有面積測度的遠交聯絡空間的體積幾何學*

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一、緒 論

本論文裏面所討論的空間，和前篇[1][1]一樣地，是有兩種結構的.

第一種結構是：對於空間 S_N 的任何可導的 K 次元流形 V_K

$$x^i = x^i(u^\alpha) \quad (i = 1, \cdots, N;\ \alpha = 1, \cdots, K) \tag{1.1}$$

定義其一區域 R 的體積爲

$$A = \int_R F(x, p)(du)^K \qquad \left(p_\alpha^i = \frac{\partial u^i}{\partial u^\alpha}\right), \tag{1.2}$$

但 $(du)^K$ 是 $du^1 du^2 \cdots du^K$ 的縮寫且函數 $F(x, p)$ 對於變換

$$\bar{x} \longleftrightarrow x, \qquad \det\left|\frac{\partial \bar{x}^i}{\partial x^j}\right| \neq 0 \tag{1.3}$$

是不變的，而對於參數變換

$$\bar{u} \longleftrightarrow u, \qquad \Delta = \det\left|\frac{\partial \bar{u}^\alpha}{\partial u^\beta}\right| > 0 \tag{1.4}$$

則受到變換

$$F\left(x, p_\beta^i \frac{\partial u^\beta}{\partial \bar{u}^\alpha}\right) = \Delta^{-1} F(x, p_\alpha^i), \tag{1.5}$$

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1）方括弧內數字是指示文末參考文獻

就是: F 關於 (1.4) 是正齊零次且重 1 的函數. 有了這一假定, 才能保證體積積分 (1.2) 與參數 u^α 的選擇無關, 而帶上流形固有的意義.

第二種結構是: 空間 S_N 有遠交聯絡, 且它的聯絡係數 Γ^i_{jk} 是依據

$$\Gamma^i_{jk} = \frac{1}{K(K+1)} H^i_{\alpha\beta}(x,p)\,|^{\alpha\beta}_{jk} \tag{1.6}$$

所定義的, 但 $H^i_{\alpha\beta}$ 表示關於 α,β 是對稱的齊次函數系統 [2] 且在流形 V_K (1.1) 的各點 (u^α) 所作的函數系統

$$P^i_{\alpha\beta} \equiv \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + H^i_{\alpha\beta}(x,p) \tag{1.7}$$

關於兩變換 (1.3) 及 (1.4) 都是張量不變的 [3].

在前篇文章內討論積分 (1.2) 的第二變差是在參數 u^α 僅受遠交變換

$$\bar{u}^\alpha = A^\alpha_\beta u^\beta + B^\alpha \quad (A^\alpha_\beta, B^\alpha \text{ 常數; } \det|A^\alpha_\beta| > 0) \tag{1.8}$$

的條件下所計算的, 所以可以看做遠交幾何學.

這篇的內容是繼前篇的問題而討論更廣泛的參數變換下的結果, 就是當 (1.4) 的變換行列式 Δ 是正常數時應有的結果. 我們定義了體積的聯絡係數 (見第二節) 及波爾特洛帯的微分 (見第三節), 於是導出聯繫方程式 (見第四節). 從此容易求出第二變差的體積幾何形式.

二、體積的聯絡係數

為明瞭函數系統 $H^i_{\alpha\beta}$ 對於變換 (1.4) 的變更情況, 必須根據 (1.7) 是張量不變的假設進行計算. 若以

$$\theta^\alpha_{\cdot\beta} = \frac{\partial u^\alpha}{\partial \bar{u}^\beta}, \tag{2.1}$$

則

$$\bar{p}^i_\alpha \equiv \frac{\partial x^i}{\partial \bar{u}^\alpha} = \theta^\gamma_{.\alpha} p^i_\gamma,$$

$$\frac{\partial^2 x^i}{\partial \bar{u}^\alpha \partial \bar{u}^\beta} = \theta^\gamma_{.\alpha} \theta^\delta_{.\beta} \frac{\partial^2 x^i}{\partial u^\gamma \partial u^\delta} + \frac{\partial^2 u^\gamma}{\partial \bar{u}^\alpha \partial \bar{u}^\beta} p^i_\gamma,$$

於是

$$\begin{aligned}\bar{P}^i_{\alpha\beta} &\equiv \frac{\partial^2 x^i}{\partial \bar{u}^\alpha \partial \bar{u}^\beta} + \bar{H}^i_{\alpha\beta}(x,\bar{p}) \\ &= \theta^\gamma_{.\alpha}\theta^\delta_{.\beta} P^i_{\gamma\delta} + \bar{H}^i_{\alpha\beta}(x,\bar{p}) - \theta^\gamma_{.\alpha}\theta^\delta_{.\beta} H^i_{\gamma\delta}(x,p) \\ &\qquad + \frac{\partial^2 u^\gamma}{\partial \bar{u}^\alpha \partial \bar{u}^\beta} \frac{\partial \bar{u}^\varrho}{\partial u^\gamma} \bar{p}^i_\varrho \\ &= \theta^\gamma_{.\alpha}\theta^\delta_{.\beta} P^i_{\gamma\delta} + \bar{H}^i_{\alpha\beta}(x,\bar{p}) - H^i_{\gamma\delta}(x,\bar{p}) \\ &\qquad + \frac{\partial^2 u^\gamma}{\partial \bar{u}^\alpha \partial \bar{u}^\beta} \frac{\partial \bar{u}^\varrho}{\partial u^\gamma} \bar{p}^i_\varrho;\end{aligned}$$

所以

$$\bar{H}^i_{\alpha\beta}(x,\bar{p}) = H^i_{\alpha\beta}(x,\bar{p}) + K^\varrho_{\alpha\beta}(x,\bar{p})\,\bar{p}^i_\varrho, \tag{2.2}$$

但

$$K^\varrho_{\alpha\beta} = -\frac{\partial \bar{u}^\varrho}{\partial u^\sigma} \frac{\partial^2 u^\sigma}{\partial \bar{u}^\alpha \partial \bar{u}^\beta}. \tag{2.3}$$

微分 (2.2) 的兩側且利用 $H^i_{\alpha\beta}$ 的齊次性, 就可導出

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j A_k + \delta^i_k A_j + p^i_\lambda B^\lambda_{jk}, \tag{2.4}$$

式中

$$\left.\begin{aligned} A_k &= \frac{1}{K(K+1)} K^\alpha_{\alpha\beta}\,|^\beta_k, \\ B^\lambda_{jk} &= \frac{1}{K(K+1)} K^\lambda_{\alpha\beta}\,|^{\alpha\beta}_{jk} \end{aligned}\right\} \tag{2.5}$$

依照 K 展空間論中所用的方法, 從 (2.4) 得知函數系統

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{\delta^i_j}{N+1}\Gamma^a_{ak} - \frac{\delta^i_k}{N+1}\Gamma^a_{ja}$$

$$- \frac{p^i_\lambda}{N-K}\left(\Gamma^a_{jk}|^\lambda_a - \frac{1}{N+1}\Gamma^a_{ak}|^\lambda_j - \frac{1}{N+1}\Gamma^a_{ja}|^\lambda_k\right) \quad (2.6)$$

在變換 (1.4) 之下是不變的. 因此, 所獲的新聯絡 Π^i_{jk} 稱爲射影聯絡.

特別當參數變換是遠交變換 (1.8) 時, 從 (2.3) 得知此時

$$K^\varrho_{\alpha\beta} = 0,$$

於是

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}$$

換言之, Γ^i_{jk} 有遠交不變的性質.

還有一種情况介於上述兩種聯絡之間, 就是體積的聯絡 V^i_{jk}. 這是依條件

$$\Delta = \text{const} \quad (2.7)$$

所定義的參數變換 (1.4) 的不變聯絡. 在這條件下,

$$K^\lambda_{\lambda\beta} = 0; \qquad A_k = 0, \quad (2.8)$$

於是

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + p^i_\lambda B^\lambda_{jk}. \quad (2.9)$$

可是

$$\bar{\Gamma}^i_{jk}|^\lambda_a = \Gamma^i_{jk}|^\lambda_a + \delta^i_a B^\lambda_{jk} + p^i_\mu B^\mu_{jk}|^\lambda_a$$

且關於 i, a 縮短; 由於 B^μ_{jk} 的齊次性

$$B^\mu_{jk}|^\lambda_a p^a_\mu = -K B^\lambda_{jk},$$

所以

$$\bar{\Gamma}^a_{jk}|^\lambda_a = \Gamma^a_{jk}|^\lambda_a + (N-K) B^\lambda_{jk}.$$

因此

$$V^i_{jk} = \Gamma^i_{jk} - \frac{1}{N-K} p^i_\lambda \Gamma^a_{jk}|^\lambda_a \quad (2.10)$$

是體積不變的聯絡係數.

三、波爾特洛蒂的微分

現在討論 S_N 的一流形 V_K

$$x^i = x^i(u^\alpha) \tag{3.1}$$

的體積積分 (1.2):

$$A = \int_R F(x,p)(du)^K \qquad \left(p^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}\right). \tag{3.2}$$

函數 F 對於參數變換 (1.4) 必須滿足 (1.5)

$$F\left(x, p^i_\beta \frac{\partial u^\beta}{\partial \bar{u}^\gamma}\right) = \det\left|\frac{\partial u^\beta}{\partial \bar{u}^\gamma}\right| \cdot F(x,p). \tag{3.3}$$

關於 $\frac{\partial u^\beta}{\partial \bar{u}^\gamma}$ 偏微分 (3.3) 的兩側且把 $\frac{\partial u^\beta}{\partial \bar{u}^\gamma} = \delta^\beta_\gamma$, 就可得到關係式

$$F(x,p)\,|^\gamma_h p^h_\beta = \delta^\gamma_\beta F(x,p). \tag{3.4}$$

在前篇所定義的聯繫方程式是以 Γ^i_{jk} 爲基礎的; 它只能適用到遠交幾何學. 爲了要在體積幾何學獲得類似的方程式, 必須重新定義任何 K 重面積原素 (p^i_α) 以它本身做支持原素時的張量不變的平行移動. 因此, 選定波爾特洛蒂 (Bortolotti) 的微分 [4]

$$\bar{\omega}^h_\alpha = dp^h_\alpha + \Gamma^h_{ki} p^h_\alpha dx^i + G^\beta_{\alpha\lambda} p^h_\beta du^\lambda, \tag{3.5}$$

式中

$$G^\beta_{\alpha\lambda} = \frac{1}{N}[H^i_{\alpha\lambda}|^\beta_i - \Gamma^i_{ih}(p^h_\alpha \delta^\beta_\lambda + p^h_\lambda \delta^\beta_\alpha)]; \tag{3.6}$$

並且依據

$$\bar{\omega}^h_\alpha = 0 \tag{3.7}$$

定義面積原素 $(p^i_\alpha) = \left(\frac{\partial x^i}{\partial u^\alpha}\right)$ 的平行移動.

從 (1.6) 易證

$$H^i_{\alpha\lambda} = \Gamma^i_{jk} p^j_\alpha p^k_\lambda, \tag{3.8}$$

於是

$$H^{i}_{\alpha\lambda}\big|^{\beta}_{i} = \Gamma^{i}_{jk}\big|^{\beta}_{i} p^{j}_{\alpha} p^{k}_{\lambda} + \Gamma^{i}_{ih}\delta^{\beta}_{\alpha} p^{h}_{\lambda} + \Gamma^{i}_{ji} p^{j}_{\alpha}\delta^{\beta}_{\lambda}$$

$$= \Gamma^{i}_{jk}\big|^{\beta}_{i} p^{j}_{\alpha} p^{k}_{\lambda} + \Gamma^{i}_{ih}(p^{h}_{\alpha}\delta^{\beta}_{\lambda} + p^{h}_{\lambda}\delta^{\beta}_{\alpha})$$

代入於 (3.6) 的結果,

$$G^{\beta}_{\alpha\lambda} = \frac{1}{N}\Gamma^{i}_{jk}\big|^{\beta}_{i} p^{j}_{\alpha} p^{k}_{\lambda}. \tag{3.9}$$

因此, 在所論的空間 S_N 波爾特洛蒂的微分是

$$\bar{\omega}_{\alpha} = d p^{h}_{\alpha} + \Gamma^{h}_{ki} p^{k}_{\alpha} d x^{i} + \frac{1}{N}\Gamma^{i}_{jk}\big|^{\beta}_{i} p^{h}_{\beta} p^{j}_{\alpha} p^{k}_{\lambda} d u^{\lambda}, \tag{3.10}$$

且平行移動的條件 (3.7) 是

$$d p^{h}_{\alpha} = -\Gamma^{h}_{ki} p_{\alpha} d x^{i} - \frac{1}{N}\Gamma^{i}_{jk}\big|^{\beta}_{i} p^{h}_{\beta} p^{j}_{\alpha} p^{k}_{\lambda} d u^{\lambda}. \tag{3.11}$$

在這平行移動之下, 函數 F 的變差等於

$$\delta F(x,p) = F_{\cdot i}\, d x^{i} + F\big|^{\alpha}_{h} d p^{h}_{\alpha}$$

$$= (F_{\cdot i} - \Gamma^{h}_{ik} p^{k}_{\alpha} F\big|^{\alpha}_{h})\, d x^{i}$$

$$- \frac{1}{N}\Gamma^{i}_{jk}\big|^{\gamma}_{i} F\big|^{\alpha}_{h} p^{h}_{\gamma} p^{j}_{\alpha} p^{k}_{\beta} d u^{\beta}.$$

可是由於 (3.4) 及週知恒等式

$$\Gamma^{i}_{jk}\big|^{\alpha}_{h} p^{j}_{\alpha} = 0, \tag{3.12}$$

得知

$$\Gamma^{i}_{jk}\big|^{i}_{\gamma} F\big|^{\alpha}_{h} p^{h}_{\gamma} p^{j}_{\alpha} = F\,\Gamma^{i}_{jk}\big|^{\alpha}_{i} p^{j}_{\alpha} = 0,$$

所以

$$\delta F(x,p) = (F_{\cdot i} - F\big|^{\alpha}_{h} p^{k}_{\alpha}\Gamma^{h}_{ik})\, d x^{i}. \tag{3.13}$$

四、 聯繫方程式及體積積分的第二變差

上述兩種結構之間假定有一聯繫: 任何一 K 重面積原素在依靠它本身做支持原素的平行移動 (3.7) 之下, 它的測度是不變的, 就是

$$\delta F(x,p)=0. \tag{4.1}$$

因此, 聯繫方程式當爲

$$F_{\cdot i}-F|^{\alpha}_{h}p^{j}_{\alpha}\Gamma^{h}_{ij}=0. \tag{4.2}$$

這關係完全與遠交幾何學同一方程式相符, 但是依據 (3.4) 及 (3.12) 容易把 (4.2) 改寫體積幾何學的形式:

$$F_{\cdot i}-F|^{\alpha}_{h}p^{j}_{\alpha}V^{h}_{ij}=0. \tag{4.3}$$

這樣, 和前篇一樣地導入向量 $\xi^i(x,t)$ 及 $\dfrac{\partial\xi^i}{\partial t}$ 關於聯絡係數 V^i_{jk} 的共變微分, 例如,

$$\xi^i_{|j}=\xi^i_{\cdot j}+V^i_{hj}\xi^h. \tag{4.4}$$

再選用記號

$$\varDelta_\alpha\xi^i=p^j_\alpha\xi^i_{|j} \tag{4.5}$$

以替前篇的 D_α, 就可改寫該文裏由公式 (7.4) 所表示的體積積分第二變差:

$$\frac{\delta^2 A}{\delta t^2}=\int_R\Big\{F|^{\alpha\beta}_{hl}\varDelta_\alpha\xi^h\varDelta_\beta\xi^l+F|^{\alpha}_{h}p^j_\alpha B^h_{\cdot ijk}\xi^i\xi^k$$

$$+F|^{\alpha}_{h}V^h_{jl}|^{\beta}_{n}\xi^l(2p^j_\alpha\varDelta_\beta\xi^n-P^n_{\alpha\beta}\xi^j)$$

$$+F|^{\alpha}_{h}\varDelta_\alpha\Big(\frac{\partial\xi^h}{\partial t}\Big)\Big\}(du)^K, \tag{4.6}$$

式中 $B^h_{\cdot ijk}$ 表示體積的曲率張量:

$$B^h_{\cdot ijk}=V^h_{ij\cdot k}-V^h_{ik\cdot j}+V^h_{ik}|^{\tau}_{m}p^n_\tau V^m_{nj}-V^h_{ij}|^{\tau}_{m}p^n_\tau V^m_{nk}+V^h_{mk}V^m_{ij}-V^h_{mj}V^m_{ik}. \tag{4.7}$$

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VOLUMENTARY GEOMETRY OF AN AFFINELY CONNECTED SPACE WITH AREAL METRIC

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This note is a sequel to a previous one in which the geometry was assumed to be affine [1]. Besides the notation newly introduced I shall use the same notation.

Let us assume that the space S_N is of affine connection

$$\Gamma^i_{jk} = \frac{1}{K(K+1)} H^i_{\alpha\beta}(x,p)\,|^{\alpha\beta}_{jk},$$

where $H^i_{\alpha\beta}$ is a homogeneous function-system symmetric in the indices α, β [2]. Further, we need to add the condition that the function-system

$$P^i_{\alpha\beta} \equiv \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + H^i_{\alpha\beta}(x,p) \qquad \left(p^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}\right)$$

constructed at each point of a differentiable variety V_K of K dimensions

$$x^i = x^i(u^\alpha)$$

should be tensor-invariant under both sorts of transformations [3]

$$\bar{x} \longleftrightarrow x, \qquad \det\left|\frac{\partial \bar{x}^i}{\partial x^j}\right| \neq 0; \qquad \bar{u} \longleftrightarrow u, \qquad \Delta \equiv \det\left|\frac{\partial \bar{u}^\alpha}{\partial u^\beta}\right| > 0.$$

Especially, under the latter the function-system $H^i_{\alpha\beta}$ is transformed to

$$\bar{H}^i_{\alpha\beta}(x,\bar{p}) = H^i_{\alpha\beta}(x,\bar{p}) + \bar{p}^i_\varrho K^\varrho_{\alpha\beta}(x,\bar{p}),$$

where we have placed

$$K^{\varrho}_{\alpha\beta} = -\frac{\partial \bar{u}^{\varrho}}{\partial u^{\sigma}}\frac{\partial^2 u^{\sigma}}{\partial \bar{u}^{\alpha}\partial \bar{u}^{\beta}}.$$

Consequently, Γ^i_{jk} is transformed to

$$\Gamma^i_{jk} = \Gamma^i_{jk} + \delta^i_j A_k + \delta^i_k A_j + p^i_\lambda B^\lambda_{jk}$$

with the abbreviation

$$A_k = \frac{1}{K(K+1)} K^{\alpha}_{\alpha\beta}|^{\beta}_{k}, \qquad B^{\lambda}_{jk} = \frac{1}{K(K+1)} K^{\lambda}_{\alpha\beta}|^{\alpha\beta}_{jk}.$$

Therefore the function-system defined by

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{\delta^i_j}{N+1}\Gamma^a_{ak} - \frac{\delta^i_k}{N+1}\Gamma^a_{ja}$$

$$-\frac{p^i_\lambda}{N-K}\left(\Gamma^a_{jk}|^{\lambda}_{a} - \frac{1}{N+1}\Gamma^a_{ak}|^{\lambda}_{j} - \frac{1}{N+1}\Gamma^a_{ja}|^{\lambda}_{k}\right)$$

is invariant under the parameter transformation.

In the case $\Delta =$ const. > 0 we obtain the volumentary connection V^i_{jk}:

$$V^i_{jk} = \Gamma^i_{jk} - \frac{1}{N-K} p^i_\lambda \Gamma^a_{jk}|^{\lambda}_{a}.$$

Consider now the "volume integral"

$$\Delta = \int_R F(x,p)(du)^K \qquad \left(p^i_a = \frac{\partial x^i}{\partial u^a}\right),$$

taken on a region R of V_K; the function F obeys the law

$$F\left(x, p^i_\sigma \frac{\partial u^\sigma}{\partial \bar{u}^\gamma}\right) = \Delta^{-1} F(x,p).$$

Then follows the relation

$$F(x,p)\,|_h^\gamma \cdot p_\beta^h \doteq \delta_\beta^\gamma F(x,p).$$

This combined with the well-known identity

$$\Gamma_{jk}^i\,|_h^\alpha p_\alpha^j = 0$$

suffices to demonstrate that

$$F\,|_h^\alpha p_\alpha^j V_{ij}^h = F\,|_h^\alpha p_\alpha^j \Gamma_{ij}^h.$$

In attempting to obtain a tensor-invariant parallelism of any K-ple areal element (p_α^i) when the element itself is taken for the supporting element of the space we adopt Bortolotti's differential [4]:

$$\bar{\omega}_\alpha^h = d p_\alpha^h + \Gamma_{ki}^h p_\alpha^k d x^i + G_{\alpha\lambda}^\beta p_\beta^h d u^\lambda,$$

where

$$G_{\alpha\lambda}^\beta = \frac{1}{N}[H_{\alpha\lambda}^i\,|_i^\beta - \Gamma_{ih}^i(\delta_\lambda^\beta p_\alpha^h + \delta_\alpha^\beta p_\lambda^h)],$$

and impose the condition that the metric function $F(x,p)$ should remain unchanged when the areal element $(p_\alpha^i) = \left(\frac{\partial x^i}{\partial u^\alpha}\right)$ of the variety V_K is subjected to the parallel transport

$$\bar{\omega}_\alpha^h = 0.$$

It is easily shown that the equations of connections are the same as in the former note, namely,

$$F_{\cdot i} - F\,|_h^\alpha p_\alpha^j \Gamma_{ij}^h = 0,$$

or in virtue of the above relation

$$F_{\cdot i} - F\,|_h^\alpha p_\alpha^j V_{ij}^h = 0.$$

Introducing the covariant derivatives of vectors $\xi^i(x,t)$ and $\frac{\partial \xi^i}{\partial t}$ by using the connection coefficients V_{jk}^i, for example,

$$\xi^i_{|j} = \frac{\partial \xi^i}{\partial x^j} + V^i_{hj}\xi^h,$$

and adopting the notation

$$\Delta_\alpha \xi^i = p^j_\alpha \xi^i_{|j},$$

we are led to the second variation of the "volume integral" of an extremal variety V_K:

$$\frac{\delta^2 A}{\delta t^2} = \int_R \Big\{ F|^{\alpha\beta}_{jk} \Delta_\alpha \xi^j \Delta^\beta \xi^k + F|^\alpha_h p^j_\alpha B^h_{\cdot ijk} \xi^i \xi^k$$

$$+ F|^\alpha_h V^h_{jl}|^\beta_n (2 p^j_\alpha \Delta_\beta \xi^n - P^n_{\alpha\beta} \xi^j) \xi^l$$

$$+ F|^\alpha_h \Delta_\alpha \left(\frac{\partial \xi^h}{\partial t}\right) \Big\} (du)^K,$$

where $B^h_{\cdot ijk}$ denotes the volumentary curvature tensor:

$$B^h_{\cdot ijk} = V^h_{ij\cdot k} - V^h_{ik\cdot j} + V^h_{ik}|^\tau_m V^m_{nj} p^n_\tau - V^h_{ij}|^\tau_m V^m_{nk} p^n_\tau + V^h_{mk} V^m_{ij} - V^h_{mj} V^m_{ik}.$$