)}=\lambda^{2n}V-V^{(n)}, \end{cases}$$

方程 (12) 可写为

$$-V_{+x}^{(n)}+[U,V_+^{(n)}]=V_{-x}^{(n)}-[U,V_-^{(n)}]. \tag{14}$$

容易验证 (14) 式的左边基元阶数大于等于 -1, 右边基元阶数小于等于 0. 因此左右两边基元阶数为 $-1,0$. 直接计算知

$-V_{+x}^{(n)}+[U,V_+^{(n)}]=[\epsilon_0b_x(0,n+1)+\epsilon_1b_x(1,n+1)-\epsilon_0q_0p(0,n+1)-\epsilon_1q_0p(1,n+1)-\epsilon_1q_1p(0,n+1)-\epsilon_0q_1p(1,n+1)+\epsilon_0r_0d(0,n+1)+\epsilon_1r_0d(1,n+1)+\epsilon_1r_1d(0,n+1)+\epsilon_0r_1d(1,n+1)]h(1,-1)+[\epsilon_0d_x(0,n+1)+\epsilon_1d_x(1,n+1)-2\epsilon_0c(0,n+1)-2\epsilon_1c(1,n+1)+2\epsilon_0q_0b(0,n+1)+2\epsilon_1q_0b(1,n+1)+2\epsilon_1q_1b(0,n+1)+2\epsilon_0q_1b(1,n+1)]e(1,-1)+[\epsilon_0p_x(0,n+1)+\epsilon_1p_x(1,n+1)+2\epsilon_0g(0,n+1)+2\epsilon_1g(1,n+1)-2\epsilon_0r_0b(0,n+1)-2\epsilon_1r_0b(1,n+1)-2\epsilon_1r_1b(0,n+1)-2\epsilon_0r_1b(1,n+1)]f(1,-1)-[2\epsilon_0d(0,n+1)+2\epsilon_1d(1,n+1)]e(0,0)+[2\epsilon_0p(0,n+1)+2\epsilon_1p(1,n+1)]f(0,0).$

取 $V^{(n)}=V_+^{(n)}$, 由零曲率方程

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0 \tag{15}$$

导出方程族

$$u_t=\begin{pmatrix} q_0\\ q_1\\ r_0\\ r_1\\ v_0\\ v_1\\ w_0\\ w_1\\ s_0\\ s_1 \end{pmatrix}_t=\begin{pmatrix} 2d(0,n+1)\\ 2d(1,n+1)\\ -2p(0,n+1)\\ -2p(1,n+1)\\ -d_x(0,n+1)+2c(0,n+1)-2q_0b(0,n+1)-2q_1b(1,n+1)\\ -d_x(1,n+1)+2c(1,n+1)-2q_0b(1,n+1)-2q_1b(0,n+1)\\ -p_x(0,n+1)-2g(0,n+1)+2r_0b(0,n+1)+2r_1b(1,n+1)\\ -p_x(1,n+1)-2g(1,n+1)+2r_0b(1,n+1)+2r_1b(0,n+1)\\ -b_x(0,n+1)+q_0p(0,n+1)+q_1p(1,n+1)-r_0d(0,n+1)-r_1d(1,n+1)\\ -b_x(1,n+1)+q_0p(1,n+1)+q_1p(0,n+1)-r_0d(1,n+1)-r_1d(0,n+1) \end{pmatrix}$$

$$=\begin{pmatrix} 0&0&0&0&0&0&2&0&0&0\\ 0&0&0&0&0&0&0&2&0&0\\ 0&0&0&0&-2&0&0&0&0&0\\ 0&0&0&0&0&-2&0&0&0&0\\ 0&0&2&0&0&0&-\partial&0&-q_0&-q_1\\ 0&0&0&2&0&0&0&-\partial&-q_1&-q_0\\ -2&0&0&0&-\partial&0&0&0&r_0&r_1\\ 0&-2&0&0&0&-\partial&0&0&r_1&r_0\\ 0&0&0&0&q_0&q_1&-r_0&-r_1&-\frac{\partial}{2}&0\\ 0&0&0&0&q_1&q_0&-r_1&-r_0&0&-\frac{\partial}{2} \end{pmatrix}\begin{pmatrix} g(0,n+1)\\ g(1,n+1)\\ c(0,n+1)\\ c(1,n+1)\\ p(0,n+1)\\ p(1,n+1)\\ d(0,n+1)\\ d(1,n+1)\\ 2b(0,n+1)\\ 2b(1,n+1) \end{pmatrix}=J_1G_{n1}$$

$$
\begin{aligned}
&=\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 2 & 0 & 0 & 0 & -\partial & -q_1 & -q_0\\
0 & 0 & 2 & 0 & 0 & 0 & -\partial & 0 & -q_0 & -q_1\\
0 & -2 & 0 & 0 & 0 & -\partial & 0 & 0 & r_1 & r_0\\
-2 & 0 & 0 & 0 & -\partial & 0 & 0 & 0 & r_0 & r_1\\
0 & 0 & 0 & 0 & q_1 & q_0 & -r_1 & -r_0 & 0 & -\frac{\partial}{2}\\
0 & 0 & 0 & 0 & q_0 & q_1 & -r_0 & -r_1 & -\frac{\partial}{2} & 0
\end{pmatrix}
\begin{pmatrix}
g(1,n+1)\\ g(0,n+1)\\ c(1,n+1)\\ c(0,n+1)\\ p(1,n+1)\\ p(0,n+1)\\ d(1,n+1)\\ d(0,n+1)\\ 2b(1,n+1)\\ 2b(0,n+1)
\end{pmatrix} = J_2G_{n2}\\
&=\begin{pmatrix}
2d(0,n+1)\\
2d(1,n+1)\\
-2p(0,n+1)\\
-2p(1,n+1)\\
2v_0a(0,n)+2v_1a(1,n)-2s_0c(0,n)-2s_1c(1,n)\\
2v_0a(1,n)+2v_1a(0,n)-2s_0c(1,n)-2s_1c(0,n)\\
-2w_0a(0,n)-2w_1a(1,n)+2s_0g(0,n)+2s_1g(1,n)\\
-2w_0a(1,n)-2w_1a(0,n)+2s_0g(1,n)+2s_1g(0,n)\\
-v_0g(0,n)-v_1g(1,n)+w_0c(0,n)+w_1c(1,n)\\
-v_0g(1,n)-v_1g(0,n)+w_0c(1,n)+w_1c(0,n)
\end{pmatrix}\\
&=\begin{pmatrix}
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -2s_0 & -2s_1 & v_0 & v_1\\
0 & 0 & 0 & 0 & 0 & 0 & -2s_1 & -2s_0 & v_1 & v_0\\
0 & 0 & 0 & 0 & 2s_0 & 2s_1 & 0 & 0 & -w_0 & -w_1\\
0 & 0 & 0 & 0 & 2s_1 & 2s_0 & 0 & 0 & -w_1 & -w_0\\
0 & 0 & 0 & 0 & -v_0 & -v_1 & w_0 & w_1 & 0 & 0\\
0 & 0 & 0 & 0 & -v_1 & -v_0 & w_1 & w_0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
p(0,n+1)\\ p(1,n+1)\\ d(0,n+1)\\ d(1,n+1)\\ g(0,n)\\ g(1,n)\\ c(0,n)\\ c(1,n)\\ 2a(0,n)\\ 2a(1,n)
\end{pmatrix} = J_3G_{n3}\\
&=\begin{pmatrix}
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -2s_1 & -2s_0 & v_1 & v_0\\
0 & 0 & 0 & 0 & 0 & 0 & -2s_0 & -2s_1 & v_0 & v_1\\
0 & 0 & 0 & 0 & 2s_1 & 2s_0 & 0 & 0 & -w_1 & -w_0\\
0 & 0 & 0 & 0 & 2s_0 & 2s_1 & 0 & 0 & -w_0 & -w_1\\
0 & 0 & 0 & 0 & -v_1 & -v_0 & w_1 & w_0 & 0 & 0\\
0 & 0 & 0 & 0 & -v_0 & -v_1 & w_0 & w_1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
p(1,n+1)\\ p(0,n+1)\\ d(1,n+1)\\ d(0,n+1)\\ g(1,n)\\ g(0,n)\\ c(1,n)\\ c(0,n)\\ 2a(1,n)\\ 2a(0,n)
\end{pmatrix} = J_4G_{n4},
\end{aligned}
\tag{16}
$$

其中 J_i $(i=1,2,3,4)$ 是 Hamilton 算子. 根据 (13) 式, 得到一个递推算子

$$L=(l_{ij})_{10\times10}, \tag{17}$$

且 L 满足 $G_{n1}=LG_{n3}$, $G_{n3}=LG_{n-1,1}$, $G_{n2}=LG_{n4}$, $G_{n4}=LG_{n-1,2}$. 这里

$l_{11}=-\dfrac{\partial}{2}+r_0\partial^{-1}q_0+r_1\partial^{-1}q_1$, $l_{12}=r_0\partial^{-1}q_1+r_1\partial^{-1}q_0$, $l_{13}=-r_0\partial^{-1}r_0-r_1\partial^{-1}r_1$,

$l_{14}=-r_0\partial^{-1}r_1-r_1\partial^{-1}r_0$, $l_{15}=r_0\partial^{-1}v_0+r_1\partial^{-1}v_1-s_0$, $l_{16}=r_0\partial^{-1}v_1+r_1\partial^{-1}v_0-s_1$,

$l_{17}=-r_0\partial^{-1}w_0-r_1\partial^{-1}w_1$, $l_{18}=-r_0\partial^{-1}w_1-r_1\partial^{-1}w_0$, $l_{19}=\dfrac{w_0}{2}$, $l_{1,10}=\dfrac{w_1}{2}$,

$l_{21}=r_0\partial^{-1}q_1+r_1\partial^{-1}q_0$, $l_{22}=-\dfrac{\partial}{2}+r_0\partial^{-1}q_0+r_1\partial^{-1}q_1$,

$l_{23}=-r_0\partial^{-1}r_1-r_1\partial^{-1}r_0,\quad l_{24}=-r_0\partial^{-1}r_0-r_1\partial^{-1}r_1,\quad l_{25}=r_0\partial^{-1}v_1+r_1\partial^{-1}v_0-s_1,$
$l_{26}=r_0\partial^{-1}v_0+r_1\partial^{-1}v_1-s_0,\quad l_{27}=-r_0\partial^{-1}w_1-r_1\partial^{-1}w_0,\quad l_{28}=-r_0\partial^{-1}w_0-r_1\partial^{-1}w_1,$
$l_{29}=\dfrac{w_1}{2},\quad l_{2,10}=\dfrac{w_0}{2},\quad l_{31}=q_0\partial^{-1}q_0+q_1\partial^{-1}q_1,\quad l_{32}=q_0\partial^{-1}q_1+q_1\partial^{-1}q_0,$
$l_{33}=\dfrac{\partial}{2}-q_0\partial^{-1}r_0-q_1\partial^{-1}r_1,\quad l_{34}=-q_0\partial^{-1}r_1-q_1\partial^{-1}r_0,\quad l_{35}=q_0\partial^{-1}v_0+q_1\partial^{-1}v_1,$
$l_{36}=q_0\partial^{-1}v_1+q_1\partial^{-1}v_0,\quad l_{37}=-q_0\partial^{-1}w_0-q_1\partial^{-1}w_1-s_0,$
$l_{38}=-q_0\partial^{-1}w_1-q_1\partial^{-1}w_0-s_1,\quad l_{39}=\dfrac{v_0}{2},\quad l_{3,10}=\dfrac{v_1}{2},$
$l_{41}=q_0\partial^{-1}q_1+q_1\partial^{-1}q_0,\quad l_{42}=q_0\partial^{-1}q_0+q_1\partial^{-1}q_1,$
$l_{43}=-q_0\partial^{-1}r_1-q_1\partial^{-1}r_0,\quad l_{44}=\dfrac{\partial}{2}-q_0\partial^{-1}r_0-q_1\partial^{-1}r_1,$
$l_{45}=q_0\partial^{-1}v_1+q_1\partial^{-1}v_0,\quad l_{46}=q_0\partial^{-1}v_0+q_1\partial^{-1}v_1,$
$l_{47}=-q_0\partial^{-1}w_1-q_1\partial^{-1}w_0-s_1,\quad l_{48}=-q_0\partial^{-1}w_0-q_1\partial^{-1}w_1-s_0,$
$l_{49}=\dfrac{v_1}{2},\quad l_{4,10}=\dfrac{v_0}{2},\quad l_{51}=1,\quad l_{52}=\cdots=l_{5,10}=0,$
$l_{61}=0,\quad l_{62}=1,\quad l_{63}=\cdots=l_{6,10}=0,\quad l_{73}=1,\quad l_{71}=l_{72}=l_{74}=\cdots=l_{7,10}=0,$
$l_{84}=1,\quad l_{81}=l_{82}=l_{83}=l_{85}=\cdots=l_{8,10}=0,$
$l_{91}=2\partial^{-1}q_0,\quad l_{92}=2\partial^{-1}q_1,\quad l_{93}=-2\partial^{-1}r_0,\quad l_{94}=-2\partial^{-1}r_1,$
$l_{95}=2\partial^{-1}v_0,\quad l_{96}=2\partial^{-1}v_1,\quad l_{97}=-2\partial^{-1}w_0,\quad l_{98}=-2\partial^{-1}w_1,$
$l_{99}=l_{9,10}=0,\quad l_{10,1}=2\partial^{-1}q_1,\quad l_{10,2}=2\partial^{-1}q_0,\quad l_{10,3}=-2\partial^{-1}r_1,$
$l_{10,4}=-2\partial^{-1}r_0,\quad l_{10,5}=2\partial^{-1}v_1,\quad l_{10,6}=2\partial^{-1}v_0,$
$l_{10,7}=-2\partial^{-1}w_1,\quad l_{10,8}=-2\partial^{-1}w_0,\quad l_{10,9}=l_{10,10}=0.$

于是, 系统 (16) 可写为

$$u_t=J_1LG_{n3}=J_1L^2G_{n-1,1}=J_1L^{2n}\begin{pmatrix} g(0,1)\\ g(1,1)\\ c(0,1)\\ c(1,1)\\ p(0,1)\\ p(1,1)\\ d(0,1)\\ d(1,1)\\ 2b(0,1)\\ 2b(1,1)\end{pmatrix}=J_2L^{2n}\begin{pmatrix} g(1,1)\\ g(0,1)\\ c(1,1)\\ c(0,1)\\ p(1,1)\\ p(0,1)\\ d(1,1)\\ d(0,1)\\ 2b(1,1)\\ 2b(0,1)\end{pmatrix}. \tag{18}$$

作为系统 (18) 的约化情形, 取 $q_0=r_0=v_0=v_1=w_0=w_1=s_0=s_1=0$, 则得到著名的 AKNS 方程族

$$u_t=\begin{pmatrix} q_1\\ r_1\end{pmatrix}_t=\begin{pmatrix} 2d(1,n+1)\\ -2p(1,n+1)\end{pmatrix}=\begin{pmatrix} 0 & 2\\ -2 & 0\end{pmatrix}\begin{pmatrix} \dfrac{\partial}{2}+r_1\partial^{-1}q_1 & -r_1\partial^{-1}r_1\\ q_1\partial^{-1}q_1 & \dfrac{\partial}{2}-q_1\partial^{-1}r_1\end{pmatrix}\begin{pmatrix} g(1,n)\\ c(1,n)\end{pmatrix}.$$

直接计算得

$$\Big\langle V,\frac{\partial U}{\partial q_0}\Big\rangle=\epsilon_0 g(0)+\epsilon_1 g(1)+(\epsilon_0 p(0)+\epsilon_1 p(1))\lambda,\quad \Big\langle V,\frac{\partial U}{\partial q_1}\Big\rangle=\epsilon_1 g(0)+\epsilon_0 g(1)+(\epsilon_1 p(0)+\epsilon_0 p(1))\lambda,$$
$$\Big\langle V,\frac{\partial U}{\partial r_0}\Big\rangle=\epsilon_0 c(0)+\epsilon_1 c(1)+(\epsilon_0 d(0)+\epsilon_1 d(1))\lambda,\quad \Big\langle V,\frac{\partial U}{\partial r_1}\Big\rangle=\epsilon_0 c(1)+\epsilon_1 c(0)+(\epsilon_0 d(1)+\epsilon_1 d(0))\lambda,$$
$$\Big\langle V,\frac{\partial U}{\partial v_0}\Big\rangle=\epsilon_0 p(0)+\epsilon_1 p(1)+\frac{\epsilon_0 g(0)+\epsilon_1 g(1)}{\lambda},\quad \Big\langle V,\frac{\partial U}{\partial v_1}\Big\rangle=\epsilon_0 p(1)+\epsilon_1 p(0)+\frac{\epsilon_1 g(0)+\epsilon_0 g(1)}{\lambda},$$
$$\Big\langle V,\frac{\partial U}{\partial w_0}\Big\rangle=\epsilon_0 d(0)+\epsilon_1 d(1)+\frac{\epsilon_0 c(0)+\epsilon_1 c(1)}{\lambda},\quad \Big\langle V,\frac{\partial U}{\partial w_1}\Big\rangle=\epsilon_0 d(1)+\epsilon_1 d(0)+\frac{\epsilon_1 c(0)+\epsilon_0 c(1)}{\lambda},$$

$$\left\langle V,\frac{\partial U}{\partial s_0}\right\rangle=2\epsilon_0 b(0)+2\epsilon_1 b(1)+2\frac{\epsilon_0 a(0)+\epsilon_1 a(1)}{\lambda},\ \left\langle V,\frac{\partial U}{\partial s_1}\right\rangle=2\epsilon_0 b(1)+2\epsilon_1 b(0)+2\frac{\epsilon_1 a(0)+\epsilon_0 a(1)}{\lambda},$$

$$\begin{aligned}\left\langle V,\frac{\partial U}{\partial \lambda}\right\rangle=&2\epsilon_0 a(0)+2\epsilon_1 a(1)-\frac{1}{\lambda^2}[2\epsilon_0 s_0 a(0)+2\epsilon_1 s_1 a(0)+2\epsilon_1 s_0 a(1)+2\epsilon_0 s_1 a(1)+\epsilon_0 w_0 c(0)\\&+\epsilon_1 w_1 c(0)+\epsilon_1 w_0 c(1)+\epsilon_0 w_1 c(1)+\epsilon_0 v_0 g(0)+\epsilon_0 v_1 g(1)+\epsilon_1 v_0 g(1)+\epsilon_1 v_1 g(0)]\\&+(2\epsilon_0 b(0)+2\epsilon_1 b(1))\lambda-\frac{1}{\lambda}[2\epsilon_0 s_0 b(0)+2\epsilon_0 s_1 b(1)+2\epsilon_1 s_0 b(1)+2\epsilon_1 s_1 b(0)+\epsilon_0 w_0 d(0)\\&+\epsilon_0 w_1 d(1)+\epsilon_1 w_0 d(1)+\epsilon_1 w_1 d(0)+\epsilon_0 v_0 p(0)+\epsilon_0 v_1 p(1)+\epsilon_1 v_0 p(1)+\epsilon_1 v_1 p(0)].\end{aligned}$$

把上面的计算结果代入迹恒等式得

$$\begin{aligned}&\frac{\delta}{\delta u}(2b(0,n+2)-2s_0 b(0,n+1)-2s_1 b(1,n+1)+w_0 d(0,n+1)+w_1 d(1,n+1)\\&\quad+v_0 p(0,n+1)+v_1 p(1,n+1))=(-2n-2+\gamma)G_{n1},\end{aligned}\tag{19}$$

$$\begin{aligned}&\frac{\delta}{\delta u}(2b(1,n+2)-2s_0 b(1,n+1)-2s_1 b(0,n+1)+w_0 d(1,n+1)+w_1 d(0,n+1)\\&\quad+v_0 p(1,n+1)+v_1 p(0,n+1))=(-2n-2+\gamma)G_{n2},\end{aligned}\tag{20}$$

$$\begin{aligned}&\frac{\delta}{\delta u}(2a(0,n+1)-2s_0 a(0,n)-2s_1 a(1,n)+w_0 c(0,n)+w_1 c(1,n)+v_0 g(0,n)+v_1 g(1,n))\\&\quad=(-2n-1+\gamma)G_{n3},\end{aligned}\tag{21}$$

$$\begin{aligned}&\frac{\delta}{\delta u}(2a(1,n+1)-2s_1 a(0,n)+2s_0 a(1,n)+w_1 c(0,n)+w_0 c(1,n)+v_0 g(1,n)+v_1 g(0,n))\\&\quad=(-2n-1+\gamma)G_{n4}.\end{aligned}\tag{22}$$

取 $n=0$ 得 $\gamma=0$. 于是我们得到系统 (18) 的满足下面关系的 4-Hamilton 结构

$$\begin{cases}\dfrac{\delta H(1,n)}{\delta u}=G_{n1},\\ H(1,n)=\dfrac{1}{2n+2}(2s_0 b(0,n+1)-2b(0,n+2)+2s_1 b(1,n+1)\\ \qquad -w_0 d(0,n+1)-w_1 d(1,n+1)-v_0 p(0,n+1)-v_1 p(1,n+1)),\end{cases}$$

$$\begin{cases}\dfrac{\delta H(2,n)}{\delta u}=G_{n2},\\ H(2,n)=\dfrac{1}{2n+2}(2s_0 b(1,n+1)-2b(1,n+2)+2s_1 b(0,n+1)\\ \qquad -w_0 d(1,n+1)-w_1 d(0,n+1)-v_0 p(1,n+1)-v_1 p(0,n+1)),\end{cases}$$

$$\begin{cases}\dfrac{\delta H(3,n)}{\delta u}=G_{n3},\\ G_{n3}=\dfrac{1}{2n+1}(2s_0 a(0,n)-2a(0,n+1)+2s_1 a(1,n)-w_0 c(0,n)\\ \qquad -w_1 c(1,n)-v_0 g(0,n)-v_1 g(1,n)),\end{cases}$$

$$\begin{cases}\dfrac{\delta H(4,n)}{\delta u}=G_{n4},\\ G_{n4}=\dfrac{1}{2n+1}(-2a(1,n+1)+2s_1 a(0,n)-2s_0 a(1,n)-w_1 c(0,n)\\ \qquad -w_0 c(1,n)-v_0 g(1,n)-v_1 g(0,n)).\end{cases}$$

系统 (18) 的 4-Hamilton 结构为

$$u_t=J_1\frac{\delta H(1,n)}{\delta u}=J_2\frac{\delta H(2,n)}{\delta u}=J_3\frac{\delta H(3,n)}{\delta u}=J_4\frac{\delta H(4,n)}{\delta u}.$$

2 广义的非线性 Schrödinger 方程和 MKdV 方程

考虑下面的 loop 代数 $\tilde{A}_2^*$:

$$\epsilon_k h_{\pm}(m),\quad \epsilon_k e_{\pm}(m),\quad k=0,1,2,\dots,s;\quad m=0,\pm1,\pm2,\dots,\tag{23}$$

其换位运算为

$$[\epsilon_k h_+(m),\epsilon_l h_-(n)]=0,[\epsilon_k h_+(m),\epsilon_l e_\pm(n)]=\begin{cases}4\epsilon_{k+l}e_\mp(m+n), & k+l\le s,\\ 4\epsilon_{k+l-s-1}e_\mp(m+n), & k+l\ge s+1,\end{cases}$$

$$[\epsilon_k h_-(m),\epsilon_l e_\pm(n)]=\begin{cases}2\epsilon_{k+l}e_\mp(m+n), & k+l\le s,\\ 2\epsilon_{k+l-s-1}e_\mp(m+n), & k+l\ge s+1,\end{cases}$$

$$[\epsilon_k e_-(m),\epsilon_l e_+(n)]=\begin{cases}2\epsilon_{k+l}h_+(m+n), & k+l\le s,\\ 2\epsilon_{k+l-s-1}h_+(m+n), & k+l\ge s+1,\end{cases}$$

其中 [17]

$$h_\pm=\begin{pmatrix}1&0&\pm1\\0&-2&0\\\pm1&0&1!\end{pmatrix},\ e_\pm=\begin{pmatrix}0&1&0\\\pm1&0&\pm1\\0&1&0\end{pmatrix},\ h_\pm(n)=\lambda^n h_\pm,\ e_\pm(n)=\lambda^n e_\pm,\ n=0,\pm1,\pm2,\dots.$$

由 $\tilde{A}_2^*$ (取 $s=1$), 给定等谱问题:

$$\psi_x=U\psi,\quad \lambda_t=0,\quad U=\epsilon_0 h_+(1)+(\epsilon_0 q_0+\epsilon_1 q_1)e_+(0)+(\epsilon_0 r_0+\epsilon_1 r_1)e_-(0). \tag{24}$$

设 $V=\sum_{m\ge0}((\epsilon_0 a(0,m)+\epsilon_1 a(1,m))h_+(-m)+(\epsilon_0 b(0,m)+\epsilon_1 b(1,m))e_+(-m)+(\epsilon_0 c(0,m)+\epsilon_1 c(1,m))e_-(-m))$, 解类似于 (12) 的方程得

$$\begin{aligned}
&a_x(0,m)=-2q_0c(0,m)-2q_1c(1,m)+2r_0b(0,m)+2r_1b(1,m),\\
&a_x(1,m)=-2q_0c(1,m)-2q_1c(0,m)+2r_0b(1,m)+2r_1b(0,m),\\
&c(0,m+1)=-\frac14 b_x(0,m)-r_0a(0,m)-r_1a(1,m),\ c(1,m+1)=-\frac14 b_x(1,m)-r_0a(1,m)-r_1a(0,m),\\
&b(0,m+1)=-\frac14 c_x(0,m)-q_0a(0,m)-q_1a(1,m),\ \ b(1,m+1)=-\frac14 c_x(1,m)-q_0a(1,m)-q_1a(0,m),\\
&a(0,0)=\alpha=\text{const.},\quad a(1,0)=\beta=\text{const.},\quad b(0,0)=b(1,0)=c(0,0)=c(1,0)=0,\\
&c(0,1)=\alpha r_0+\beta r_1,\quad c(1,1)=\alpha r_1+\beta r_0,\quad b(0,1)=\alpha q_0+\beta q_1,\\
&b(1,1)=\alpha q_1+\beta q_0,\ \ a(1,1)=a(0,1)=0.
\end{aligned}\tag{25}$$

直接计算有

$$\begin{aligned}&-(\lambda^nV)_{+x}+[U,(\lambda^nV)_+]\\&\quad=-4(\epsilon_0 b(0,n+1)+\epsilon_1 b(1,n+1))e_-(0)-4(\epsilon_0 c(0,n+1)+\epsilon_1 c(1,n+1))e_+(0).\end{aligned}\tag{26}$$

于是, 由零曲率方程得

$$\begin{aligned}
u_t&=\begin{pmatrix}q_0\\q_1\\r_0\\r_1\end{pmatrix}_t=\begin{pmatrix}\frac14 b_x(0,n)+r_0a(0,n)+r_1a(1,n)\\ \frac14 b_x(1,n)+r_0a(1,n)+r_1a(0,n)\\ \frac14 c_x(0,n)+q_0a(0,n)+q_1a(1,n)\\ \frac14 c_x(1,n)+q_0a(1,n)+q_1a(0,n)\end{pmatrix}\\
&=\begin{pmatrix}
\frac{\partial}{16}+\frac12 r_0\partial^{-1}r_0+\frac12 r_1\partial^{-1}r_1 & \frac12 r_0\partial^{-1}r_1+\frac12 r_1\partial^{-1}r_0 & \frac12 r_0\partial^{-1}q_0+\frac12 r_1\partial^{-1}q_1 & \frac12 r_0\partial^{-1}q_1+\frac12 r_1\partial^{-1}q_0\\
\frac12 r_0\partial^{-1}r_1+\frac12 r_1\partial^{-1}r_0 & \frac{\partial}{16}+\frac12 r_0\partial^{-1}r_0+\frac12 r_1\partial^{-1}r_1 & \frac12 r_0\partial^{-1}q_1+\frac12 r_1\partial^{-1}q_0 & \frac12 r_0\partial^{-1}q_0+\frac12 r_1\partial^{-1}q_1\\
\frac12 q_0\partial^{-1}r_0+\frac12 q_1\partial^{-1}r_1 & \frac12 q_0\partial^{-1}r_1+\frac12 q_1\partial^{-1}r_0 & -\frac{\partial}{16}+\frac12 q_0\partial^{-1}q_0+\frac12 q_1\partial^{-1}q_1 & \frac12 q_0\partial^{-1}q_1+\frac12 q_1\partial^{-1}q_0\\
\frac12 q_0\partial^{-1}r_1+\frac12 q_1\partial^{-1}r_0 & \frac12 q_0\partial^{-1}r_0+\frac12 q_1\partial^{-1}r_1 & \frac12 q_0\partial^{-1}q_1+\frac12 q_1\partial^{-1}q_0 & -\frac{\partial}{16}+\frac12 q_0\partial^{-1}q_0+\frac12 q_1\partial^{-1}q_1
\end{pmatrix}\\
&\cdot\begin{pmatrix}4b(0,n)\\4b(1,n)\\-4c(0,n)\\-4c(1,n)\end{pmatrix}=J_1F_{n1}\\
&=\frac14\begin{pmatrix}
2r_0\partial^{-1}r_1+2r_1\partial^{-1}r_0 & \frac{\partial}{4}+2r_0\partial^{-1}r_0+2r_1\partial^{-1}r_1 & 2r_0\partial^{-1}q_1+2r_1\partial^{-1}q_0 & 2r_0\partial^{-1}q_0+2r_1\partial^{-1}q_1\\
\frac{\partial}{4}+2r_0\partial^{-1}r_0+2r_1\partial^{-1}r_1 & 2r_0\partial^{-1}r_1+2r_1\partial^{-1}r_0 & 2r_0\partial^{-1}q_0+2r_1\partial^{-1}q_1 & 2r_0\partial^{-1}q_1+2r_1\partial^{-1}q_0\\
2q_0\partial^{-1}r_1+2q_1\partial^{-1}r_0 & 2q_0\partial^{-1}r_0+2q_1\partial^{-1}r_1 & 2q_0\partial^{-1}q_1+2q_1\partial^{-1}q_0 & -\frac{\partial}{4}+2q_0\partial^{-1}q_0+2q_1\partial^{-1}q_1\\
2q_0\partial^{-1}r_0+2q_1\partial^{-1}r_1 & 2q_0\partial^{-1}r_1+2q_1\partial^{-1}r_0 & -\frac{\partial}{4}+2q_0\partial^{-1}q_0+2q_1\partial^{-1}q_1 & 2q_0\partial^{-1}q_1+2q_1\partial^{-1}q_0
\end{pmatrix}\\
&\cdot\begin{pmatrix}4b(1,n)\\4b(0,n)\\-4c(1,n)\\-4c(0,n)\end{pmatrix}=J_2F_{n2},
\end{aligned}\tag{27}$$

这里 J_1 和 J_2 是 Hamilton 算子. 根据 (25) 式, 我们得到一个递推算子

$$L=(l_{ij})_{4\times 4}, \tag{28}$$

其中

$$\begin{aligned}
&l_{11}=2q_0\partial^{-1}r_0+2q_1\partial^{-1}r_1,\quad l_{12}=2q_0\partial^{-1}r_1+2q_1\partial^{-1}r_0,\\
&l_{13}=\frac{\partial}{4}+2q_0\partial^{-1}q_0+2q_1\partial^{-1}q_1,\quad l_{14}=2q_0\partial^{-1}q_1+2q_1\partial^{-1}q_0,\\
&l_{21}=2q_0\partial^{-1}r_1+2q_1\partial^{-1}r_0,\quad l_{22}=2q_0\partial^{-1}r_0+2q_1\partial^{-1}r_1,\\
&l_{23}=2q_0\partial^{-1}q_1+2q_1\partial^{-1}q_0,\quad l_{24}=-\frac{\partial}{4}+2q_0\partial^{-1}q_0+2q_1\partial^{-1}q_1,\\
&l_{31}=\frac{\partial}{4}+2r_0\partial^{-1}r_0+2r_1\partial^{-1}r_1,\quad l_{32}=2r_0\partial^{-1}r_1+2r_1\partial^{-1}r_0,\\
&l_{33}=2r_0\partial^{-1}q_0+2r_1\partial^{-1}q_1,\quad l_{34}=2r_0\partial^{-1}q_1+2r_1\partial^{-1}q_0,\\
&l_{41}=2r_0\partial^{-1}r_1+2r_1\partial^{-1}r_0,\quad l_{42}=\frac{\partial}{4}+2r_0\partial^{-1}r_0+2r_1\partial^{-1}r_1,\\
&l_{43}=2r_0\partial^{-1}q_0+2r_1\partial^{-1}q_1,\quad l_{44}=2r_0\partial^{-1}q_1+2r_1\partial^{-1}q_0.
\end{aligned}$$

因此, 系统 (27) 可写为

$$u_t=J_1L^{n-1}\begin{pmatrix}4\alpha q_0+4\beta q_1\\4\alpha q_1+4\beta q_0\\-4\alpha r_0-4\beta r_1\\-4\alpha r_1-4\beta r_0\end{pmatrix}=J_2L^{n-2}\begin{pmatrix}4\alpha q_1+4\beta q_0\\4\alpha q_0+4\beta q_1\\-4\alpha r_1-4\beta r_0\\-4\alpha r_0-4\beta r_1\end{pmatrix}. \tag{29}$$

通过计算知

$$\begin{aligned}
&\left\langle V,\frac{\partial U}{\partial q_0}\right\rangle=4\epsilon_0 b(0)+4\epsilon_1 b(1),\quad \left\langle V,\frac{\partial U}{\partial q_1}\right\rangle=4\epsilon_1 b(0)+4\epsilon_0 b(1),\\
&\left\langle V,\frac{\partial U}{\partial r_0}\right\rangle=-4\epsilon_0 c(0)-4\epsilon_1 c(1),\quad \left\langle V,\frac{\partial U}{\partial r_1}\right\rangle=-4\epsilon_1 c(0)-4\epsilon_0 c(1),\\
&\left\langle V,\frac{\partial U}{\partial \lambda}\right\rangle=8(\epsilon_0 a(0)+\epsilon_1 a(1)),
\end{aligned}$$

其中

$$a(0)=\sum_{m\geq 0}a(0,m)\lambda^{-m},\quad a(1)=\sum_{m\geq 0}a(1,m)\lambda^{-m},\ldots.$$

把上面的计算结果代入迹恒等式, 有

$$\frac{\delta}{\delta u}\left(\left\langle V,\frac{\partial U}{\partial\lambda}\right\rangle\right)=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}\left\langle V,\frac{\partial U}{\partial q_0}\right\rangle\\\left\langle V,\frac{\partial U}{\partial q_1}\right\rangle\\\left\langle V,\frac{\partial U}{\partial r_0}\right\rangle\\\left\langle V,\frac{\partial U}{\partial r_1}\right\rangle\end{pmatrix}. \tag{30}$$

比较 (30) 式中 λ^{-2n-1} 的系数, 得

$$\frac{\delta}{\delta u}(8a(0,n+1))=(-n+\gamma)F_{n1}, \tag{31}$$

$$\frac{\delta}{\delta u}(8a(1,n+1))=(-n+\gamma)F_{n2}. \tag{32}$$

由 (25) 式中的初值知 $\gamma=0$. 于是, 我们得到了系统 (29) 的 Hamilton 结构

$$u_t=J_1\frac{\delta H(1,n)}{\delta u}=J_2\frac{\delta H(2,n)}{\delta u}, \tag{33}$$

这里 $H(1,n)=-\frac{8a(0,n+1)}{n}$, $H(2,n)=-\frac{8a(1,n+1)}{n}$.

取 $q=r_1=0$, 系统 (29) 约化为

$$u_t=\begin{pmatrix} q_0 \\ r_0 \end{pmatrix}_t=\begin{pmatrix} \frac{\partial}{16}+\frac{1}{2}r_0\partial^{-1}r_0 & -\frac{1}{2}r_0\partial^{-1}q_0 \\ \frac{1}{2}q_0\partial^{-1}r_0 & \frac{\partial}{16}-\frac{1}{2}q_0\partial^{-1}q_0 \end{pmatrix}\begin{pmatrix} -2q_0\partial^{-1}r_0 & \frac{\partial}{4}-2q_0\partial^{-1}q_0 \\ \frac{\partial}{4}+2r_0\partial^{-1}r_0 & 2r_0\partial^{-1}q_0 \end{pmatrix}$$

$$\cdot\begin{pmatrix} 4b(0,n-1) \\ -4c(0,n-1) \end{pmatrix}=\tilde{J}\tilde{L}\begin{pmatrix} 4b(0,n-1) \\ -4c(0,n-1) \end{pmatrix}. \tag{34}$$

在系统 (34) 中取 $n=2$, 得到耦合的 Burgers 方程

$$\begin{cases} q_{0t}=-\frac{\alpha}{16}r_{0xx}+\frac{\alpha}{4}r_0(q_0^2-r_0^2), \\ r_{0t}=-\frac{\alpha}{16}q_{0xx}+\frac{\alpha}{4}q_0(q_0^2-r_0^2). \end{cases}$$

取 $n=3$, 系统 (34) 就约化为耦合 KdV 方程

$$\begin{cases} q_{0t}=\frac{\alpha}{64}q_{0xxx}-\frac{\alpha}{16}q_{0x}(q_0^2-r_0^2)-\frac{\alpha}{8}q_0(q_0q_{0x}-r_0r_{0x}), \\ r_{0t}=\frac{\alpha}{64}r_{0xxx}-\frac{\alpha}{16}r_{0x}(q_0^2-r_0^2)-\frac{\alpha}{8}r_0(q_0q_{0x}-r_0r_{0x}). \end{cases}$$

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