压电陶瓷中圆币形裂纹在横向剪力下的机-电耦合行为 [1)]

王子昆　郑百林
(西安交通大学工程力学系，西安 710049)

摘要　以弹性位移分量和电势函数为基本未知量时，横观各向同性压电介质三维问题的场方程可化为四个联立的二阶线性偏微分方程组．本文导出了用四个调和函数表示位移分量及电势函数的表达式，即得到了该场方程的势函数通解．作为通解的应用举例，文中求解了圆币形裂纹受横向剪切作用的问题，得到了裂尖附近应力场及电位移场的解析表达式．结果表明，在横向剪切载荷下圆币形裂纹的尖端场及应力、电位移强度因子均具有明显的机 - 电耦合性质，而应力和电位移分量在裂尖仍具有 $-1/2$ 的奇异性．

关键词　压电介质，三维问题，势函数通解，圆币形裂纹，机 - 电耦合

引 言

压电材料因具有正、逆压电效应被广泛用于制做各种换能器及传感器，并在电子、激光、超声、水声、微声、红外、导航及生物等高新科技领域得到重要应用．压电陶瓷是一种横观各向同性压电材料，因其具有良好的压电性应用尤为广泛．但压电陶瓷的固有弱点是力学性能上的脆性，在工作状态下由机械或电载荷引起的应力集中会导致裂纹的产生和扩展，最终将造成压电部件的失效．事实上，压电陶瓷本身也往往预先存在微裂纹、分层、空洞或夹杂等缺陷，它们也是引起部件失效的重要根源．要提高压电器件的工作性能并对其可靠工作寿命进行预估，就必须从机 - 电耦合的观点上对压电陶瓷材料的损伤和断裂过程作理论分析和精确的定量描述．Deeg 和 Pak 先后研究了压电材料平面及反平面裂纹问题，得到了裂尖应力及电位移的奇异行为 [1,2].Sosa 和 Pak 用特征函数展开法讨论了裂纹端线与压电体横观各向同性对称轴重合的特殊问题，分析了电场对裂尖应力场的影响 [3]．进而 Sosa 还提出了带有缺陷的压电陶瓷材料平面问题的一般解法，为分析各种平面裂纹问题提供了有用的工具 [4]．然而，实际应用的压电元件往往具有各种不同的几何形状并处在复杂的载荷及约束条件下；同时压电体中的裂纹在许多情况下用圆币形或椭圆形来模拟更为符合实际．因而发展压电材料三维问题的有效解法以及分析三维裂纹在机 - 电耦合效应下的行为是很有理论及应用意义的．文 [5] 已给出了横观各向同性压电介质空间轴对称问题的势函数通解，并求解了圆币形裂纹受轴对称拉伸载荷的问题，

1) 国家自然科学基金资助项目．
1993–08–21 收到．

得到了裂尖附近应力场及电位移场的解析表达式．本文推广文 [6] 的方法，得到了横观各向同性压电介质三维问题的势函数通解．并借助这个通解得到了圆币形裂纹在横向剪切作用下裂尖场的解答．事实上，灵活运用上述通解还可分析讨论压电陶瓷中的空洞、夹杂及其它三维裂纹问题．

1 控制微分方程

在不计体积力和不存在自由电荷的情况下，压电介质三维理论的控制方程为

$$\sigma_{ij,j}=0 \tag{1}$$

$$D_{i,i}=0 \tag{2}$$

$$\sigma_{ij}=c_{ijkl}\varepsilon_{kl}-e_{kij}E_k \tag{3}$$

$$D_i=e_{ikl}\varepsilon_{kl}+\varepsilon_{ik}E_k \tag{4}$$

$$\varepsilon_{ij}=\frac{1}{2}(u_{j,i}+u_{i,j}) \tag{5}$$

$$E_i=-\phi_{,i} \tag{6}$$

以上各式中， $i,\ j,\ k,\ l=1,\ 2,\ 3$; σ_{ij}, D_i, ε_{ij}, u_i, E_i 和 ϕ 分别为应力、电位移、应变、位移、电场分量和电势函数． c_{ijkl}, ε_{ij} 和 e_{kij} 分别为弹性、介电和压电常数．在最一般的情况下，独立的材料常数 c_{ijkl} 为 21 个， ε_{ij} 为 6 个， e_{ijk} 为 18 个，共计为 45 个．若为横观各向同性压电体，上述常数依次为 5, 2, 3 个，共计为 10 个．对于后者，取直角坐标系 $oxyz$, 使 xoy 平面与材料各向同性面重合，将方程 (1)—(6) 写为分量形式，将 (5), (6) 两式代入 (3), (4) 式，再将结果代入 (1), (2) 式，可得

$$\left.\begin{aligned}
&c_{11}\frac{\partial^2u}{\partial x^2}+\frac{1}{2}(c_{11}-c_{12})\frac{\partial^2u}{\partial y^2}+c_{44}\frac{\partial^2u}{\partial z^2}+\frac{1}{2}(c_{11}+c_{12})\frac{\partial^2v}{\partial x\partial y}\\
&\qquad+(c_{13}+c_{44})\frac{\partial^2w}{\partial x\partial z}+(e_{15}+e_{31})\frac{\partial^2\phi}{\partial x\partial z}=0\\
&\frac{1}{2}(c_{11}-c_{12})\frac{\partial^2v}{\partial x^2}+c_{11}\frac{\partial^2v}{\partial y^2}+c_{44}\frac{\partial^2v}{\partial z^2}+\frac{1}{2}(c_{11}+c_{12})\frac{\partial^2u}{\partial x\partial y}\\
&\qquad+(c_{13}+c_{44})\frac{\partial^2w}{\partial y\partial z}+(e_{15}+e_{31})\frac{\partial^2\phi}{\partial y\partial z}=0\\
&c_{44}\left(\frac{\partial^2w}{\partial x^2}+\frac{\partial^2w}{\partial y^2}\right)+c_{33}\frac{\partial^2w}{\partial z^2}+(c_{13}+c_{44})\left(\frac{\partial^2u}{\partial x\partial z}+\frac{\partial^2v}{\partial y\partial z}\right)\\
&\qquad+e_{15}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+e_{33}\frac{\partial^2\phi}{\partial z^2}=0\\
&e_{15}\left(\frac{\partial^2w}{\partial x^2}+\frac{\partial^2w}{\partial y^2}\right)+e_{33}\frac{\partial^2w}{\partial z^2}+(e_{15}+e_{31})\left(\frac{\partial^2u}{\partial x\partial z}+\frac{\partial^2v}{\partial y\partial z}\right)\\
&\qquad-\varepsilon_{11}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)-\varepsilon_{33}\frac{\partial^2\phi}{\partial z^2}=0
\end{aligned}\right\} \tag{7}$$

上式即为以弹性位移分量 $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$ 和电势函数 $\phi(x,y,z)$ 为基本未知量时横观各向同性压电介质三维问题的场方程．

2 势函数通解

方程 (7) 是四个二阶偏微分方程联立的方程组，因结构形式复杂，直接积分求解是极为困难的．仿照弹性力学中的惯用方法，若引进势函数即可望将方程 (7) 转化为熟知的偏微分方程，从而得到问题的所谓势函数通解．为此可设

$$u=\frac{\partial\psi}{\partial x}-\frac{\partial x}{\partial y}\,,\quad v=\frac{\partial\psi}{\partial y}+\frac{\partial\chi}{\partial x}\,,\quad w=k_1\frac{\partial\psi}{\partial z}\,,\quad \phi=k_2\frac{\partial\psi}{\partial z} \tag{8}$$

上式中 $\psi(x,y,z)$ 和 $\chi(x,y,z)$ 是新引进的势函数， k_1 和 k_2 是待定常数．将式 (8) 代入式 (7) 经运算整理后可以发现，方程 (7) 的求解将转化为以下各式的积分

$$\frac{1}{2}(c_{11}-c_{12})\Big(\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}\Big)+c_{44}\frac{\partial^2\chi}{\partial z^2}=0 \tag{9}$$

$$\left.\begin{aligned}&c_{11}\Big(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\Big)+[c_{44}+(c_{13}+c_{44})k_1+(e_{15}+e_{31})k_2]\frac{\partial^2\psi}{\partial z^2}=0\\&(c_{13}+c_{44}+c_{44}k_1+e_{15}k_2)\Big(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\Big)+(c_{33}k_1+e_{33}k_2)\frac{\partial^2\psi}{\partial z^2}=0\\&(e_{15}+e_{31}+e_{15}k_1-\varepsilon_{11}k_2)\Big(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\Big)+(e_{33}k_1-\varepsilon_{33}k_2)\frac{\partial^2\psi}{\partial z^2}=0\end{aligned}\right\} \tag{10}$$

对于一般的三维压电问题，式 (10) 中的 $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}$ 和 $\frac{\partial^2\psi}{\partial z^2}$ 都不会恒等于零，在此限制条件下，上列三个关于函数 $\psi(x,y,z)$ 的方程只有当它们彼此恒等时才会有非平凡解，即必须有

$$\left.\begin{aligned}&\frac{c_{44}+(c_{13}+c_{44})k_1+(e_{15}+e_{31})k_2}{c_{11}}=\frac{c_{33}k_1+e_{33}k_2}{c_{13}+c_{44}+c_{44}k_1+e_{15}k_2}=\lambda\\&\frac{c_{44}+(c_{13}+c_{44})k_1+(e_{15}+e_{31})k_2}{c_{11}}=\frac{e_{33}k_1-\varepsilon_{33}k_2}{e_{15}+e_{31}+e_{15}k_1-\varepsilon_{11}k_2}=\lambda\end{aligned}\right\} \tag{11}$$

由上式中消去 k_1 和 k_2, 可得关于 λ 的代数方程为

$$A\lambda^3+B\lambda^2+C\lambda+D=0 \tag{12}$$

其中

$$\begin{aligned}A=&\ e_{15}^2+c_{44}\varepsilon_{11}\\B=&\ (2e_{15}^2c_{13}-c_{44}e_{31}^2+2e_{15}e_{31}c_{13}-2e_{15}c_{11}e_{33}+\varepsilon_{11}c_{13}^2\\&+2c_{13}c_{44}\varepsilon_{11}-c_{33}c_{11}\varepsilon_{11}-c_{44}c_{11}\varepsilon_{33})/c_{11}\\C=&\ \Big[(e_{15}+e_{31})^2c_{33}-2(c_{13}+c_{44})(e_{15}+e_{31})e_{33}-\frac{e_{15}(e_{15}+e_{31})c_{44}c_{33}}{c_{13}+c_{44}}\\&+\frac{e_{15}(e_{15}+e_{31})c_{44}c_{11}}{c_{13}+c_{44}}+c_{33}c_{44}\varepsilon_{11}+c_{11}e_{33}^2\\&-(c_{13}+c_{44})^2\varepsilon_{33}+(c_{44}^2+c_{11}c_{33})\varepsilon_{33}\Big]/c_{11}\\D=&\ -(c_{44}e_{33}^2+c_{11}c_{33}\varepsilon_{33})/c_{11}\end{aligned}$$

方程 (12) 的三个根将被记作 $\lambda_j (j=1,2,3)$，并假定它们互不相等，同时在今后的讨论中将设定 λ_1 是实的正数，λ_2 和 λ_3 既可以是正实数，也可以是具有正的实部的共轭复数．k_{1j} 和 k_{2j} 是与 λ_j 对应的常数，它们可由式 (11) 确定．这样将存在三个可能的势函数 ψ_j，每一个必须满足的方程是

$$\frac{\partial^2\psi_j}{\partial x^2}+\frac{\partial^2\psi_j}{\partial y^2}+\lambda_j\frac{\partial^2\psi_j}{\partial z^2}=0 \qquad (j=1,2,3) \tag{13}$$

若令 $\chi(x,y,z)=\psi_4(x,y,z)$，并定义

$$\lambda_4=\frac{2c_{44}}{c_{11}-c_{12}}$$

则方程 (9) 可写为

$$\frac{\partial^2\psi_4}{\partial x^2}+\frac{\partial^2\psi_4}{\partial y^2}+\lambda_4\frac{\partial^2\psi_4}{\partial z^2}=0 \tag{14}$$

现将 $\nabla_i^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\lambda_i\frac{\partial^2}{\partial z^2}$ 称为直角坐标系 (x,y,z) 下的准调和算子，则方程 (7) 的势函数可用四个准调和函数表示，它们满足方程

$$\frac{\partial^2\psi_i}{\partial x^2}+\frac{\partial^2\psi_i}{\partial y^2}+\lambda_i\frac{\partial^2\psi_i}{\partial z^2}=0 \qquad (i=1,2,3,4) \tag{15}$$

进一步，若令

$$\lambda_i=1/s_i^2\ , \qquad z_i=s_i z \qquad (i=1,2,3,4) \tag{16}$$

则有 $\nabla_i^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\lambda_i\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z_i^2}$，于是方程 (15) 成为

$$\frac{\partial^2\psi_i}{\partial x}+\frac{\partial^2\psi_i}{\partial y^2}+\frac{\partial^2\psi_i}{\partial z_i^2}=0 \qquad (i=1,2,3,4) \tag{17}$$

因而四个势函数又可用 (x,y,z_i) 坐标系下的调和函数来表示．

根据以上讨论，式 (8) 现在可写为

$$\left.\begin{aligned} u&=\frac{\partial}{\partial x}(\psi_1+\psi_2+\psi_3)-\frac{\partial\psi_4}{\partial y}\\ v&=\frac{\partial}{\partial y}(\psi_1+\psi_2+\psi_3)+\frac{\partial\psi_4}{\partial x}\\ w&=k_{11}\frac{\partial\psi_1}{\partial z}+k_{12}\frac{\partial\psi_2}{\partial z}+k_{13}\frac{\partial\psi_3}{\partial z}\\ \phi&=k_{21}\frac{\partial\psi_1}{\partial z}+k_{22}\frac{\partial\psi_2}{\partial z}+k_{23}\frac{\partial\psi_3}{\partial z} \end{aligned}\right\} \tag{18}$$

对于许多实际问题，采用柱坐标进行分析是较为方便的，由直角坐标与柱坐标的关

系，很容易将 (18) 式写成柱坐标的形式，即

$$\left.\begin{aligned}
u_r &= \frac{\partial}{\partial r}(\psi_1+\psi_2+\psi_3) - \frac{1}{r}\frac{\partial \psi_4}{\partial \theta} \\
v_\theta &= \frac{1}{r}\frac{\partial}{\partial \theta}(\psi_1+\psi_2+\psi_3) + \frac{\partial \psi_4}{\partial r} \\
w_z &= k_{11}\frac{\partial \psi_1}{\partial z} + k_{12}\frac{\partial \psi_2}{\partial z} + k_{13}\frac{\partial \psi_3}{\partial z} \\
\phi &= k_{21}\frac{\partial \psi_1}{\partial z} + k_{22}\frac{\partial \psi_2}{\partial z} + k_{23}\frac{\partial \psi_3}{\partial z}
\end{aligned}\right\} \tag{19}$$

与上式对应的应力及电位移分量的表达式为

$$\left.\begin{aligned}
\sigma_r =\ & \left(c_{11}\frac{\partial^2}{\partial r^2} + c_{12}\frac{1}{r}\frac{\partial}{\partial r} + c_{12}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\psi_1+\psi_2+\psi_3) \\
& +(c_{13}k_{1j}+e_{31}k_{2j})\frac{\partial^2 \psi_j}{\partial z^2} + c_{11}\frac{1}{r^2}\frac{\partial \psi_4}{\partial \theta} + c_{12}\frac{1}{r}\frac{\partial^2 \psi_4}{\partial r\partial \theta} \\
\sigma_\theta =\ & \left(c_{12}\frac{\partial^2}{\partial r^2} + c_{11}\frac{1}{r}\frac{\partial}{\partial r} + c_{11}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\psi_1+\psi_2+\psi_3) \\
& +(c_{13}k_{ij}+e_{31}k_{2j})\frac{\partial^2 \psi_j}{\partial z^2} + c_{12}\frac{1}{r^2}\frac{\partial \psi_4}{\partial \theta} + c_{11}\frac{1}{r}\frac{\partial^2 \psi_4}{\partial r\partial \theta} \\
\sigma_z =\ & c_{13}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\psi_1+\psi_2+ psi_3) + (c_{33}k_{1j}+e_{33}k_{2j})\frac{\partial^2 \psi_j}{\partial z^2} \\
\tau_{zr} =\ & (c_{44}+c_{44}k_{1j}+e_{15}k_{2j})\frac{1}{r}\frac{\partial^2 \psi_j}{\partial r\partial \theta} + c_{44}\frac{\partial^2 \psi_4}{\partial r\partial z} \\
\tau_{z\theta} =\ & (c_{44}+c_{44}k_{1j}+e_{15}k_{2j})\frac{\partial^2 \psi_j}{\partial r\partial z} - c_{44}\frac{1}{r}\frac{\partial^2 \psi_4}{\partial \theta\partial z} \\
\tau_{r\theta} =\ & (c_{11}-c_{12})\left(\frac{1}{r}\frac{\partial^2}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial}{\partial \theta}\right)(\psi_1+\psi_2+\psi_3) \\
& +\frac{1}{2}(c_{11}-c_{12})\left(\frac{\partial^2 \psi_4}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2 \psi_4}{\partial \theta^2}\right) \\
D_r =\ & (e_{15}+e_{15}k_{1j}-\varepsilon_{11}k_{2j})\frac{\partial^2 \psi_j}{\partial r\partial z} - e_{15}\frac{1}{r}\frac{\partial^2 \psi_4}{\partial \theta\partial z} \\
D_\theta =\ & (e_{15}+e_{15}k_{1j}-\varepsilon_{11}k_{2j})\frac{1}{r}\frac{\partial^2 \psi_j}{\partial \theta\partial z} + e_{15}\frac{\partial^2 \psi_4}{\partial r\partial z} \\
D_z =\ & e_{31}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\psi_1+\psi_2+\psi_3) + (e_{33}k_{1j}-\varepsilon_{33}k_{2j})\frac{\partial^2 \psi_j}{\partial z^2}
\end{aligned}\right\} \tag{20}$$

上式中 $j=1,2,3$, 重复指标表示求和. 可以看出，当 λ_2 和 λ_3 为共轭复数时， k_{12} 和 k_{13} 以及 k_{22} 和 k_{23} 也是共轭复数，而 ψ_2 和 ψ_3 将是共轭的复函数，故由式 (19) 及 (20) 所求得的位移分量、电势函数、应力及电位移分量均为实数.

3 横向剪切作用下的圆币形裂纹

3.1 问题的提法

如图 1 所示，设一无限大横观各向同性压电介质 (例如压电陶瓷) 中含一半径为

a 的圆币形裂纹，裂纹面与材料各向同性面重．以裂纹面中心为坐标原点，z 轴垂直各向同性面．设裂纹上下面作用着大小相等方向相反的均布剪应力 τ_0，不失一般性可取剪应力的方向平行 x 轴．为了确定裂纹尖端附近的应力及电位移场，考虑到问题的反对称性质，只须讨论 $z \geq 0$ 的上半空间即可，而且边界条件可列出如下

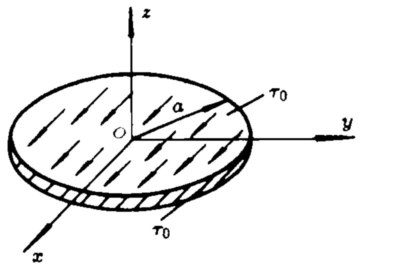

图 1　横向剪切作用下的圆币形裂纹
Fig.1 A penny-shape crack subjected to laterally distributed shearing forces

$$\sigma_z|_{z=0} = 0\,,\quad D_z|_{z=0} = 0 \qquad (|x| < \infty\,,\quad |y| < \infty) \tag{21}$$

$$\tau_{zr}|_{z=0} = -\tau_0 \cos\theta\,,\qquad \tau_{z\theta}|_{z=0} = \tau_0 \sin\theta \qquad (0 \leq r \leq a) \tag{22}$$

$$u_r|_{z=0} = 0\,,\qquad v_\theta|_{z=0} = 0 \qquad (r > a) \tag{23}$$

因无限远处无外力及电载荷存在，故当 $\sqrt{r^2+z^2} \to \infty$ 时应有

$$\sigma_r = \sigma_\theta = \sigma_z = \tau_{zr} = \tau_{z\theta} = \tau_{r\theta} = D_r = D_x = D_\theta = 0 \tag{24}$$

我们的目的就是利用前面已提出的势函数通解根据定解条件 (21)—(24) 式去寻求上述问题的解答．

3.2 势函数的选择

若将式 (13) 写为柱坐标形式并利用 (16) 式，则有

$$\frac{\partial^2\psi_j}{\partial r^2} + \frac{1}{r}\frac{\partial\psi_j}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi_j}{\partial\theta^2} = -\frac{1}{s_j^2}\frac{\partial^2\psi_j}{\partial z^2} \qquad (j = 1,2,3)$$

将上式代入 (20) 式中 σ_z 及 D_z 的表达式，得

$$\left.\begin{aligned} \sigma_z &= a_1\frac{\partial^2\psi_1}{\partial z_1^2} + a_2\frac{\partial^2\psi_2}{\partial z_2^2} + a_3\frac{\partial^2\psi_3}{\partial z_3^2} \\ D_z &= b_1\frac{\partial^2\psi_1}{\partial z_1^2} + b_2\frac{\partial^2\psi_2}{\partial z_2^2} + b_3\frac{\partial^2\psi_3}{\partial z_3^2} \end{aligned}\right\} \tag{25}$$

其中

$$\begin{aligned} a_j &= -c_{13} + (c_{33}k_{1j} + e_{33}k_{2j})s_j^2 \\ b_j &= -e_{31} + (e_{33}k_{1j} + \varepsilon_{33}k_{2j})s_j^2 \end{aligned} \qquad (\text{不求和})$$

将 (25) 式代入 (21) 式，有

$$\left.\begin{aligned} a_1\Big[\frac{\partial^2\psi_1}{\partial z_1^2}\Big]_{z=0} + a_2\Big[\frac{\partial^2\psi_3}{\partial z_2^2}\Big]_{z=0} + a_3\Big[\frac{\partial^2\psi_3}{\partial z_3^2}\Big]_{z=0} = 0 \\ b_1\Big[\frac{\partial^2\psi_1}{\partial z_1^2}\Big]_{z=0} + b_2\Big[\frac{\partial^2\psi_2}{\partial z_2^2}\Big]_{z=0} + b_3\Big[\frac{\partial^2\psi_3}{\partial z_3^2}\Big]_{z=0} = 0 \end{aligned}\right\}$$

上式规定了势函数 $\psi_j(j=1,2,3)$ 之间的相互关系，联立求解可得

$$\left[\frac{\partial^2\psi_2}{\partial z_2^2}\right]_{z=0}=K_1\left[\frac{\partial^2\psi_1}{\partial z_1^2}\right]_{z=0},\qquad \left[\frac{\partial^2\psi_3}{\partial z_3^2}\right]_{z=0}=K_2\left[\frac{\partial^2\psi_1}{\partial z_1^2}\right]_{z=0} \tag{26}$$

这里

$$K_1=\frac{a_3b_1-a_1b_3}{a_2b_3-a_3b_2},\qquad K_2=\frac{a_2b_1-a_1b_2}{a_2b_3-a_3b_2}$$

可见为了满足边界条件 (21) 式或 (26) 式，本问题的势函数可按如下方式选取

$$\left.\begin{aligned}\psi_1(r,\theta,z_1)&=f_1(r,\theta,z_1)\\ \psi_2(r,\theta,z_2)&=K_1f_1(r,\theta,z_2)\\ \psi_3(r,\theta,z_3)&=K_2f_1(r,\theta,z_3)\\ \psi_4(r,\theta,z_4)&=f_2(r,\theta,z_4)\end{aligned}\right\} \tag{27}$$

若函数 $f_1(r,\theta,z_j)(j=1,2,3)$ 和 $f_2(r,\theta,z_4)$ 分别满足方程

$$\left.\begin{aligned}&\frac{\partial^2 f_1}{\partial r^2}+\frac{1}{r}\frac{\partial f_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f_1}{\partial\theta^2}+\frac{\partial^2 f_1}{\partial z_j^2}=0\qquad (j=1,2,3)\\ &\frac{\partial^2 f_2}{\partial r^2}+\frac{1}{r}\frac{\partial f_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f_2}{\partial\theta^2}+\frac{\partial^2 f_2}{\partial z_4^2}=0\end{aligned}\right\} \tag{28}$$

则式 (27) 给出的 $\psi_i(r,\theta,z_i)$ 即可作为本问题的势函数解答.

3.3 对偶积分方程组的导出及解答

为了进一步由其余边界条件确定势函数 $f_1(r,\theta,z_j)$ 和 $f_2(r,\theta,z_4)$, 考虑到式 (20) 及 (22), 取

$$\left.\begin{aligned}f_1(r,\theta,z_j)&=\sum_{m=0}^{\infty}F_{1m}(r,z_j)\cos m\theta\\ f_2(r,\theta,z_4)&=\sum_{m=0}^{\infty}F_{2m}(r,z_4)\sin m\theta\end{aligned}\right\} \tag{29}$$

将上式依次代入 (28) 式，得

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}+\frac{\partial^2}{\partial z_j^2}\right)F_{1m}(r,z_j)=0$$
$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{m^2}{r^2}+\frac{\partial^2}{\partial z_4^2}\right)F_{2m}(r,z_4)=0$$

对以上两式作 m 阶 Hankel 变换，并令

$$\left.\begin{aligned}H_{1m}(\xi,z_j)&=\int_0^\infty rF_{1m}(r,z_j)J_m(\xi r)\mathrm{d}r\\ H_{2m}(\xi,z_4)&=\int_0^\infty rF_{2m}(r,z_4)J_m(\xi r)\mathrm{d}r\end{aligned}\right\} \tag{30}$$

即得

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z_j^2}-\xi^2\right)H_{1m}(\xi,z_j)=0$$
$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z_4^2}-\xi^2\right)H_{2m}(\xi,z_4)=0$$

以上两个方程的通解为

$$H_{1m}(\xi,z_j)=P_{1m}(\xi)e^{-\xi z_j}+Q_{1m}(\xi)e^{\xi z_j}$$
$$H_{2m}(\xi,z_4)=P_{2m}(\xi)e^{-\xi z_4}+Q_{2m}(\xi)e^{\xi z_4}$$

由无穷远处所有场量为零的条件 (24) 式可知，应有 $Q_{1m}(\xi)=Q_{2m}(\xi)=0$, 故有

$$H_{1m}(\xi,z_j)=P_{1m}(\xi)e^{-\xi z_j}$$
$$H_{2m}(\xi,z_4)=P_{2m}(\xi)e^{-\xi z_4}$$

将以上两式分别代入 (30) 式，由 Hankel 反变换可得

$$\left.\begin{aligned}F_{1m}(r,z_j)&=\int_0^\infty\xi P_{1m}(\xi)J_m(\xi r)e^{-\xi z_j}\mathrm{d}\xi\\F_{2m}(r,z_4)&=\int_0^\infty\xi P_{2m}(\xi)J_m(\xi r)e^{-\xi z_4}\mathrm{d}\xi\end{aligned}\right\}\tag{31}$$

将式 (31) 代入式 (29), 再将结果代入式 (27). 由式 (20) 中 τ_{zr}, $\tau_{z\theta}$ 的表达式可得

$$\left.\begin{aligned}\tau_{zr}=&-\frac{1}{2}\sum_{m=0}^{\infty}\cos m\theta\int_0^\infty\xi^3\Big\{[J_{m-1}(\xi r)-J_{m+1}(\xi r)]\Big[B_1P_{1m}(\xi)e^{-\xi z_1}\\&+K_1B_2P_{1m}(\xi)e^{-\xi z_2}+K_2B_3P_{1m}(\xi)e^{-\xi z_3}\Big]-c_{44}[J_{m-1}(\xi r)\\&+J_{m+1}(\xi r)]P_{2m}(\xi)e^{-\xi z_4}\Big\}\mathrm{d}\xi\\\tau_{z\theta}=&\frac{1}{2}\sum_{m=0}^{\infty}\sin m\theta\int_0^\infty\xi^3\Big\{[J_{m-1}(\xi r)+J_{m+1}(\xi r)]\Big[B_1P_{1m}(\xi)e^{-\xi z_1}\\&+K_1B_2P_{1m}(\xi)e^{-\xi z_2}+K_2B_3P_{1m}(\xi)e^{-\xi z_3}\Big]-c_{44}[J_{m-1}(\xi r)\\&-J_{m+1}(\xi r)]P_{2m}(\xi)e^{-\xi z_4}\Big\}\mathrm{d}\xi\end{aligned}\right\}\tag{32}$$

其中

$$B_j=c_{44}+c_{44}k_{1j}+e_{15}k_{2j}\qquad(j=1,2,3)$$

类似地由式 (19) 可得

$$
\left.\begin{aligned}
u_r = \frac{1}{2}\sum_{m=0}^{\infty}\cos m\theta\int_0^{\infty}\xi^3\Big\{[J_{m-1}(\xi r) - J_{m+1}(\xi r)]\Big[P_{1m}(\xi)e^{-\xi z_1} \\
+K_1P_{1m}(\xi)e^{-\xi z_2} + K_2P_{1m}(\xi)e^{-\xi z_3}\Big] - [J_{m-1}(\xi r) \\
+J_{m+1}(\xi r)]P_{2m}(\xi)e^{-\xi z_4}\Big\}\mathrm{d}\xi \\
v_\theta = -\frac{1}{2}\sum_{m=0}^{\infty}\sin m\theta\int_0^{\infty}\xi^3\Big\{[J_{m-1}(\xi r) + J_{m+1}(\xi r)]\Big[P_{1m}(\xi)e^{-\xi z_1} \\
+K_1P_{1m}(\xi)e^{-\xi z_2} + K_2P_{1m}(\xi)e^{-\xi z_3}\Big] + [J_{m-1}(\xi r) \\
-J_{m+1}(\xi r)]P_{2m}(\xi)e^{-\xi z_4}\Big\}\mathrm{d}\xi
\end{aligned}\right\} \tag{33}
$$

将式 (32) 和 (33) 分别代入边界条件 (22) 和 (23) 式，可以看出除 $m = 1$ 的项应保留外，其余均应为零，经运算整理将有

$$
\left.\begin{aligned}
\int_0^{\infty}\xi^3[(B_1 + K_1B_2 + K_2B_3)P_{11}(\xi) - c_{44}P_{21}(\xi)]J_0(\xi r)\mathrm{d}\xi = 2\tau_0 \\
\int_0^{\infty}\xi^3[(B_1 + K_1B_2 + K_2B_3)P_{11}(\xi) + c_{44}P_{21}(\xi)]J_2(\xi r)\mathrm{d}\xi = 0
\end{aligned}\right\} \quad (0 \le r \le a) \tag{34}
$$

以及

$$
\left.\begin{aligned}
\int_0^{\infty}\xi^2\{[J_0(\xi r) - J_2(\xi r)](1 + K_1 + K_2)P_{11}(\xi) - [J_0(\xi r) + J_2(\xi r)]P_{21}(\xi)\}\mathrm{d}\xi \\
\int_0^{\infty}\xi^2\{[J_0(\xi r) + J_2(\xi r)](1 + K_1 + K_2)P_{11}(\xi) - [J_0(\xi r) - J_2(\xi r)]P_{21}(\xi)\}\mathrm{d}\xi
\end{aligned}\right\} \quad (r > a) \tag{35}
$$

为使以上结果简化，现引进待求函数 $G(\xi)$ 及 $Q(\xi)$，并令

$$
P_{11}(\xi) = \frac{G(\xi) + Q(\xi)}{(1 + K_1 + K_2)\xi^2}, \qquad P_{21}(\xi) = \frac{G(\xi) - Q(\xi)}{\xi^2} \tag{36}
$$

将 (36) 式代入式 (34) 和 (35)，最后得对偶积分方程组如下

$$
\left.\begin{aligned}
&\int_0^{\infty}[\gamma Q(\xi) + \mu G(\xi)]\xi J_0(\xi r)\mathrm{d}\xi = 2\tau_0 && (0 \le r \le a) \\
&\int_0^{\infty}Q(\xi)J_0(\xi r)\mathrm{d}\xi && (r > a) \\
&\int_0^{\infty}[\mu Q(\xi) + \gamma G(\xi)]\xi J_2(\xi r)\mathrm{d}\xi = 0 && (0 \le r \le a) \\
&\int_0^{\infty}G(\xi)J_2(\xi r)\mathrm{d}\xi = 0 && (r > a)
\end{aligned}\right\} \tag{37}
$$

其中

$$
\gamma = \frac{B_1 + K_1B_2 + K_2B_3}{1 + K_1 + K_2} + c_{44}, \qquad \mu = \frac{B_1 + K_1B_2 + K_2B_3}{1 + K_1 + K_2} - c_{44}
$$

对偶积分方程组 (37) 的解为

$$Q(\xi)=\frac{4\tau_0}{\sqrt{2\pi}\gamma}\xi^{-1/2}J_{3/2}(\xi)\ ,\quad G(\xi)=0$$

将这个结果代入式 (36), 得

$$\left.\begin{aligned}P_{11}(\xi)&=\frac{4\tau_0}{\sqrt{2\pi}\gamma(1+K_1+K_2)}\xi^{-5/2}J_{3/2}(\xi)\\P_{21}(\xi)&=-\frac{4\tau_0}{\sqrt{2\pi}\gamma}\xi^{-5/2}J_{3/2}(\xi)\end{aligned}\right\}\tag{38}$$

至此，问题的势函数解答即被完全确定.

3.4 圆币形裂纹尖端附近的奇异场

在式 (29) 和 (31) 中取 $m=1$, 并将式 (38) 代入，再将所得结果代入式 (27). 由式 (20) 经复杂运算可得裂纹尖端附近应力及电位移分量的解答如下

$$\left.\begin{aligned}\tau_{zr}&=\frac{4\tau_0\cos\theta}{\pi\gamma(1+K_1+K_2)}\left[B_1(2\delta_1)^{-1/2}+K_1B_2(2\delta_2)^{-1/2}+K_2B_3(2\delta_3)^{-1/2}\right]\cos\frac{\varphi}{2}\\\tau_{z\theta}&=-\frac{4\tau_0c_{44}\sin\theta}{\pi\gamma}(2\delta_4)^{-1/2}\cos\frac{\varphi}{2}\\\sigma_z&=\frac{4\tau_0\cos\theta}{\pi\gamma(1+K_1+K_2)}\left[a_1(2\delta_1)^{-1/2}+K_1a_2(2\delta_2)^{-1/2}+K_2a_3(2\delta_3)^{-1/2}\right]\sin\frac{\varphi}{2}\\D_r&=\frac{4\tau_0\cos\theta}{\pi\gamma(1+K_1+K_2)}\left[D_1(2\delta_1)^{-1/2}+K_1D_2(2\delta_2)^{-1/2}+K_2D_3(2\delta_3)^{-1/2}\right]\cos\frac{\varphi}{2}\\D_\theta&=-\frac{4\tau_0e_{15}\sin\theta}{\pi\gamma}(2\delta_4)^{-1/2}\cos\frac{\varphi}{2}\\D_z&=\frac{4\tau_0\cos\theta}{\pi\gamma(1+K_1+K_2)}\left[b_1(2\delta_1)^{-1/2}+K_1b_2(2\delta_2)^{-1/2}+K_2b_3(2\delta_3)^{-1/2}\right]\sin\frac{\varphi}{2}\end{aligned}\right\}\tag{39}$$

其中

$$D_j=e_{15}+e_{15}k_{1j}-\varepsilon_{11}k_{2j}\qquad(j=1,2,3)$$

在以上运算中曾用了坐标变量的无量纲形式. 即令 $\rho=r/a$, $Z=z/a$. 如图 2 所示，设 $\delta=(r^2+z^2)^{1/2}/a$ 及 φ 为裂尖极坐标，坐标平面为垂直裂纹面的径向平面. 由 $z_i=s_iz$ 及 $Z=\delta\sin\varphi$ 可知应有 $\delta_i=s_i\delta(i=1,2,3,4)$ 还应当说明的是在得到式 (39) 的运算过程中还应用了如下的有关公式 [7,8]

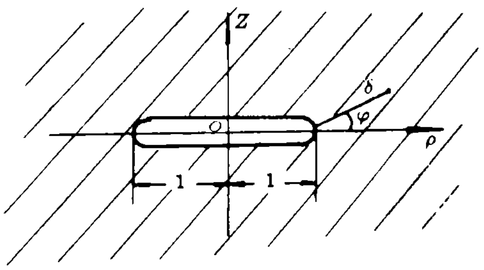

图 2　裂纹尖端附近的极坐标
Fig.2　The polar coordinates near the crack tip

$$J_{3/2}(\xi)=\sqrt{\frac{2}{\pi}}\xi^{-1/2}(\xi^{-1}\sin\xi-\cos\xi)$$
$$\int_0^\infty \cos\xi J_0(\xi r)e^{-\xi z_i}\mathrm{d}\xi=(2\delta_i)^{-1/2}\cos\varphi/2$$
$$\int_0^\infty \cos\xi J_1(\xi r)e^{-\xi z_i}\mathrm{d}\xi=-(2\delta_i)^{-1/2}\sin\varphi/2$$
$$\cdots\cdots$$

最后，由应力强度因子及电位移强度因子的定义可以求得

$$k_{\mathrm{II}}^M=\lim_{\rho\to1^+}\{\sqrt{2\pi(\rho-1)}\tau_{zr}(\rho,\theta,0)\}=\frac{4\tau_0\cos\theta}{\sqrt{\pi}\gamma(1+K_1+K_2)}\left(\frac{B_1}{\sqrt{S_1}}+\frac{K_1B_2}{\sqrt{S_2}}+\frac{K_2B_3}{\sqrt{S_3}}\right) \tag{40}$$

$$k_{\mathrm{III}}^M=\lim_{\rho\to1^+}\{\sqrt{2\pi(\rho-1)}\tau_{z\theta}(\rho,\theta,0)\}=-\frac{4\tau_0c_{44}\sin\theta}{\sqrt{\pi}\gamma}\frac{1}{\sqrt{s_4}} \tag{41}$$

$$k_{\mathrm{II}}^E=\lim_{\rho\to1^+}\{\sqrt{2\pi(\rho-1)}D_r(\rho,\theta,0)\}=\frac{4\tau_0\cos\theta}{\sqrt{\pi}\gamma(1+K_1+K_2)}\left(\frac{D_1}{\sqrt{S_1}}+\frac{K_1D_2}{\sqrt{S_2}}+\frac{K_2D_3}{\sqrt{S_3}}\right) \tag{42}$$

$$k_{\mathrm{III}}^E=\lim_{\rho\to1^+}\{\sqrt{2\pi(\rho-1)}D_\theta(\rho,\theta,0)\}=-\frac{4\tau_0e_{15}\sin\theta}{\sqrt{\pi}\gamma}\frac{1}{\sqrt{S_4}} \tag{43}$$

以上结果表明，横观各向同性压电介质中的圆币形裂纹在横向剪切作用下，裂尖应力场及电位移场均具有 $\delta^{-1/2}$ 的奇异性. 同时，裂尖附近应力场及电移场，以及应力强度因子 k_{II}^M，k_{III}^M 和电位移强度因子 k_{II}^E，k_{III}^E，均具有明显的机 - 电耦合行为. 以上结果为预测压电体中圆币形裂纹的扩展规律提供了理论依据.

参 考 文 献

1 Deeg WF. The analysis of dislocation, crack, and inclusion problems in piezoelectric solids. Ph D Thesis, Stanford University, 1980

2 Pak YE. Crack extension force in a piezoelectric material. *J Appl Mech*, 1992, 57(3): 647–653

3 Sosa HA, Pak YE. Three-dimensional eigen-function analysis of a crack in a piezoelectric material. *Int J Solids Structures*, 1990, 26(1): 1–15

4 Sosa HA. Plane problems in piezoelectric media with defects. *Int J Solids Structures*, 1991, 28(4): 491–505

5 Wang Zikun. Penny-shaped crack in transversely isotropic piezoelectric materials *Acta Mechanica Sinica*, 1994, 10(1): 49–60

6 Chen WT. On some problems in transversely isotropic elastic materials. *J Appl Mech*, 1966, 32(2): 347–355

7 Sneddon IN. Fourier Transforms. New York: McGraw-Hill, 1951
8 范天佑. 断裂力学基础. 江苏科学技术出版社, 1978

MECHANICAL – ELECTRIC COUPLING BEHAVIOUR OF A PENNY – SHAPE CRACK IN PIEZOELECTRIC CERAMICS SUBJECTED TO LATERAL SHEARING FORCE

Wang Zikun Zheng Bailin
(Department of Engineering Mechanics, Xi'an Jiaotong University, Xi'an 710049, China)

Abstract The field equations for three-dimensional problems in transversely isotropic piezoelectric medium may be reduced to four second order linear partial differential equations, in which the elastic displacement components and electric potential functions are unknowns. In this paper, expressions for displacement components and electric potential functions are derived by using four harmonic functions, they form the so-called general solutions of the field equations for three-dimensional problems. As an example, a penny-shape crack subjected to laterally distributed shearing forces is solved and analytic expressions for the crack-tip stress and electric displacement fields are obtained. It is found that the crack-tip fields, stress and electric displacement intensity factors have obvious mechanical-electric coupling behaviour, and the stress and electric displacement components at crack tip have $r^{-1/2}$ singularity.

Key words piezoeledtric medium, three-dimensional problem, general solution, potential function, penny-shape crack, mechanical-electric coupling