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线弹性孔洞问题的一种半解析解

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摘要 在研究材料的损伤行为中，常常要考虑到孔洞效应. 本文利用一组能满足内椭圆孔应力自由边界条件的罗朗级数，建立与外边界相适应的泛函，通过变分求泛函驻值或极值来求得相应问题，推广到双周期排列孔洞模型可得到损伤孔洞模型的等效杨氏模量和等效泊松比.

关键词 数学弹性力学，变分法，损伤力学

前 言

随着材料科学的发展，在一些弹塑性和黏弹性材料中孔洞的扩展和分布的影响，引起了越来越多的注意. Becker 和 Needleman[1] 利用双周期排列的孔洞模型研究弹塑性材料孔洞的扩展规律； Brison 和 Knauss[2,3] 则利用相同模型预测黏弹体的模量特性. 本文利用穆斯海利什维里给出的复变函数公式[4] 和 M. Isida[5] 提出的一组罗朗级数，通过建立相应泛函，得到变分方程，给出了线弹性孔洞问题的半解析解法. 需要指出的是，对应于具体的不同边界条件的孔洞问题必须建立相适应的泛函. 本文将用这一方法讨论一般含孔洞有限板和用于研究损伤的双周期排列孔洞问题，并得到后一种情况的等效杨氏模量 $\bar{E}$, 等效泊松比 $\bar{\nu}$ 及它们所受孔洞大小和形状的影响.

1 基本公式

为了讨论方便，文中把应力、位移无量纲化. 这样根据文 [4] 得到应力、位移表达式

$$\left.\begin{aligned}
\sigma_x &= \mathrm{Re}[2\phi(z) - \bar{z}\phi''(z) - \psi''(z)]\cdot\sigma\\
\sigma_y &= \mathrm{Re}[2\phi'(z) + \bar{z}\phi''(z) + \psi''(z)]\cdot\sigma\\
\tau_{xy} &= \mathrm{Im}[\bar{z}\phi''(z) + \psi''(z)]\cdot\sigma\\
u &= \frac{\sigma\cdot d}{2G}\cdot\mathrm{Re}[\gamma\phi(z) - z\cdot\overline{\phi'(z)} - \psi'(z)]\\
v &= \frac{\sigma\cdot d}{2G}\cdot\mathrm{Im}[\gamma\phi(z) - z\cdot\overline{\phi'(z)} - \psi'(z)]
\end{aligned}\right\} \tag{1}$$

$$\sigma_n - \mathrm{i}\tau_{nt} = 2\mathrm{Re}\Big\{[\phi'(z)] - [\bar{z}\phi''(z) + \psi''(z)]e^{2\mathrm{i}\alpha}\Big\}\cdot\sigma$$

其中 σ 是无量纲化参量， d 是位移无量纲化因子， $z = x + \mathrm{i}y = z/d$, $Z = X + \mathrm{i}Y$ 是真实位移点， $\phi(z)$ 和 $\psi(z)$ 是未知解析函数， G, ν 分别是剪切模量和泊松比，

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$\gamma=\dfrac{3-\nu}{1+\nu}$(平面应力问题) 或 $\gamma=3-4\nu$(平面应变问题),α 为边界法线方向和 x 轴夹角.

一般情况下，解析函数 $\phi(z)$, $\psi(z)$ 取

$$\phi(z)=\sum_{n=0}^{\infty}\left[F_n\cdot z^{-(n+1)}+M_n\cdot z^{(n+1)}\right] \tag{2.1}$$

$$\psi(z)=\sum_{n=1}^{\infty}D_n z^{-n}+\sum_{n=0}^{\infty}K_n\cdot z^{(n+1)}-D_0\ln z \tag{2.2}$$

式中系数是复数. 当孔洞问题几何对称且边界条件也对称时，式中系数均是实数，而且 (2.1) 中只存在奇次幂项， (2.2) 中只存在偶次幂项和 D_0 项. 于是 (2.1), (2.2) 可简化成

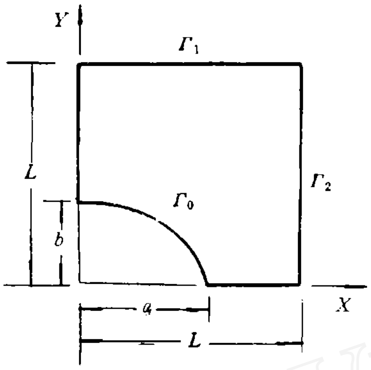

图 1

Fig.1

$$\phi(z)=\sum_{n=0}^{\infty}\left[F_{2n}z^{-(2n+1)}+M_{2n}\cdot z^{(2n+1)}\right]$$

$$\psi(z)=\sum_{n=1}^{\infty}D_{2n}z^{-2n}+\sum_{n=0}^{\infty}K_{2n}z^{(2n+2)}-D_0\ln z \tag{3}$$

当如图 1 所示的椭圆孔内边应力自由时即 $\sigma_n-i\tau_{nt}=0$, 以上两式中的 F_{2n}, M_{2n}, D_{2n}, K_{2n}, D_0 满足一定关系 (具体表达式见附录 A), 这样 (3) 式只有 M_{2n}, K_{2n} 两个未知数.

把 (3) 式代入 (1) 中，利用 F_{2n}, D_{2n} 和 M_{2n}, K_{2n} 的关系，可得

$$\left.\begin{aligned}
\sigma_x &= \sigma\cdot\mathrm{Re}\Big\{\sum_{k=0}^{\infty}[(2H_{k1}-\bar z\cdot H_{k2}-H_{k4})M_{2k}\\
&\quad+(2\cdot G_{k1}-\bar zG_{k2}-G_{k4})\cdot K_{2k}]\Big\}\\
\sigma_y &= \sigma\cdot\mathrm{Re}\Big\{\sum_{k=0}^{\infty}[(2\cdot H_{k1}+\bar z\cdot H_{k2}+H_{k4})\cdot M_{2k}\\
&\quad+(2\cdot G_{k1}+\bar zG_{k2}+G_{k4})\cdot K_{2k}]\Big\}\\
\tau_{xy} &= \sigma\cdot\mathrm{Im}\Big\{\sum_{k=0}^{\infty}[(2H_{k2}\cdot\bar z+H_{k4})\cdot M_{2k}\\
&\quad+(2G_{k2}\cdot\bar z+G_{k4})\cdot K_{2k}]\Big\}\\
u &= \frac{\sigma\cdot d}{2G}\mathrm{Re}\Big\{\sum_{k=0}^{\infty}[(\gamma\cdot H_{k5}-z\overline{H}_{k1}-H_{k3})\cdot M_{2k}\\
&\quad+(\gamma G_{k5}-z\cdot\overline{G}_{k1}-G_{k3})K_{2k}]\Big\}\\
v &= \frac{\sigma\cdot d}{2G}\mathrm{Im}\Big\{\sum_{k=0}^{\infty}[(\gamma\cdot H_{k5}-z\cdot H_{k1}-H_{k3})\cdot M_{2k}\\
&\quad+(\gamma G_{k5}-z\cdot G_{k1}-G_{k3})K_{2k}]\Big\}
\end{aligned}\right\} \tag{4}$$

(4) 中的 H_{k1}, H_{k2}, $\cdots$, G_{k5} 都是 z 和 k 的已知函数，具体表达式见附录 B. 如果其中的 M_{2k}, K_{2k} 被确定了，就得到整个应力场和位移场. 下面将针对两个具体问题确定 M_{2k}, K_{2k}.

2 含孔有限板平面问题

含椭圆孔有限板问题如图 2, 取边界条件

$$\left.\begin{aligned} &\Gamma_0: \frac{X^2}{a^2}+\frac{Y^2}{b^2}=1, && \sigma_n=\sigma_t=0 \\ &\Gamma_1: Y_0=L, && \sigma_y=\bar{\sigma}_y, \sigma_t=0 \\ &\Gamma_2: X_0=L, && \sigma_x=\sigma_t=0 \end{aligned}\right\} \tag{5}$$

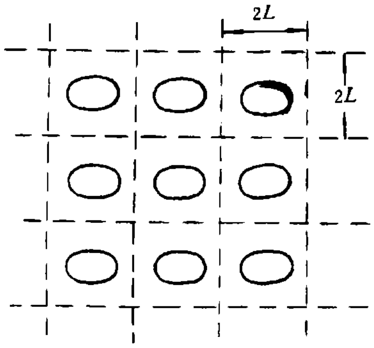

图 2
Fig.2

由最小势能原理

$$\begin{aligned} \frac{1}{4}\delta\Pi &= \int_\Omega \delta A \mathrm{d}\Omega - \int_\Gamma \bar{T}_i \delta u i \mathrm{d}s \\ &= \int_\Gamma T i \delta u i \mathrm{d}s - \int_\Gamma \bar{T}_i \delta u i \mathrm{d}s \\ &\quad - \int_\Omega \sigma_{ij,j} \delta u i \mathrm{d}\Omega = 0 \end{aligned} \tag{6}$$

式中 Ω, Γ 分别是图 1 所示的域和外边界，$\bar{T}_i$ 是已知边界力，又因所取的解析函数 $\varphi(z)$, $\psi(z)$ 保证了平衡方程和内孔应力自由边界条件的满足，且不考虑体力，从而

$$\begin{aligned} \frac{1}{4}\delta\Pi = & \int_{\Gamma_1} \sigma_y \delta v \mathrm{d}s + \int_{\Gamma_1} \tau_{xy} \delta u \mathrm{d}s + \int_{\Gamma_2} \sigma_x \delta u \mathrm{d}s \\ & + \int_{\Gamma_2} \tau_{xy} \delta v \mathrm{d}s - \int_{\Gamma_1} \bar{\sigma}_y \mathrm{d}s = 0 \end{aligned} \tag{7}$$

将 (4) 式代上式得

$$\delta\Pi = \frac{\partial\Pi}{\partial M_{2p}}\delta M_{2p} + \frac{\partial\Pi}{\partial K_{2p}}\delta K_{2p} = 0, \quad p=0,1,\cdots,M-1$$

其中 M_{2p}, K_{2p} 是线性无关的，则

$$\frac{\partial\Pi}{\partial M_{2p}} = 0, \qquad \frac{\partial\Pi}{\partial K_{2p}} = 0 \tag{8}$$

由 (8) 即可得到 $2M$ 阶线性方程组，其矩阵形式为

$$\begin{bmatrix} [A_1(p,k)] & [C_1(p,k)] \\ [B_1(p,k)] & [D_1(p,k)] \end{bmatrix} \begin{Bmatrix} M_{2k} \\ K_{2k} \end{Bmatrix} = \begin{Bmatrix} E_1(p) \\ E_2(p) \end{Bmatrix} \tag{9}$$

A_1, B_1, C_1, D_1 分别是 $M\times M$ 阶子阵，E_1, E_2 是 M 阶向量，它们的具体表达式见附录 C. 以下将用 (9) 求解具体问题.

设含椭圆孔无限大板受单拉情况，且取 $\sigma = \bar{\sigma}_y$, $\nu = 0.3$, $d = L$, $a = 2.0$, $b = 1.5$, $L = 20.0$, $M = 40$. 计算 (7) 式中积分时用 5 结点高斯积分法，这样所得到的解，经过检验，其满足边界情况较好，相对误差小于甚至远小于 10^{-3}.

另外，还就椭圆孔环向应力 σ_c, $Y = 0$ 时应力集中曲线 $\sigma_y \sim X$, $X = 0$ 时 $\sigma_y \sim Y$ 曲线与文 [4] 中给出的解析解分别作了比较，见图 3 到图 4. 可见二者吻合得较好. 而且当取 $a = 2.0$, $b = 1.0$, $X_0 = 4.0$, $Y_0 = 20.0$ 时，本文方法得到的在孔右端的应力集中系数是 6.46, 与西田正孝 [6] 用实验给出的 6.4 符合得很好，可见本文方法是可靠的.

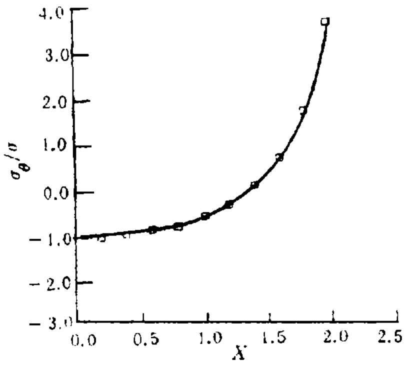

图 3 σ_θ-X 曲线

Fig.3 Curve of σ_θ-X

□ solution of this paper

—— analytic solution

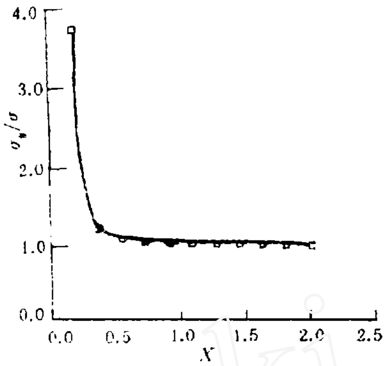

图 4 σ_y-X 曲线

Fig.4 Curve of σ_y-X

□ solution of this paper

—— analytic solution

3 双周期孔洞排列问题及等效模量的求解

在研究弹塑性材料含孔洞的损伤行为中， Becker 和 Needleman 提出了双周期排列的孔洞模型如图 2 所示. Brison 和 Knauss[2,3] 还定义了表征该模型材料宏观力学特性的等效杨氏模量 $\bar{E}$ 和等效泊松比 $\bar{\nu}$, 与前面的分析一致，取如图 1 所示研究，边界条件为

$$\left.\begin{array}{lll} \Gamma_0: \dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} = 1\,, & \sigma_n = \sigma_t = 0 & \\ \Gamma_1: Y = L\,, & u_n = \bar{u}_n, & \sigma_t = 0 \\ \Gamma_2: X = L\,, & u_n = u_0\,, & \sigma_t = 0 \end{array}\right\} \tag{10}$$

定义 $\bar{E}$ 、 $\bar{\nu}$

$$\left.\begin{array}{lll} \bar{E}_2 = \dfrac{\int_{\Gamma_2} \sigma_n \mathrm{d}s}{\bar{u}_n}\,, & \bar{\nu}_{12} = -\dfrac{u_0}{\bar{u}_n}\,, & (\bar{u}_n \text{在 } \Gamma_2 \text{ 边}) \\ \bar{E}_1 = \dfrac{\int_{\Gamma_1} \sigma_n \mathrm{d}s}{\bar{u}_n}\,, & \bar{\nu}_{21} = -\dfrac{u_0}{\bar{u}_n}\,, & (\bar{u}_n \text{在 } \Gamma_1 \text{ 边}) \end{array}\right\} \tag{11}$$

(10) 和 (11) 中 $\bar{u}_n$ 是已知位移， u_0 是未知位移，它使 $\int_{\Gamma_2} \sigma_x \mathrm{d}s = 0$, “$n$”, “$t$” 分别指

边界法向、切向.

设泛函

$$\frac{1}{4}\Pi=\int_{\Omega}B(\sigma_{ij})\mathrm{d}\Omega-\int_{\Gamma_1}\sigma_n\cdot\bar{u}_n\mathrm{d}s-\int_{\Gamma_2}\sigma_n\cdot u_0\mathrm{d}s-\int_{\Gamma_1+\Gamma_2}\sigma_t u_t\mathrm{d}s \tag{12}$$

其中 $B(\sigma_{ij})$ 是变形余能，上述问题的变分原理可描述为：

在所有可能的应力 σ_{ij} 和应变 ε_{ij} 中，真实的应力和应变应使 $\delta\pi=0$. 证明如下，由 (12) 式取变分可得

$$\begin{aligned}\frac{1}{4}\delta\Pi=&\int_{\Omega}\delta B(\sigma_{ij})\mathrm{d}\Omega-\int_{\Gamma_1}\delta\sigma_n\cdot\bar{u}_n\mathrm{d}s-\int_{\Gamma_2}u_0\cdot\delta\sigma_n\mathrm{d}s\\&-\delta u_0\int_{\Gamma_2}\sigma_n\mathrm{d}s-\int_{\Gamma_1+\Gamma_2}(\sigma_t\delta u_t+\delta\sigma_t\cdot u_t)\mathrm{d}s\end{aligned} \tag{13}$$

由于文中所给函数 $\varphi(z)$, $\psi(z)$, 保证了平衡方程和 Γ_0 边自由应力条件的满足，则

$$\int_{\Omega}\delta B(\sigma_{ij})\mathrm{d}\Omega=\int_{\Gamma_1+\Gamma_2}(u_n\delta\sigma_n+u_t\delta\sigma_t)\mathrm{d}s \tag{14}$$

把式 (14) 代入式 (13) 可得

$$\begin{aligned}\frac{1}{4}\delta\pi=&\int_{\Gamma_1}(u_n-\bar{u}_n)\delta\sigma_n\mathrm{d}s+\int_{\Gamma_2}(u_n-u_0)\delta\sigma_n\mathrm{d}s\\&-\int_{\Gamma_1+\Gamma_2}\sigma_t\delta u_t\mathrm{d}s-\delta u_0\int_{\Gamma_2}\sigma_n\mathrm{d}s\end{aligned} \tag{15}$$

从上式可知，只有满足式 (10) 中所取边界条件时，才能保证 $\delta\Pi=0$, 也就说明了泛函式 (12) 是与式 (10) 对应的.

把式 (4) 代入上式，且 $\delta\Pi=0$, 则

$$\delta\Pi=\frac{\partial\Pi}{\partial M_{2p}}\delta M_{2p}+\frac{\partial\Pi}{\partial K_{2p}}\delta K_{2p}+\frac{\partial\Pi}{\partial u_0}\delta u_0=0$$

式中 $p=0,1,2,\cdots,M-1$, 且 M_{2p}, K_{2p}, u_0 是线性无关的，则

$$\frac{\partial\Pi}{\partial M_{2p}}=0\,,\quad\frac{\partial\Pi}{\partial K_{2p}}=0\,,\quad\frac{\partial\Pi}{\partial u_0}=0 \tag{16}$$

由式 (16) 即可得 $2M+1$ 阶线性方程组，其矩阵形式为

$$\begin{bmatrix}[A_{a1}(p,k)] & [C_{c1}(p,k)] & [F_{f1}(p)]\\ [B_{b1}(p,k)] & [D_{d1}(p,k)] & [F_{f2}(p)]\\ [F_{f1}(k)]^{\mathrm{T}} & [F_{f2}(k)]^{\mathrm{T}} & 0\end{bmatrix}\begin{Bmatrix}M_{2k}\\K_{2k}\\u_0\end{Bmatrix}=\begin{Bmatrix}E_{e1}(p)\\E_{e2}(p)\\0\end{Bmatrix} \tag{17}$$

式中 A_{a1}, B_{b1}, C_{c1}, D_{d1} 是 $M\times M$ 阶子阵， F_{f1}, F_{f2}, E_{e1}, E_{e2} 是 M 阶向量，具体表达式见附录 D.

计算中取 $a/L=0.5$, $b/L=0.375$, $M=40$, 式 (1) 中取 $\sigma=E$, $d=L$, $\nu=0.3$, 计算过程与第 2 部分一样，得到了令人感兴趣的 Γ_1 和 Γ_2 上正应力分布曲线见图 5 和图 6. 计算结果还表明本文结果满足边界条件情况较好，其相对误差小于 10^{-5}. 文中还就 a, b, L 取不同值算出了一系列的等效模量和等效泊松比见表 1— 表 3, 由此观察孔洞大小和形状对 $\bar{E}$, $\bar{\nu}$ 的影响. 从表中可知，椭圆越扁即 b/a 越小，等效模量越大，$\bar{\nu}_{12}$ 也越大，但 $\bar{\nu}_{21}$ 越小；孔越小，等效模量越大. 值得注意的是，孔洞的作用使等效泊松比变大，但 Γ_2 边界的作用却使它变小. 当孔洞作用大过 Γ_2 边界作用时，等效泊松比就大于材料泊松比 (文中取 0.3), 否则就小于材料泊松比.

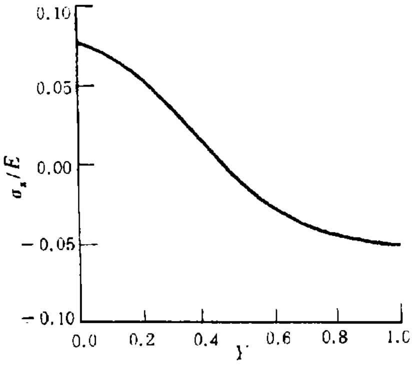

图 5 Γ_2 边 σ_x-Y 曲线
Fig.5 Curve of σ_x-Y on Γ_2

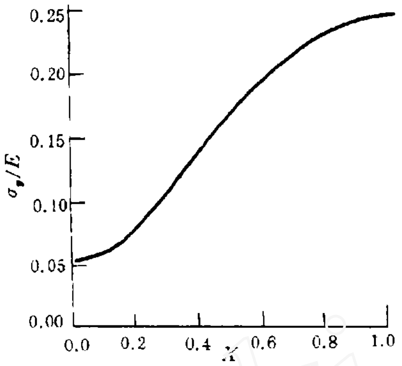

图 6 Γ_1 边 σ_y-X 曲线
Fig.6 Curve of σ_y-X on Γ_1

表 1 $b/a=0.9999$ 时 $\bar{E}_1$, $\bar{E}_2$, $\bar{\nu}_{12}$, $\bar{\nu}_{21}$

Table 1 $\bar{E}_1$, $\bar{E}_2$, $\bar{\nu}_{12}$, $\bar{\nu}_{21}$ when $b/a=0.9999$

L/a	$\bar{E}_1$	$\bar{\nu}_{12}$	$\bar{E}_2$	$\bar{\nu}_{21}$
1.75	0.5419	0.2461	0.5417	0.2460
2.0	0.6171	0.2666	0.6169	0.2664
2.5	0.7225	0.2867	0.7224	0.2867
3.0	0.7914	0.2949	0.7913	0.2948
3.5	0.8384	0.2984	0.8383	0.2984
4.0	0.8716	0.3000	0.8715	0.3000
5.0	0.9140	0.3010	0.9139	0.3010
10.	0.9770	0.3006	0.9770	0.3006

表 2 $b/a=0.85$ 时 $\bar{E}_1$, $\bar{E}_2$, $\bar{\nu}_{12}$, $\bar{\nu}_{21}$

Table 2 $\bar{E}_1$, $\bar{E}_2$, $\bar{\nu}_{12}$, $\bar{\nu}_{21}$ when $b/a=0.85$

L/a	$\bar{E}_1$	$\bar{\nu}_{12}$	$\bar{E}_2$	$\bar{\nu}_{21}$
1.75	0.6917	0.2730	0.5537	0.2439
2.0	0.6849	0.2867	0.6280	0.2629
2.5	0.7753	0.2988	0.7318	0.2820
3.0	0.8331	0.3027	0.7990	0.2904
3.5	0.8719	0.3039	0.8447	0.2944
4.0	0.8989	0.3040	0.8769	0.2965
5.0	0.9329	0.3034	0.9177	0.2984
10.	0.9823	0.3012	0.9781	0.2998

表 3 $b/a=0.75$ 时 $\bar{E}_1$，$\bar{E}_2$，$\bar{\nu}_{12}$，$\bar{\nu}_{21}$

Table 3 $\bar{E}_1$, $\bar{E}_2$, $\bar{\nu}_{12}$, $\bar{\nu}_{21}$ when $b/a=0.75$

L/a	$\bar{E}_1$	$\bar{\nu}_{12}$	$\bar{E}_2$	$\bar{\nu}_{21}$
1.75	0.6715	0.2886	0.5621	0.2416
2.0	0.7297	0.2981	0.6356	0.2597
2.5	0.8096	0.3054	0.7381	0.2785
3.0	0.8598	0.3069	0.8044	0.2871
3.5	0.8930	0.3066	0.8479	0.2915
4.0	0.9160	0.3059	0.8805	0.2941
5.0	0.9445	0.3045	0.9203	0.2966
10.	0.9855	0.3013	0.9789	0.2990

4 结束语

原则上讲，只要结合广义变分原理和选择合理的解析函数，就能得到不仅仅限于孔洞问题的半解析解，但是盲目地选择广义变分原理和不合理的解析函数将使计算量增大，并且计算精度降低．本文针对具体问题建立了仅存在局部边界积分的泛函，并使用了满足内孔边界自由的解析函数，通过对泛函驻值的求解替代了求解复杂边界条件的微分方程组，从而减少了许多不必要的工作量．另外，文中的方法事实上并不仅仅适用于对称情况，当不对称时，只需在附录 A 中加入 F_{2n+1}，D_{2n+1} 和 M_{2n+1}，K_{2n+1} 的关系就行了．

从文所运用的方法和给出的结果可以看到，关键就是要建立一个合理的泛函和选用合适的解析函数．一旦做到上述两点要得到半解析解并不困难，且这种思想具有一定广泛性，特别是针对一些几何形状相对简单的问题．

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附录 A

式 (3) 中 F_{2n}，M_{2n}，D_{2n}，K_{2n}，D_0 满足

$$F_{2n}=-\sum_{p=0}^{\infty}\lambda^{2n+2p+2}\left(Q_{2p}^{2n}\cdot K_{2p}+S_{2p}^{2n}\cdot M_{2p}\right)$$

$$D_{2n}=-\sum_{p=0}^{\infty}\lambda^{2n+2p+2}\left(P_{2p}^{2n}\cdot K_{2p}+R_{2p}^{2n}\cdot M_{2p}\right),\quad n=0,1,2,\cdots$$

其中

$$
\begin{aligned}
P_{2p}^{0} = {} & (1-\varepsilon^2)^{p+1}\frac{p+1}{2^{2p}}\cdot C_{2p+1}^{p} \\
R_{2p}^{0} = {} & P_{2p}^{2p}(R^2+R^{-2})/2 \\
P_{2p}^{2n} = {} & \frac{(1-\varepsilon^2)^{n+p+1}}{2^{2p}}\Bigg[4(p+1)C_{2p+1}^{p}\left(1+\frac{1}{n}\right)A_{n,1}+\sum_{m=0}^{\min(p+1,n)}C_{2p+2}^{p-m+1} \\
& \cdot\Big\{(2p+1)R^{4m}-1\Big\}A_{n-m,2m}\Bigg] \\
R_{2p}^{2n} = {} & \frac{(1-\varepsilon^2)^{n+p+1}}{2^{2p+1}}\Bigg[(p+1)C_{2p+1}^{p}(R^2+R^{-2})\cdot\left(1+\frac{1}{n}\right)A_{n,1} \\
& +\sum_{m=0}^{\min(n,p)}C_{2p+1}^{p}\cdot R^{4m+2}\cdot A_{n-m,2m+1}\cdot 2p\Bigg] \\
Q_{2p}^{2n} = {} & \frac{(1-\varepsilon^2)^{n+p+1}}{2^{2p}}(p+1)\sum_{m=0}^{\min(p,n)}C_{2p+1}^{p-1}\cdot R^{4m+2}\cdot A_{n-m,2m+1} \\
S_{2p}^{2n} = {} & \frac{(1-\varepsilon^2)^{n+p+1}}{2^{2p+1}}\Bigg[(p+1)C_{2p+1}^{p}\cdot A_{n,1}+\sum_{m=0}^{\min(n,p)}C_{2p+1}^{p-m}\cdot A_{n-m,2m+1} \\
& +2(2p+1)\sum_{m=1}^{\min(n+1,p)}C_{2p}^{p-m}\cdot R^{4m}\cdot A_{n-m+1,2m}\Bigg] \\
\lambda = {} & a/d\,,\quad R=\sqrt{\frac{a+b}{a-b}} \\
\varepsilon = {} & b/a\,, \\
A_{n-m,2m} = {} & \frac{m}{2^{2n}\cdot n}\cdot C_{2n}^{n-m} \\
A_{n-m,2m+1} = {} & \frac{2m+1}{2^{2n+1}\cdot(2n+1)}\cdot C_{2n+1}^{n-m} \\
C_{n}^{m} = {} & \frac{n!}{m!(n-m)!}
\end{aligned}
$$

时，椭圆孔边应力自由的边界条件就恒满足了 (详见文 [5]).

附录 B

$$
\begin{aligned}
H_{k1} &= \sum_{n=0}^{\infty}(2n+1)\cdot\lambda^{2n+2k+2}\cdot S_{2k}^{2n}\cdot z^{-(2n+2)}+(2k+1)\cdot z^{2k} \\
G_{k1} &= \sum_{n=0}^{\infty}(2n+1)\cdot\lambda^{2n+2k+2}\cdot Q_{2k}^{2n}\cdot z^{-(2n+2)} \\
H_{k2} &= -\sum_{n=0}^{\infty}(2n+1)(2n+2)\cdot\lambda^{2n+2k+2}\cdot z^{-(2n+3)}Q_{2k}^{2n}+2k\cdot(2k+1)\cdot z^{2k-1} \\
G_{k2} &= -\sum_{n=0}^{\infty}(2n+1)(2n+2)\cdot\lambda^{2n+2k+2}\cdot z^{-(2n+3)}\cdot Q_{2k}^{2n} \\
H_{k3} &= -\sum_{n=0}^{\infty}2n\cdot\lambda^{2n+2k+2}\cdot R_{2k}^{2n}\cdot z^{-(2n+1)}-\lambda^{2k+2}\cdot R_{2k}^{0}\cdot\frac{1}{z} \\
G_{k3} &= -\sum_{n=0}^{\infty}2n\cdot\lambda^{2n+2k+2}\cdot P_{2k}^{2n}\cdot z^{-(2n+1)}-\lambda^{2k+2}\cdot P_{2k}^{0}\cdot\frac{1}{z}
\end{aligned}
$$

$$H_{k4} = -\sum_{n=0}^{\infty} 2n(2n+1) \cdot \lambda^{2n+2k+2} \cdot R_{2k}^{2n} \cdot z^{-(2n+2)} + \lambda^{2k+2} \cdot R_{2k}^{0} \cdot \frac{1}{z^2}$$

$$G_{k4} = \sum_{n=0}^{\infty} 2n(2n+1) \cdot \lambda^{2n+2k+2} P_{2k}^{2n} \cdot z^{-(2n+2)} + \lambda^{2k+2} \cdot P_{2k}^{0} \cdot \frac{1}{z^2} + (2k+1)(2k+2) \cdot z^{2k}$$

$$H_{k5} = -\sum_{n=0}^{\infty} \lambda^{2n+2k+2} \cdot S_{2k}^{2n} \cdot z^{-(2n+1)} + z^{2k+1}$$

$$G_{k5} = -\sum_{n=0}^{\infty} \lambda^{2n+2k+2} \cdot Q_{2k}^{2n} \cdot z^{-(2n+1)}$$

式中符号意义与附录 A 一致.

附录 C

$$A_1(p,k) = \int_{\Gamma_1} \left(\frac{\partial \sigma_y}{\partial M_{2k}} \cdot \frac{\partial v}{\partial M_{2p}} + \frac{\partial \tau_{xy}}{\partial M_{2k}} \cdot \frac{\partial u}{\partial M_{2p}} \right) \mathrm{d}s + \int_{\Gamma_2} \left(\frac{\partial \sigma_x}{\partial M_{2k}} \cdot \frac{\partial u}{\partial M_{2p}} + \frac{\partial \tau_{xy}}{\partial M_{2k}} \cdot \frac{\partial v}{\partial M_{2p}} \right) \mathrm{d}s$$

$$C_1(p,k) = \int_{\Gamma_1} \left(\frac{\partial \sigma_y}{\partial K_{2k}} \cdot \frac{\partial v}{\partial M_{2p}} + \frac{\partial \tau_{xy}}{\partial K_{2k}} \cdot \frac{\partial u}{\partial M_{2p}} \right) \mathrm{d}s + \int_{\Gamma_2} \left(\frac{\partial \sigma_x}{\partial K_{2k}} \cdot \frac{\partial u}{\partial M_{2p}} + \frac{\partial \tau_{xy}}{\partial K_{2k}} \cdot \frac{\partial v}{\partial M_{2p}} \right) \mathrm{d}s$$

$$B_1(p,k) = \int_{\Gamma_1} \left(\frac{\partial \sigma_y}{\partial M_{2k}} \cdot \frac{\partial v}{\partial K_{2p}} + \frac{\partial \tau_{xy}}{\partial M_{2k}} \cdot \frac{\partial u}{\partial K_{2p}} \right) \mathrm{d}s + \int_{\Gamma_2} \left(\frac{\partial \sigma_x}{\partial M_{2k}} \cdot \frac{\partial u}{\partial K_{2p}} + \frac{\partial \tau_{xy}}{\partial M_{2k}} \cdot \frac{\partial v}{\partial K_{2p}} \right) \mathrm{d}s$$

$$D_1(p,k) = \int_{\Gamma_1} \left(\frac{\partial \sigma_y}{\partial K_{2k}} \cdot \frac{\partial v}{\partial K_{2p}} + \frac{\partial \tau_{xy}}{\partial K_{2k}} \cdot \frac{\partial u}{\partial K_{2p}} \right) \mathrm{d}s + \int_{\Gamma_2} \left(\frac{\partial \sigma_x}{\partial K_{2k}} \cdot \frac{\partial u}{\partial K_{2p}} + \frac{\partial \tau_{xy}}{\partial K_{2k}} \cdot \frac{\partial v}{\partial K_{2p}} \right) \mathrm{d}s$$

$$E_1(p) = \int_{\Gamma_1} \bar{\sigma}_y \cdot \frac{\partial v}{\partial M_{2p}} \mathrm{d}s$$

$$E_2(p) = \int_{\Gamma_2} \bar{\sigma}_y \cdot \frac{\partial v}{\partial K_{2p}} \mathrm{d}s$$

式中 $p, k = 0, 1, 2, \cdots, M-1$.

附录 D

$$A_{a1}(p,k) = \int_{\Gamma_1} \left(\frac{\partial v}{\partial M_{2k}} \cdot \frac{\partial \sigma_y}{\partial M_{2p}} - \frac{\partial u}{\partial M_{2k}} \cdot \frac{\partial \tau_{xy}}{\partial M_{2p}} \right) \mathrm{d}s + \int_{\Gamma_2} \left(\frac{\partial u}{\partial M_{2k}} \cdot \frac{\partial \sigma_x}{\partial M_{2p}} - \frac{\partial v}{\partial M_{2k}} \cdot \frac{\partial \tau_{xy}}{\partial M_{2p}} \right) \mathrm{d}s$$

$$C_{c1}(p,k) = \int_{\Gamma_1} \left(\frac{\partial v}{\partial K_{2k}} \cdot \frac{\partial \sigma_y}{\partial M_{2p}} - \frac{\partial u}{\partial K_{2k}} \cdot \frac{\partial \tau_{xy}}{\partial M_{2p}} \right) \mathrm{d}s + \int_{\Gamma_2} \left(\frac{\partial u}{\partial K_{2k}} \cdot \frac{\partial \sigma_x}{\partial M_{2p}} - \frac{\partial v}{\partial K_{2k}} \cdot \frac{\partial \tau_{xy}}{\partial M_{2p}} \right) \mathrm{d}s$$

$$B_{b1}(p,k)=\int_{\Gamma_1}\left(\frac{\partial v}{\partial M_{2k}}\cdot\frac{\partial\sigma_y}{\partial K_{2p}}-\frac{\partial u}{\partial M_{2k}}\cdot\frac{\partial\tau_{xy}}{\partial K_{2p}}\right)\mathrm{d}s+\int_{\Gamma_2}\left(\frac{\partial u}{\partial M_{2k}}\cdot\frac{\partial\sigma_x}{\partial K_{2p}}-\frac{\partial v}{\partial M_{2k}}\cdot\frac{\partial\tau_{xy}}{\partial K_{2p}}\right)\mathrm{d}s$$

$$D_{d1}(p,k)=\int_{\Gamma_1}\left(\frac{\partial v}{\partial K_{2k}}\cdot\frac{\partial\sigma_y}{\partial K_{2p}}-\frac{\partial u}{\partial K_{2k}}\cdot\frac{\partial\tau_{xy}}{\partial K_{2p}}\right)\mathrm{d}s+\int_{\Gamma_2}\left(\frac{\partial u}{\partial K_{2k}}\cdot\frac{\partial\sigma_x}{\partial K_{2p}}-\frac{\partial v}{\partial K_{2k}}\cdot\frac{\partial\tau_{xy}}{\partial K_{2p}}\right)\mathrm{d}s$$

$$E_{f1}(p)=\int_{\Gamma_2}\frac{\partial\sigma_x}{\partial M_{2p}}\mathrm{d}s$$

$$E_{f2}(p)=\int_{\Gamma_2}\frac{\partial\sigma_x}{\partial K_{2p}}\mathrm{d}s$$

$$E_{e1}(p)=\int_{\Gamma_1}\bar{u}_n\cdot\frac{\partial\sigma_y}{\partial M_{2p}}\mathrm{d}s$$

$$E_{e2}(p)=\int_{\Gamma_1}\bar{u}_n\cdot\frac{\partial\sigma_y}{\partial K_{2p}}\mathrm{d}s$$

式中 $p,k=0,1,2,\cdots,m-1$.

A VARIATIONAL METHOD TO SOLUTION OF LINEAR ELASTIC PROBLEM WITH VOID

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Abstract This paper utilizes two Laurrent's series which satisfy the conditions of no traction on inner elliptical boundary, and sets up functions. It turns out that the other boundary conditions can be satisfied provided that the variation of the functions equals zero. With the variational approaching, the elastic problem with void is resovled. Furthermore, this method is applied to the model of periodical distributed void, the effective modulus and the effctive Poisson's ratios have been obtained.

Key words mathematical elasticity, variational method, continuous damage mechanics