含孔软铁磁材料 Mindlin 板中弹性波散射问题[1)]

胡 超　韩 刚　黄文虎
(哈尔滨工业大学航天工程与力学系，哈尔滨 150001)

摘要　基于考虑磁弹相互作用的 Mindlin 板弯曲波动方程，采用波函数展开法，分析研究了含孔软铁磁材料 Mindlin 板中弹性波散射与动应力集中问题，给出了问题的分析解和数值算例. 通过分析发现：磁感应强度对动弯矩集中系数和动剪力集中系数有增加的作用，特别是在低频的情况下.

关键词　Mindlin 厚板，弹性波散射，磁弹耦合动力学，软铁磁材料，动应力集中

引　言

平板中弹性波导与动应力集中是固体结构力学研究中的重要课题. 结构中的开孔直接影响着结构的承载能力与使用寿命，因此，国内外有许多专家和学者从事这方面的理论分析与实验研究 [1~3].

在分析计算动应力集中系数或动应力强度因子时, 经典薄板理论具有局限性. Mindlin 提出的厚板理论 [4]，考虑了转动惯性和剪切变形的影响，弥补了经典薄板理论的不足，在工程分析计算中可以获得满意的结果. Pao Yih-Hsing 等采用 Mindlin 厚板理论，对含圆孔平板弹性波散射问题进行了研究，并给出了分析解和数值结果 [5,6].

随着现代科学技术的发展，在超导核电站、磁悬浮列车上开始采用物理和力学性能比较好的铁磁材料结构. 处于电磁场中的铁磁材料结构，裂纹或孔洞附近的应力可能会增大，因而影响结构的承载能力和使用寿命. 据有关文献资料 [7]，铁磁弹性结构的动态特性明显地受到磁场的影响. 日本一些学者曾基于磁弹性动力学理论 [8,9]，研究了含裂纹铁磁材料 Mindlin 板中波散射与动弯矩强度因子等问题. 方法是采用 Fourier 变换，把混合边值问题归结为 Fredholm 积分方程求解.

本文将基于考虑磁弹相互作用的 Mindlin 板弯曲波动方程，采用波函数展开法，分析研究含圆孔软铁磁材料平板中弹性波散射与动应力集中问题，给出问题的解析解和数值算例.

1　软铁磁材料 Mindlin 板弯曲波动方程

设软铁磁弹性板的厚度为 $2s$. x 轴和 y 轴为板中面的坐标轴，z 轴为厚度方向. 板放在垂直入射的静态均匀磁场中，磁感应强度为 $\boldsymbol{B}_0$.

设整个磁场物理量分为两部分，一部分为基本物理量状态，即刚性状态，另一部分为微扰物理量状态. 总的磁场可描述为

$$\left.\begin{aligned} \boldsymbol{B} &= \boldsymbol{B}_0 + \boldsymbol{b} \\ \boldsymbol{M} &= \boldsymbol{M}_0 + \boldsymbol{m} \\ \boldsymbol{H} &= \boldsymbol{H}_0 + h \end{aligned}\right\} \tag{1}$$

式中，$\boldsymbol{B}, \boldsymbol{M}, \boldsymbol{H}$ 分别为磁感应强度、磁化强度和磁场强度矢量，下标 0 表示不变磁场的物理量，小写字母表示相应物理量的微扰量. 刚体状态下的各磁场物理量为

当 $|z| > s$ 时

$$\left.\begin{aligned} B_{0z}^e &= B_0 \\ H_{0z}^e &= \frac{B_0}{\mu_0} \\ M_{0z}^e &= 0 \end{aligned}\right\} \tag{2a}$$

当 $|z| \le s$ 时

$$\left.\begin{aligned} B_{0z} &= B_0 \\ H_{0z} &= \frac{B_0}{\mu_0\mu_r} \\ M_{0z} &= \frac{\chi B_0}{\mu_0\mu_r} \end{aligned}\right\} \tag{2b}$$

式中，B_{0z}, H_{0z}, M_{0z} 分别表示 $\boldsymbol{B}_0, \boldsymbol{H}_0, \boldsymbol{M}_0$ 在 z 方向上的量，上标 e 为板外的物理量值；μ_0 是真空

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1) 国家自然科学基金资助项目 (19972018).

磁导率， $\mu_0 = 4\pi\times10^{-7}\,\mathrm{N/A^2}$; μ_r 是相对磁导率，$\mu_r = 1+\chi$.

在本研究问题中，对于板中的磁场，假设

$$\left.\begin{aligned}&\nabla\cdot\boldsymbol{b}=0\\&\nabla\times\boldsymbol{h}=0\end{aligned}\right\}\tag{3}$$

引进磁位势函数 φ，这样，微扰状态下的方程 (3) 满足

$$\left.\begin{aligned}&\boldsymbol{h}=\nabla\varphi\\&\nabla^2\varphi=0\end{aligned}\right\}\tag{4}$$

忽略磁致伸缩效应，当 $|M_{0z}\partial\boldsymbol{u}/\partial z|<|m|$ 时，软铁磁材料 Mindlin 板弯曲波动方程 [8,9]

$$\nabla\cdot\boldsymbol{\sigma}+\frac{2\chi B_0}{\mu_r}\frac{\partial}{\partial z}\nabla\varphi=\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}\tag{5}$$

在 $|z|=s$ 平板边界时，线性化后的边值条件为

$$\left.\begin{aligned}&\sigma_{zx}=-\frac{\chi B_0}{\mu_r}\frac{\partial\varphi}{\partial x},\qquad\sigma_{zy}=-\frac{\chi B_0}{\mu_r}\frac{\partial\varphi}{\partial y}\\&\sigma_{zz}=\frac{\chi(\chi-2)}{\mu_r}\Big[\frac{B_0^2}{2\mu_0\mu_r}+B_0\frac{\partial\varphi}{\partial z}\Big]\end{aligned}\right\}\tag{6}$$

根据 Mindlin 板理论，在直角坐标系下位移分量 u_x,u_y,u_z 的表达式为

$$u_x=z\psi_x(x,y,t),\ \ u_y=z\psi_y(x,y,t),\ \ u_z=W(x,y,t)\tag{7}$$

式中， W 表示板的法向位移，而 ψ_x 和 ψ_y 分别表示关于 x 轴和 y 轴上的法线转动. 这样，板内弯矩 $M_{xx}^m, M_{yy}^m, M_{xy}^m$ 和剪力 Q_{xx}, Q_{yy} 可描述为

$$\left.\begin{aligned}&M_{xx}^m=\int_{-s}^{s}z\sigma_{xx}\mathrm{d}z=D\Big(\frac{\partial\psi_x}{\partial x}+v\frac{\partial\psi_y}{\partial y}\Big)\\&M_{yy}^m=\int_{-s}^{s}z\sigma_{yy}\mathrm{d}z=D\Big(\frac{\partial\psi_y}{\partial y}+v\frac{\partial\psi_x}{\partial x}\Big)\\&M_{xy}^m=M_{yx}^m=\int_{-s}^{s}z\sigma_{xy}\mathrm{d}z=\\&\qquad\frac{(1-v)}{2}D\Big(\frac{\partial\psi_y}{\partial x}+\frac{\partial\psi_x}{\partial y}\Big)\\&Q_{xx}=\int_{-s}^{s}\sigma_{xz}\mathrm{d}z=2\kappa Gs\Big(\frac{\partial W}{\partial x}+\psi_x\Big)\\&Q_{yy}=\int_{-s}^{s}\sigma_{yz}\mathrm{d}z=2\kappa Gs\Big(\frac{\partial W}{\partial y}+\psi_y\Big)\end{aligned}\right\}\tag{8}$$

式中， κ 是剪切折算系数， $\kappa=\pi^2/12$； D 为平板的抗弯刚度， $D=2Es^3/3(1-\nu^2)$.

利用方程 (5)~(8)，可以得到如下表达式

$$\left.\begin{aligned}&\frac{\partial M_{xx}^m}{\partial x}+\frac{\partial M_{yx}^m}{\partial y}-Q_{xx}+L_{xx}=\frac{2}{3}\rho s^3\frac{\partial^2\psi_x}{\partial t^2}\\&\frac{\partial M_{xy}^m}{\partial x}+\frac{\partial M_{yy}^m}{\partial y}-Q_{yy}+L_{yy}=\frac{2}{3}\rho s^3\frac{\partial^2\psi_y}{\partial t^2}\\&\frac{\partial Q_{xx}}{\partial x}+\frac{\partial Q_{yy}}{\partial y}+q=2\rho s\frac{\partial^2 W}{\partial t^2}\end{aligned}\right\}\tag{9}$$

施加到板内的合外力矩 L_{xx}, L_{yy}, 可用如下式子表示

$$\left.\begin{aligned}&L_{xx}=-\frac{\chi B_0 s}{\mu_r}\Big[\frac{\partial\varphi(s)}{\partial x}+\frac{\partial\varphi(-s)}{\partial x}\Big]+\\&\qquad\frac{2\chi B_0}{\mu_r}\int_{-s}^{s}z\frac{\partial^2\varphi}{\partial x\partial z}\mathrm{d}z\\&L_{yy}=-\frac{\chi B_0 s}{\mu_r}\Big[\frac{\partial\varphi(s)}{\partial y}+\frac{\partial\varphi(-s)}{\partial y}\Big]+\\&\qquad\frac{2\chi B_0}{\mu_r}\int_{-s}^{s}z\frac{\partial^2\varphi}{\partial y\partial z}\mathrm{d}z\end{aligned}\right\}\tag{10}$$

作用在板表面上的外载 q 可描述为

$$q=\frac{\chi(\chi-2)B_0}{\mu_r}\Big[\frac{\partial\varphi(s)}{\partial z}-\frac{\partial\varphi(-s)}{\partial z}\Big]+\frac{2\chi B_0}{\mu_r}\int_{-s}^{s}\frac{\partial^2\varphi}{\partial z^2}\mathrm{d}z\tag{11}$$

2 弹性波散射问题的分析求解

把方程 (7), (8) 代入到方程 (9) 中，可得到在磁场作用下 Mindlin 板的运动方程

$$\left.\begin{aligned}&D\Big[\frac{\partial^2\psi_x}{\partial x^2}+\frac{(1-v)}{2}\frac{\partial^2\psi_x}{\partial y^2}+\frac{(1+v)}{2}\frac{\partial^2\psi_y}{\partial x\partial y}\Big]-\\&\qquad C\Big(\frac{\partial W}{\partial x}+\psi_x\Big)=\rho J\frac{\partial^2\psi_x}{\partial t^2}-L_{xx}\\&D\Big[\frac{\partial^2\psi_y}{\partial y^2}+\frac{(1-v)}{2}\frac{\partial^2\psi_y}{\partial x^2}+\frac{(1+v)}{2}\frac{\partial^2\psi_x}{\partial x\partial y}\Big]-\\&\qquad C\Big(\frac{\partial W}{\partial y}+\psi_y\Big)=\rho J\frac{\partial^2\psi_y}{\partial t^2}-L_{yy}\\&C\Big(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}+\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}\Big)=\\&\qquad 2s\rho\frac{\partial^2 W}{\partial t^2}-q\end{aligned}\right\}\tag{12}$$

式中， $C=2\kappa Gs$； G 是 Lame 常数， $G=E/2(1+\nu)$； J 为平板的转动惯量， $J=2s^3/3$.

z 方向的扰动电磁波势函数与板的横向位移函数成正比. 板内 z 方向上的磁位势函数为

$$\varphi=a_1\cosh(k_1 z)W\quad(|z|\le s)\tag{13}$$

研究软铁磁材料 Mindlin 板，圆孔的半径为 a，圆心位于坐标原点．磁感应强度为 $\boldsymbol{B}_0$ 的磁场垂直作用于平板．不失一般性，设入射波沿 x 轴正方向传播，其数学表达式为

$$F_1^{(\mathrm{i})}=F_0\mathrm{e}^{\mathrm{i}(k_1x-\omega t)},\quad F_2^{(\mathrm{i})}=0,\quad f^{(\mathrm{i})}=0 \tag{14}$$

式中，F_0 为入射波的幅值；k_1,ω 为弹性波入射波数和圆频率．

电磁波场沿 z 方向，电磁波势函数与板的横向位移函数成正比．板内 z 方向上形成驻波模，而在板外 z 方向上形成辐射模．经分析可知，磁位势函数可描述为

$$\left.\begin{aligned}&\varphi^{(\mathrm{i})}=[a_1\cosh(k_1z)+a_2\sinh(k_1z)]W^{(\mathrm{i})}\\&\qquad\qquad(|z|\le s)\\&\varphi^{(\mathrm{ei})}=a_3\exp(-k_1z)W^{(\mathrm{i})}\quad(z>s)\\&\varphi^{(\mathrm{ei})}=a_4\exp(k_1z)W^{(\mathrm{i})}\quad(z<-s)\end{aligned}\right\} \tag{15}$$

式中，a_1,a_2,a_3,a_4 为电磁波模式系数，可由问题的边界条件确定．由边界条件可得

$$\left.\begin{aligned}&a_1=\frac{\chi B_0}{\mu_0\mu_r\varDelta},\quad a_2=0\\&a_3=a_4=-a_1\mu_r\exp(k_1s)\sinh(k_1s)\end{aligned}\right\} \tag{16}$$

其中，$\varDelta=\mu_r\sinh(k_1s)+\cosh(k_1s)$．

因此，板内入射的电磁场与入射的弹性场的关系

$$\varphi^{(\mathrm{i})}=\frac{\chi B_0}{\mu_0\mu_r\varDelta}\cosh(k_1z)W^{(\mathrm{i})} \tag{17}$$

为获得问题解析解 [10]，引进 3 个函数 F,f,g，设

$$\left.\begin{aligned}&\varPsi_x=\frac{\partial F}{\partial x}+\frac{\partial f}{\partial y}\\&\varPsi_y=\frac{\partial F}{\partial y}-\frac{\partial f}{\partial x}\\&W=\frac{1}{I}\Big[\frac{D}{C}\nabla^2F-\Big(1+\frac{\rho J}{C}\frac{\partial^2}{\partial t^2}\Big)F\Big]\\&L_x=\frac{\partial g}{\partial x}\\&L_y=\frac{\partial g}{\partial y}\end{aligned}\right\} \tag{18}$$

式中

$$\left.\begin{aligned}&I=1-\frac{2\chi B_0}{Ck_1\mu_r}a_1[k_1s\cosh(k_1s)-2\sinh(k_1s)]\\&g(x,y)=\frac{2\chi B_0}{k_1\mu_r}a_1[k_1s\cosh(k_1s)-\\&\qquad 2\sinh(k_1s)]W(x,y,t)\end{aligned}\right\} \tag{19}$$

将方程 (19) 代入到方程 (11) 中，可得如下表达式

$$q=\frac{2\chi^2B_0}{\mu_r}a_1k_1\sinh(k_1s)W(x,y,t) \tag{20}$$

把式 (18) 代入到式 (12) 中，利用复变函数的解析函数理论，可得方程式 [10]

$$\left.\begin{aligned}&D\nabla^2\nabla^2F+\Big[\frac{D}{C}\frac{2\chi^2B_0}{\mu_r}a_1k_1\sinh(k_1s)-\\&\quad C(1-I)-2s\rho\frac{D}{C}\frac{\partial^2}{\partial t^2}-\rho J\frac{\partial^2}{\partial t^2}\Big]\nabla^2F-\\&\quad\Big[\frac{2\chi^2B_0}{\mu_r}a_1k_1\sinh(k_1s)-2s\rho\frac{\partial^2}{\partial t^2}\Big]\cdot\\&\quad\Big(1+\frac{\rho J}{C}\frac{\partial^2}{\partial t^2}\Big)F=0\\&\frac{(1-\nu)}{2}D\nabla^2f-Cf=\rho J\frac{\partial^2f}{\partial t^2}\end{aligned}\right\} \tag{21}$$

方程 (21) 可写为如下形式

$$\left.\begin{aligned}&\nabla^2F_1+k_1^2F_1=0\\&\nabla^2F_2-k_2^2F_2=0\\&\nabla^2f-k_3^2f=0\end{aligned}\right\} \tag{22}$$

其中，k_1 和 k_2 应满足频散方程式

$$\begin{aligned}&k^4-\Big[\frac{2(1+\nu)\chi^3}{\kappa s\mu_0\mu_r^2\varDelta}\frac{B_0^2}{E}k_1\sinh(k_1s)-\frac{C(1-I)}{D}+\\&\quad\frac{8s^2}{\pi^2(1-\nu)}k_0^4+\frac{1}{3}s^2k_0^4\Big]k^2-\\&\quad\Big[\frac{3(1-\nu^2)\chi^3}{s^3\mu_0\mu_r^2\varDelta}\frac{B_0^2}{E}k_1\sinh(k_1s)+k_0^4\Big]\cdot\\&\quad\Big(1-\frac{8s^4}{3\pi^2(1-\nu)}k_0^4\Big)=0\end{aligned} \tag{23a}$$

而

$$k_3^2=\frac{2(C-\rho J\omega^2)}{D(1-\nu)}=\frac{\pi^2}{4}\frac{1}{s^2}-\frac{2}{3(1-\nu)}k_0^4s^2 \tag{23b}$$

式中，$k_0=\Big[\dfrac{2\rho s\omega^2}{D}\Big]^{1/4}$ 对应经典薄板的入射波波数．

这样，方程 (22) 的一般解可描述为

$$\left.\begin{aligned}&F=F_1+F_2=\sum_{n=-\infty}^{+\infty}A_n\mathrm{H}_n^{(1)}(k_1r)\mathrm{e}^{\mathrm{i}n\theta}+\\&\qquad\sum_{n=-\infty}^{+\infty}B_n\mathrm{K}_n(k_2r)\mathrm{e}^{\mathrm{i}n\theta}\\&f=\sum_{n=-\infty}^{+\infty}C_n\mathrm{K}_n(k_3r)\mathrm{e}^{\mathrm{i}n\theta}\end{aligned}\right\} \tag{24}$$

其中，A_n, B_n, C_n 为由开孔边界条件确定的弹性波模式系数；$\mathrm{H}_n^{(1)}(\cdot)$ 为第 1 类 Hankel 函数；$\mathrm{K}_n(\cdot)$ 为修正 Bessel 函数.

3 入射波的激发和总弹性波场

设平板中的弹性波入射场为

$$\left.\begin{aligned} &F_1^{(\mathrm{i})} = F_0\mathrm{e}^{\mathrm{i}(k_1x-\omega t)} \\ &F_2^{(\mathrm{i})} = 0 \\ &f^{(\mathrm{i})} = 0 \end{aligned}\right\} \tag{25}$$

板中孔洞产生的弹性波散射场的数学表达式为

$$\left.\begin{aligned} F^{(\mathrm{s})} = &\sum_{n=-\infty}^{+\infty} A_n\mathrm{H}_n^{(1)}(k_1r)\mathrm{e}^{\mathrm{i}(n\theta-\omega t)} + \\ &\sum_{n=-\infty}^{+\infty} B_n\mathrm{K}_n(k_2r)\mathrm{e}^{\mathrm{i}(n\theta-\omega t)} \\ f^{(\mathrm{s})} = &\sum_{n=-\infty}^{+\infty} C_n\mathrm{K}_n(k_3r)\mathrm{e}^{\mathrm{i}(n\theta-\omega t)} \end{aligned}\right\} \tag{26}$$

这样，平板中孔附近的总弹性波场应由入射场与散射场叠加而成，其表达式为

$$\left.\begin{aligned} &F = F^{(\mathrm{i})} + F^{(\mathrm{s})} \\ &f = f^{(\mathrm{i})} + f^{(\mathrm{s})} \end{aligned}\right\} \tag{27}$$

4 边值条件与模式系数 A_n, B_n, C_n 的确定

以广义位移形式给出的边值条件可描述为

$$\psi_{\mathrm{n}} = \bar{\psi}_{\mathrm{n}}, \quad \psi_{\mathrm{t}} = \bar{\psi}_{\mathrm{t}}, \quad W_{\mathrm{n}} = \overline{W}_{\mathrm{n}} \tag{28}$$

式中，n,t 分别为边界的法线和切线方向.

用广义内力形式给出的边值条件可描述为

$$M_{\mathrm{n}}^m = \overline{M}_{\mathrm{n}}^m, \quad M_{\mathrm{nt}}^m = \overline{M}_{\mathrm{nt}}^m, \quad Q_{\mathrm{n}} = \overline{Q}_{\mathrm{n}} \tag{29}$$

把入射场和散射场的表达式代入到总波场表达式中，满足边界条件，可得确定模式系数的方程

$$\boldsymbol{\Gamma X} = \boldsymbol{Y} \tag{30}$$

其中

$$\Gamma_{11} = k_1^2\mathrm{H}_n''^{(1)}(k_1a) - \nu n^2\mathrm{H}_n^{(1)}(k_1a) + \nu k_1\mathrm{H}_n'^{(1)}(k_1a)$$

$$\Gamma_{12} = k_2^2\mathrm{K}_n''(k_2a) - \nu n^2\mathrm{K}_n(k_2a) + \nu k_2\mathrm{K}_n'(k_2a)$$

$$\Gamma_{13} = \mathrm{i}n(1-\nu)[k_3\mathrm{K}_n'(k_3a) - \mathrm{K}_n(k_3a)]$$

$$\Gamma_{21} = 2\mathrm{i}n[k_1\mathrm{H}_n'^{(1)}(k_1a) - \mathrm{H}_n^{(1)}(k_1a)]$$

$$\Gamma_{22} = 2\mathrm{i}n[k_2\mathrm{K}_n'(k_2a) - \mathrm{K}_n(k_2a)]$$

$$\Gamma_{23} = k_3\mathrm{K}_n'(k_3r) - k_3^2\mathrm{K}_n''(k_3r) - n^2\mathrm{K}_n(k_3r)$$

$$\begin{aligned}\Gamma_{31} = &k_1^3\mathrm{H}_n'''^{(1)}(k_1a) + k_1^2\mathrm{H}_n''^{(1)}(k_1a) - k_1n^2\mathrm{H}_n'^{(1)}\cdot \\ &(k_1a) + 2n^2\mathrm{H}_n^{(1)}(k_1a) + \left[\frac{C}{D}\left(\frac{\rho J}{C}\omega^2 - 1 + I\right) - 1\right]\cdot \\ &k_1\mathrm{H}_n'^{(1)}(k_1a)\end{aligned}$$

$$\begin{aligned}\Gamma_{32} = &k_2^3\mathrm{K}_n'''(k_2a) + k_2^2\mathrm{K}_n''(k_2a) - k_2n^2\mathrm{K}_n'(k_2a) + \\ &2n^2\mathrm{K}_n(k_2a) + \left[\frac{C}{D}\left(\frac{\rho J}{C}\omega^2 + I - 1\right) - 1\right]k_2\mathrm{K}_n'(k_2a)\end{aligned}$$

$$\Gamma_{33} = \frac{IC}{D}\mathrm{K}_n(k_3a)\mathrm{i}n$$

$$\boldsymbol{X} = (A_n, B_n, C_n)^{\mathrm{T}}, \quad \boldsymbol{Y} = (Y_1, Y_2, Y_3)^{\mathrm{T}}$$

$$Y_1 = -\mathrm{i}^n[k_1^2\mathrm{J}_n''(k_1a) - \nu n^2\mathrm{J}_n(k_1a) + \nu k_1\mathrm{J}_n'(k_1a)]$$

$$Y_2 = -2n\mathrm{i}^{n+1}F_0[k_1\mathrm{J}_n'(k_1a) - \mathrm{J}_n(k_1a)]$$

$$\begin{aligned}Y_3 = -\mathrm{i}^n\Big\{&k_1^3\mathrm{J}_n'''(k_1a) + k_1^2\mathrm{J}_n''(k_1a) - n^2k_1\mathrm{J}_n'(k_1a) + \\ &2n^2\mathrm{J}_n(k_1a) + \left[\frac{C}{D}\left(\frac{\rho J}{C}\omega^2 + I - 1\right) - 1\right]k_1\mathrm{J}_n'(k_1a)\Big\}\end{aligned}$$

5 动应力集中系数

由动应力集中系数的定义可知，平板孔洞附近动应力集中系数的表达式可描述为

$$DMCF = |M_\theta^m/M_0^m|, \quad DQCF = |Q_\theta/Q_0| \tag{31}$$

其中，M_θ^m, Q_θ 分别为孔洞周边上任一点的环向动弯矩和动剪力，动弯矩和动剪力分别为

$$\left.\begin{aligned} M_\theta^m = &D\Big(\frac{1}{r^2}\frac{\partial^2F}{\partial\theta^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \nu\frac{\partial^2F}{\partial r^2} - \\ &\frac{1}{r}(1-\nu)\frac{\partial^2f}{\partial r\partial\theta} + (1-\nu)\frac{1}{r^2}\frac{\partial f}{\partial\theta}\Big) \\ Q_\theta = &C\left(\frac{1}{r}\frac{\partial W}{\partial\theta} + \psi_\theta\right) = \\ &\frac{1}{I}\left[D\frac{1}{r}\nabla^2\frac{\partial F}{\partial\theta} - \frac{1}{r}\left(C + \rho J\frac{\partial^2}{\partial t^2}\right)\frac{\partial F}{\partial\theta}\right] + \\ &C\left(\frac{1}{r}\frac{\partial F}{\partial\theta} - \frac{\partial f}{\partial r}\right) \end{aligned}\right\} \tag{32}$$

而 M_0^m, Q_0 分别为无磁场作用时，入射波的动弯矩

和动剪力幅值. 对于沿 x 轴传播的弯曲波可有

$$\left.\begin{aligned} &M_0^m = D\tilde{k}_1^2 F_0 \\ &Q_0 = C\tilde{k}_1 \mathrm{i}\left\{1-\frac{1}{I}\left[\tilde{k}_1^2\frac{D}{C}+\left(1-\frac{\rho J}{\omega^2}\right)\right]\right\}F_0 \end{aligned}\right\} \tag{33}$$

式中, $\tilde{k}_1^2 = \frac{1}{2}k_0^4\left[R+U+\sqrt{(R-U)^2+4k_0^{-4}}\right]$, $R=\frac{s^2}{3}$, $U=\frac{D}{C}=\frac{8s^2}{\pi^2(1-\nu)}$, $\tilde{k}_1$ 为无磁场时的入射波数.

6 算例与讨论

研究含孔 Mindlin 板无穷远处弹性波沿 x 轴入射, 磁感应强度为 B_0 的磁场垂直于板面的情况 (如图 1 所示). 分析计算时, 板的材料性质取为 $E = 196\,\mathrm{GPa}$, $\nu = 0.30$, 磁化系数 $\chi = 70$[8].

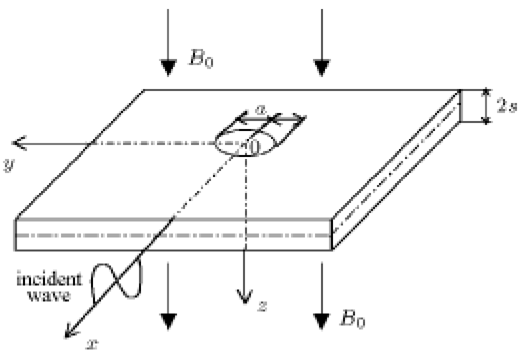


图 1　沿 x 轴正向经过 Mindlin 板开孔的弯曲波与磁场示意图
Fig.1　Sketch of elastic wave through Mindlin plates with cutouts

磁感应强度的平方与弹性模量比 $B_0^2/E = 1.3265\times10^{-11}\,\mathrm{A^2/N}$, $5.3061\times10^{-11}\,\mathrm{A^2/N}$, $11.939\times10^{-11}\,\mathrm{A^2/N}$ 分别对应于 $B_0 = 1\,\mathrm{T}, 2\mathrm{T}, 3\mathrm{T}$(T 表示磁感应强度单位特斯拉). 图 2, 图 3 给出了在不同磁感应强度的平方与弹性模量比 B_0^2/E 情况下, 孔径与半板厚比 a/s 一定, 动弯矩与动剪力集中系数 $(\theta=\pi/2)$ 随入射波频率比 ω/ω_0 的变化曲线.

图 4, 图 5 给出了在不同孔径与半板厚比 a/s 情况下, 磁感应强度的平方与弹性模量比 B_0^2/E 一定, 动弯矩与动剪力集中系数 $(\theta=\pi/2)$ 随入射波频率比 ω/ω_0 的变化曲线. 图 6～图 8 给出了在不同孔径与半板厚比 a/s 情况下, 磁感应强度的平方与弹性模量比 B_0^2/E 和入射波频率比 ω/ω_0 一定, 孔洞周边动弯矩集中系数变化的曲线. 图 9～图 11 给出了在不同孔径与半板厚比 a/s 情况下, 磁感应强度的平

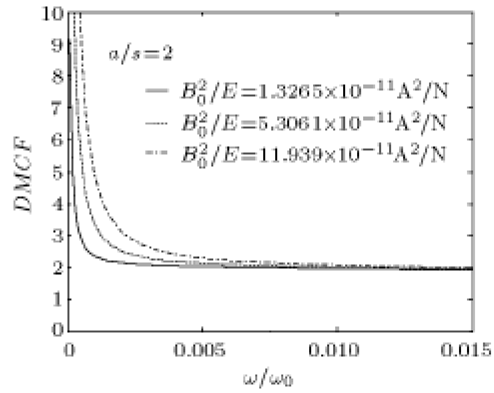


图 2　动弯矩集中系数随频率的变化 $(\theta=\pi/2)$
Fig.2　Dynamic moment concentration factors vs. dimensionless frequency $(\theta=\pi/2)$

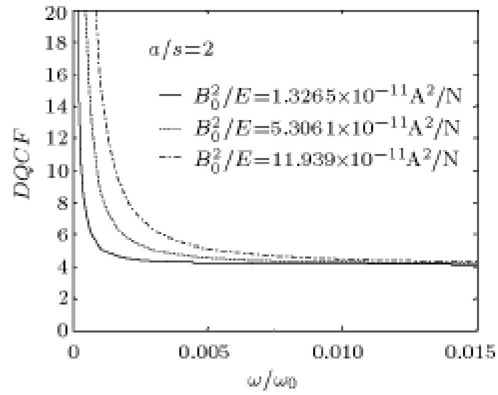


图 3　动剪力集中系数随频率的变化 $(\theta=\pi/2)$
Fig.3　Dynamic shear concentration factors vs. dimensionless frequency $(\theta=\pi/2)$

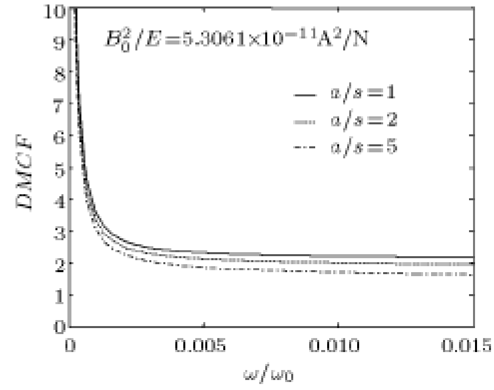


图 4　动弯矩集中系数随频率的变化 $(\theta=\pi/2)$
Fig.4　Dynamic moment concentration factors vs. dimensionless frequency $(\theta=\pi/2)$

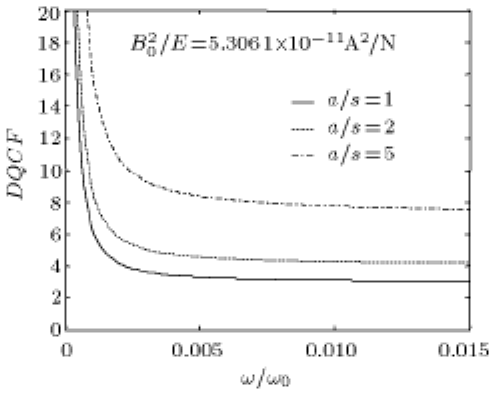


图 5 动弯矩集中系数随频率的变化 ($\theta = \pi/2$)

Fig.5 Dynamic moment concentration factors vs. dimensionless frequency ($\theta = \pi/2$)

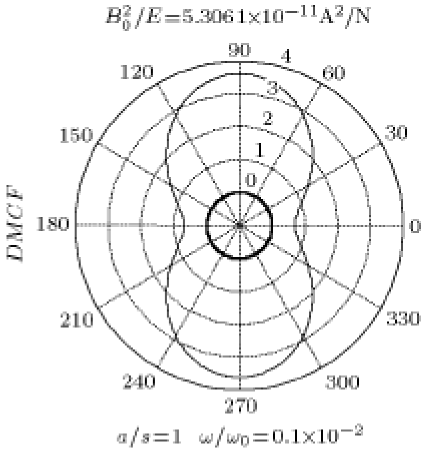


图 6 孔周边动弯矩集中系数

Fig.6 Dynamic moment factors

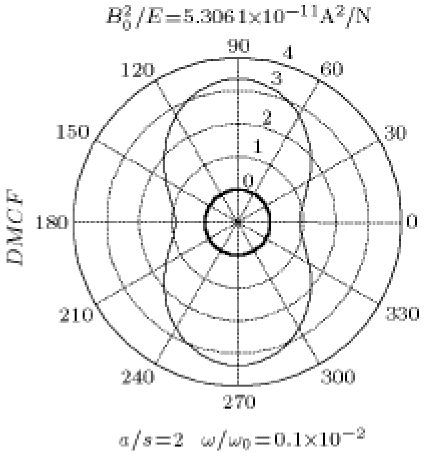


图 7 孔周边动弯矩集中系数

Fig.7 Dynamic moment factors

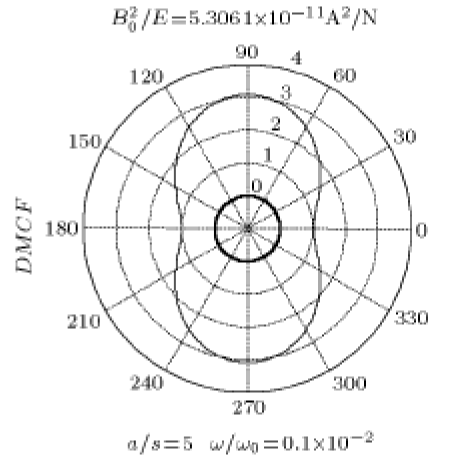


图 8 孔周边动弯矩集中系数

Fig.8 Dynamic moment factors

方与弹性模量比 B_0^2/E 和入射波频率比 ω/ω_0 一定，孔边动剪力集中系数的变化曲线.

由图 2 和图 3 可以看出，a/s 一定情况下，动弯矩集中系数和动剪力集中系数 ($\theta = \pi/2$) 随磁感应强度的平方与弹性模量比 B_0^2/E 的增加而增加. 当入射波频率比 $\omega/\omega_0 < 0.3 \times 10^{-2}$ 时，动弯矩集中系数和动剪力集中系数受磁场的影响比较明显，数值比较大. 当入射波频率比 $\omega/\omega_0 > 0.3 \times 10^{-2}$ 继续增加时，动弯矩集中系数和动剪力集中系数迅速减小，说明磁场的作用在低频时比较明显.

由图 4 和图 5 可以看出，当 B_0^2/E 一定的情况下，动弯矩集中系数 ($\theta = \pi/2$) 随 a/s 增加而减小. 动剪力集中系数 ($\theta = \pi/2$) 随 a/s 增加而增加.

由图 6~ 图 8 可以看出，当 B_0^2/E 和 ω/ω_0 一定的情况下，孔洞周边动弯矩集中系数随 a/s 增加而减小. 由图 9~ 图 11 可以看出，当 B_0^2/E 和 ω/ω_0

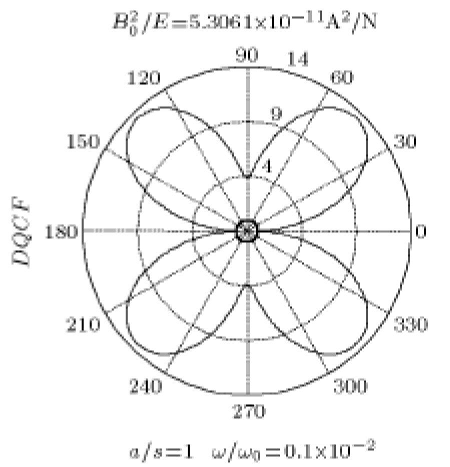
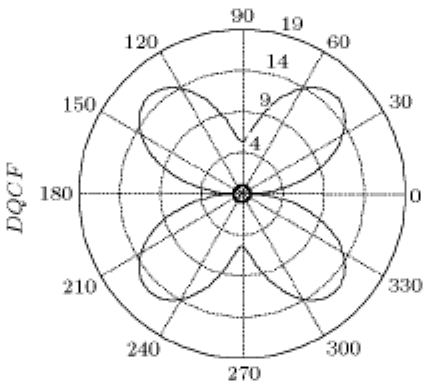


图 9 孔周边动剪力集中系数

Fig.9 Dynamic shear factors

$B_0^2/E=5.3061\times10^{-11}\mathrm{A}^2/\mathrm{N}$

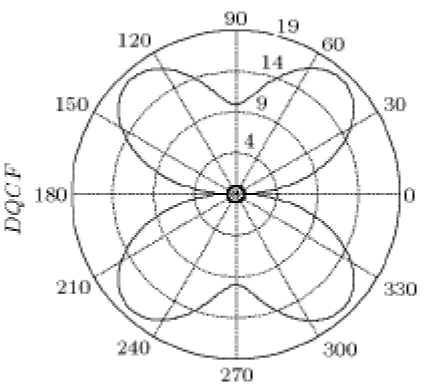


$a/s=2\quad \omega/\omega_0=0.1\times10^{-2}$

图 10　孔周边动剪力集中系数

Fig.10　Dynamic shear factors

$B_0^2/E=5.3061\times10^{-11}\mathrm{A}^2/\mathrm{N}$



$a/s=5\quad \omega/\omega_0=0.1\times10^{-2}$

图 11　孔周边动剪力集中系数

Fig.11　Dynamic shear factors

一定的情况下，孔边动剪力集中系数随 a/s 增加而增加.

总之，动弯矩和动剪力集中系数随入射频率的增加而减小. 磁场的作用使动弯矩和动剪力集中系数增加，尤其是在低频时增加得比较明显. 随着入射波频率的增加磁场作用效果逐渐减弱. 高频时，磁场对动弯矩和动剪力集中系数作用效果不明显.

本文基于考虑磁弹相互作用的 Mindlin 板弯曲波动方程，采用波函数展开法，分析研究了含孔软铁磁材料厚板中弹性波散射与动应力集中问题. 采用解析函数理论，将描述平板弯曲波动的总阶数为六阶的偏微分方程组化成了三个二阶 Helmholtz 型微分方程，最终获得了问题的分析解. 通过动弯矩与动剪力集中系数计算发现：磁场强度对动弯矩集中系数和动剪力集中系数有增加的作用.

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SCATTERING OF FLEXURAL WAVES IN MINDLIN'S PLATES OF SOFT FERROMAGNETIC MATERIALS WITH A CUTOUT [1)]

Hu Chao Han Gang Huang Wenhu
(*Department Aerospace Engineering & Mechanics, Harbin Institute of Technology, Harbin* 150001, *China*)

Abstract The problem of elastic waveguide and dynamic stress concentrations in plates with a cutout is the important subject in solid mechanics. The cutout in structures has influence directly on the loading capacity and the lifetime of structures, therefore, some researchers have devoted to theoretical analysis and experimental research in the world.

Considered dynamic stress concentration or intensity factors, the classical theory of thin plate has disadvantage. Thick plate theory proposed by Mindlin made up for the shortage classical theory of thin plate including the effect of transverse shear deformation and rotator inertia. The satisfying result is gained in engineering. In the 1960's, with wave function expansion method, Pao Yih-Hsing first studied the problem of the flexural wave scattering and dynamic stress concentrations in Mindlin's thick plates with circular cavity and gave an analytical solution and numerical results.

With the development of modern science and technology, the ferromagnetic materials have been applied to superconduct nuclear power station and magnetic levitation trains. It has better physical and mechanical property. The stress on the contour of a cavity or crack in ferromagnetic materials may be increase in a uniform magnetic field. It has a influence on the carrying capacity and the lifetime of structures. According to the many references, the dynamical behavior of ferromagnetic elastic structures can be significantly affected by the presence of a uniform magnetic field.

Based on the theory of magneto-elastic interaction, Japanese researchers analyzed scattering of flexural wave and the dynamic bending moment intensity factors in cracked Mindlin plates of ferromagnetic materials and gave numerical results. They used Fourier transforms to reduce the mixed boundary value problem to a Fredholm integral equation that can be solved numerically.

In this paper, based on the equation of wave motion in Mindlin's plate of magneto-elastic interaction, using wave function expansion method, the scattering of flexural wave and dynamic stress concentrations in a plate of ferromagnetic materials with a cutout are investigated. According to analysis and numerical results, the magnetic induction intensity has great influence on the dynamic stress concentration factors at low frequency.

Key words Mindlin's thick plate, scattering of elastic waves, dynamics of magneto-elastic interaction, soft ferromagnetic materials, dynamic stress concentration

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