Control of a finite dam when the input process is either spectrally positive Lévy or spectrally positive Lévy reflected at its infimum

Mohamed Abdel-Hameed
Department of Statistics
College of Business and Economics
UAE University

Abstract

Bae *et al.* [6] consider the problem of optimal control of a finite dam using $P^M_{\lambda,\tau}$ policies, assuming that the input process is a compound Poisson process with a negative drift. Lam and Lou [8] treat the case where the input is a Wiener process with a reflecting boundary at its infimum, with drift term $\mu \geq 0$, using the long-run average and total discounted cost criteria. Attia [4] obtains results similar to those of Lam and Lou, through simpler and more direct methods. Zuckermann [12] considers $P^M_{\lambda,0}$ policies when the input process is a is Wiener process with drift term $\mu \geq 0$. The techniques used by the above mentioned authors involve solving systems of differential or integral equations. In this paper we use the theory and methods of scale functions of Lévy processes to unify and extend the results of these authors.

Keywords: $P^M_{\lambda,\tau}$ policies; spectrally positive Lévy processes; spectrally positive Lévy processes reflected at its infimum; scale functions; exit times; α-potentials; total discounted and long-run-average costs.
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1. Introduction and summary

Suppose that a dam has capacity V. Its water input $I = (I_t, t \geq 0)$, is assumed to be a Lévy process with drift μ, variance σ^2, and the water is released at one of two rates 0 or M units per unit of time. We consider $P^M_{\lambda,\tau}$ policies in which the water release rate is assumed to be zero until the water crosses level λ, $(0 < \lambda < V)$, when the water is released at rate M until it reaches level τ, $(0 \leq \tau < \lambda)$. Once level τ is reached, the release rate remains zero until level λ is reached again, and the cycle is repeated. We deal with the cases where the input process is spectrally positive Lévy, and spectrally positive Lévy reflected at its infimum. In both cases the content process is a delayed regenerative process with regeneration points being the times of successive visits to state τ. During a given cycle, the dam's water content is a Lévy process with coefficients μ and σ^2, and it remains so until it crosses level λ; from then until it drops to level τ again the content level behaves like a Lévy process reflected at V with coefficients $\mu^* = \mu - M$, σ^2, denoted by $I^* = (I_t^*, t \geq 0)$. At any time, the release rate can be increased from 0 to M with a starting cost $K_1 M$, or decreased from M to zero with a closing cost $K_2 M$. Moreover, for each unit of output, a reward R is received. Furthermore, there is a penalty cost which accrues at a rate f , where f a bounded measurable function. For the first case we extend the results of Zuckerman [12] who assumed that $\tau = 0$ and $f = 0$. Our results in the second case extend the results of Lam and Lou [8] and Attia [4], where they assumed that the input process is a Wiener process reflected at its infimum. They also extend those of Bae *et al.* [6], who consider the case where the water input is a compound Poisson process with negative drift. Lee and Ahn [9] consider the long-run average cost case, for the $P_{\lambda,0}^M$ policy, when the water input is a compound Poisson process. Abdel-Hameed [1] treats the case where the water input is a compound Poisson process with a positive drift. He obtains the total discounted as well as the long-run average costs. Bae *et al.* [5] consider the $P_{\lambda,0}^M$ policy in assessing the workload of an M/G/1 queuing system. The techniques used by in [12], [8], [4], and [6] involve solving systems of differential or integral equations. In this paper we use the theory and methods of scale functions of Lévy processes, an approach not uses by researchers in this area before.

In Section 2 we define the input processes and discuss their properties. In Section 3 we obtain formulas needed for computing the cost functionals. In Section 4, we discuss the cost functionals using the total discounted as well as the long-run average cost cases. In section 5 we discuss the special cases where the input process is a Gaussian process, a Gaussian process reflected at its infimum and a spectrally positive Lévy process of bounded variation.

2. Spectrally positive Lévy processes and scale functions

In this section we give some basic definitions; describe spectrally positive Lévy processes and discuss some of their characteristics. The reader is referred to [7] for a more detailed discussion of the definitions and results mentioned in this section.

For any process $Y = \{Y_t, t \geq 0\}$ with state space E, any Borel set $A \subset E$ and any functional f, $E_y(f)$ denotes the expectation of f conditional on $Y_0 = y$, $P_y(A)$ denotes the corresponding probability measure and $\mathbf{I}_A(\ )$ is the indicator function of the set A. In the sequel we will write indifferently P_0 or P and E_0 or E. Throughout, we let $R = (-\infty, \infty)$, $R_+ = [0, \infty)$, $N = \{1, 2, ...\}$ and $N_+ = \{0, 1, ...\}$. For $x, y \in R$, we define $x \vee y = x \max y$ and $x \wedge y = x \min y$. For every $t \geq 0$, we define $\underline{Y_t} = \inf_{0 \leq s \leq t} (Y_s, \wedge 0)$, $\bar{Y}_t = \sup_{0 \leq s \leq t} (Y_s, \vee 0)$.

We will use the term ”increasing” to mean ”non-decreasing” throughout this paper.

Definition 1. A Lévy process $L = \{L_t, t \geq 0\}$ with state space R is said to be spectrally positive Lévy process, if it has no negative jumps.

It follows that, for each $\theta \in R_+, x \in R$,

$$E[e^{-\theta\ L\ t}] = e^{t\phi(\theta)},$$

where

$$\phi(\theta) = -a\theta + \frac{\theta^2\sigma^2}{2} - \int_0^\infty (1 - e^{-\theta x} - \theta x \mathbf{I}_{\{x<1\}})\upsilon(dx). \quad (2.1)$$

The terms $a \in R$, $\sigma^2 \in R_+$ are the drift and variance of the spectrally positive Lévy process, respectively. The Lévy measure υ is a positive measure on $(0, \infty)$ satisfying $\int_0^\infty (x^2 \wedge 1)\upsilon(dx) < \infty$.

The function ϕ is known as the Lévy exponent, and it is strictly convex and tends to infinity as θ tends to infinity. For $\alpha \in R_+$, we define

$$\eta(\alpha) = \sup\{\theta : \phi(\theta) = \alpha\} \ \ (2.2),$$

the largest root of the equation $\phi(\theta) = \alpha$. It is seen that this equation has at most two roots, one of which is the zero root. Note that, $E(L_1\ ) = \int_1^\infty x\upsilon(dx) + \mu$. Furthermore, $\lim_{t\to\infty} L_t\ \ = \infty$ if and only if $E(L_1\ ) > 0$, and $\lim_{t\to\infty} L_t\ \ = -\infty$ if and only if $E(L_1\ ) < 0$. Also, if $E(L_1\ ) = 0$, then $\lim_{t\to\infty} L_t$ does not exist. Furthermore, $\eta(0) > 0$, if and only if $E(L_1\ ) > 0$.

An important case is when the process L is of bounded variations, i.e., $\sigma^2 = 0$ and $\int_0^\infty (x \wedge 1)\upsilon(dx) < \infty$. Let

$$\zeta = -a + \int_0^1 x\upsilon(dx).$$

In this case we can write

$$\phi(\theta) = \zeta\theta - \int_0^\infty (1 - e^{-\theta x})\upsilon(dx), \ (2.3)$$

where necessarily ζ is strictly positive.

Definition 2. A Lévy process is said to be spectrally negative if it has no positive jumps.

Definition 3. For any spectrally positive Lévy input process L, we let $\hat{L} = -\ L$.

It is clear that L is spectrally positive if and only if the process $\hat{L}$ is spectrally negative.

We now introduce tools, which will be central in the rest of this paper.

Definition 4. For any spectrally positive Lévy process with Lévy exponent ϕ and for $\alpha \geq 0$, the $\alpha-$ *scale function* $W^{\alpha} : R \twoheadrightarrow R_{+}$, $W^{\alpha}(x) = 0$ for every $x < 0$, and on $[0, \infty)$ it is defined as the unique right continuous increasing function such that

$$\int_0^{\infty} e^{-\beta x} W^{(\alpha)}(x) dx = \frac{1}{\phi(\beta) - \alpha}, \ \beta > \eta(\alpha) \quad (2.4)$$

We will denote W^0 by W throughout. For $\alpha \geq 0$, we have (see (8.24) of [7])

$$W^{(\alpha)}(x) = \sum_{k=0}^{\infty} \alpha^k W^{*(k+1)}(x), \quad (2.5)$$

where $W^{*(k)}$ is the kth convolution of W with itself.

It follows that $W^{(\alpha)}(0+) = 0$ if and only if the process L is of unbounded variation. Furthermore, $W^{(\alpha)}$ is right and left differentiable on $(0, \infty)$. By $W_{+}^{(\alpha)'}(x)$, we will denote the right derivative of $W^{(\alpha)}$ in x.

The adjoint $\alpha-$ scale function associated *with* $W^{(\alpha)}$ (denoted by $Z^{(\alpha)}$) is defined as follows:

Definition 5. For $\alpha \geq 0$, the *adjoint* $\alpha-$ *scale* $Z^{(\alpha)} : R \rightarrow [1, \infty)$ is defined as

$$Z^{(\alpha)}(x) = 1 + \alpha \int_0^x W^{\alpha}(y) dy. \quad (2.6)$$

It follows that as $x \rightarrow \infty$, for $\alpha > 0$, $W^{(\alpha)}(x) \sim \frac{e^{\eta(\alpha)x}}{\phi'(\eta(\alpha)\ )}$ and $\frac{Z^{\alpha}(x)}{W^{(\alpha)}(x)} \sim \frac{\alpha}{\eta(\alpha)}$.

3. Basic results

For each $t \in R_{+}$, let Z_t be the dam content at time t , $Z = \{Z_t, t \in R_{+}\}$. We define the following sequence of stopping times :

$$\begin{aligned} \hat{T}_0 &= \inf\{t \geq 0 : Z_t \geq \lambda\}, \quad \overset{*}{T}_0 = \inf\{t \geq \hat{T}_0 : Z_t = \tau\}, \\ \hat{T}_n &= \inf\{t \geq \overset{*}{T}_{n-1} : Z_t \geq \lambda\}, \ \overset{*}{T}_n = \inf\{t \geq \hat{T}_n : Z_t = \tau\}, \ n = 1, 2, ... \ (3.1) \end{aligned}$$

It follows that the process Z is a delayed regenerative process with regeneration points $\{\overset{*}{T}_n, n = 0, 1, ...\}$.

We define the bivariate process $B = (Z, R)$, where for $t \geq 0$ R_t is the release rate (0 or M) at time t. The process B has as its state space the pair of line segments

$$S = [(l, \lambda) \times \{0\}] \cup [(\tau, V] \times \{M\}],$$

where l is the lower bound of the state space of the input process I.

The penalty cost rate function is given by

$$f(z) = \begin{cases} g(z), & (z, r) \in (l, \lambda) \times \{0\} \\ g^*(z), & (z, r) \in (\tau, V] \times \{M\} \end{cases} \quad (3.2)$$

where $g : (l, \lambda) \to R_+$ and $g^* : (\tau, V] \to R_+$ are bounded measurable function.

For $\alpha \in R_+$, let the $C_\alpha(x, \lambda, 0)$ and $C_\alpha(x, \tau, M)$ be the expected discounted penalty costs during the interval $[0, \hat{T}_0)$, and during the interval $[\hat{T}_0, \overset{*}{T}_0)$ starting at x, respectively. It follows that, for $x \in [\tau, V]$

$$C_\alpha(x, \lambda, 0) = E_x \int_0^{\hat{T}_0} e^{-\alpha t} g(I_t) dt,$$

and for $x \in [\lambda, V]$

$$C_\alpha(x, \tau, M) = E_x \int_0^{\overset{*}{T}_0} e^{-\alpha t} g^*(I_t^*) dt. \quad (3.3)$$

The functionals (3.3), $E_x[e^{-\alpha \hat{T}_0}]$, $E_x[\hat{T}_0]$, $E_x[e^{-\alpha \overset{*}{T}_0}]$, $E_x[\overset{*}{T}_0]$, which we aim to evalaute, are needed to obtain the total discounted and the long-run average costs associated with the $P^M_{\lambda,\tau}$ policy, discussed in Section 4.

For any $a \in R$, we define $T_a^+ = \inf\{t \geq 0 : I_t \geq a\}$, $T_a^- = \inf\{t \geq 0 : I_t \leq a\}$, $\top_a^+ = \inf\{t \geq 0 : \hat{I}_t \geq a\}$ and $\top_a^- = \inf\{t \geq 0 : \hat{I}_t \leq a\}$. We note that $\hat{T}_0 = T_\lambda^+$ almost everywhere.

To derive $C_\alpha(x, \lambda, 0)$, $E_x[e^{-\alpha \hat{T}_0}]$, $E_x[\hat{T}_0]$ we define the process obtained by killing the process I at $\hat{T}_0$, as follows:

$$X_t = \{I_t, t < \hat{T}_0\}. \; (3.4)$$

It is known that this killed process is a strong Markov process, with state space (l, λ).

For any Borel set $A \subset (l,\lambda)$, and $t \in R_+$, the probability transition function of this process is given as follows

$$P_t(x,A) = P_x(I_t \in A, t < \hat{T}_0)$$

and for each $\alpha \in R_+$ its $\alpha-$potential is defined as follows

$$U^\alpha(x,A) = \int_0^\infty P_t(x,A)e^{-\alpha t}dt = E_x \int_0^{\hat{T}_0} e^{-\alpha t}\boldsymbol{I}_{\{I_t \in A\}}dt. \quad (3.5)$$

We note that for $x < \lambda$

$$C_\alpha(x,\lambda,0) = U^\alpha g(x). \quad (3.6)$$

The following Theorem will be used extensively throughout this paper.

Theorem 1. Let $S = \{S_t, t \geq 0\}$ be a strong Markov process. Define, $\mathcal{G} = \{\sigma(S_u\ , u \leq t)\}_{t \geq 0}$, τ to be any stopping time with respect to $\mathcal{G}$. Let Y be the process obtained by killing the process S at time τ, denote the state space of this process by E, and let U^α be its $\alpha-$potential. Then, for $x \in E$

$$E_x[e^{-\alpha\tau}] = 1 - \alpha U^\alpha \mathbf{I}_E(x). \quad (3.7)$$

Proof. From the definition of U^α, for any bounded measurable function f whose domain is E, we have

$$U^\alpha f(x) \quad = \quad E_x[\int_0^\tau e^{-\alpha t}f(S_t)dt] = \int_E f(y)U^\alpha(x,dy)\cdot$$

Taking f to be identically equal to one, we have

$$\frac{1 - E_x[e^{-\alpha\tau}]}{\alpha} = U^\alpha \mathbf{I}_E(x).$$

The required result is immediate from the last equation above. ■

First we consider the case where the input process is a spectrally positive Lévy process.

Proposition 1. For $\alpha \geq 0$, $a \leq \lambda$ the α-potential $(\overset{(1)}{U^\alpha})$ of the process I killed at $T = \hat{T}_0 \wedge T_a^-$ is absolutely continuous with respect to the Lebesgue measure on $[a,\lambda]$ and a version of its density is given by

$$\overset{(1)}{u^\alpha}(x,y) = W^{(\alpha)}(\lambda - x)\frac{W^{(\alpha)}(y-a)}{W^{(\alpha)}(\lambda - a)} - W^{(\alpha)}(y-x), \quad x,y \in [a,\lambda]. \qquad (3.8)$$

Proof. For $A \subset [a, \lambda]$

$$\begin{aligned}
\overset{(1)}{U^{\alpha}}(x, A) &= E_x \int_0^{T} e^{-\alpha t} \mathbf{I}_{\{I_t \in A\}} dt \\
&= E_{-x} \int_0^{\top^-_{-\lambda} \wedge \top^+_{-a}} e^{-\alpha t} \mathbf{I}_{\{\hat{I}_t \in -A\}} dt \\
&= E_{\lambda - x} \int_0^{\top^-_0 \wedge \top^+_{\lambda - a}} e^{-\alpha t} \mathbf{I}_{\{\hat{I}_t \in \lambda - A\}} dt \\
&= \int_{(\lambda - A)} [W^{(\alpha)}(\lambda - x) \frac{W^{(\alpha)}(\lambda - a - y)}{W^{(\alpha)}(\lambda - a)} - W^{(\alpha)}(y - x)] dy,
\end{aligned}$$

where the last equation follows from Theorem 8.7 of [7], this establishes our assertion. ■

Corollary 1. For $\alpha \geq 0$ the α-potential (U^{α}) of the process X is absolutely continuous with respect to the Lebesgue measure on $(-\infty, \lambda]$ and a version of its density is given by

$$u^{\alpha}(x, y) = W^{(\alpha)}(\lambda - x) e^{-\eta(\alpha)(\lambda - y)} - W^{(\alpha)}(y - x), \quad x, y \in (-\infty, \lambda]. \quad (3.9)$$

Proof. The proof follows from (3.8) by letting $a \to -\infty$ and since, for $\alpha \geq 0$, $W^{(\alpha)}(x) \sim \frac{e^{\eta(\alpha)x}}{\phi'(\eta(\alpha)\ )}$ as $x \to \infty$. ■

We are now in a position to find $E_x[e^{-\alpha \hat{T}_0}]$ and $E_x[\hat{T}_0]$.

Proposition 2. (i) For $\alpha \geq 0$ and $x \leq \lambda$ we have

$$E_x[e^{-\alpha \hat{T}_0}] = Z^{(\alpha)}(\lambda - x) - \frac{\alpha}{\eta(\alpha)} W^{(\alpha)}(\lambda - x). \quad (3.10)$$

(ii) For $x \leq \lambda$ we have

$$\begin{aligned}
E_x[\hat{T}_0] &= \frac{W(\lambda - x)}{\eta(0)} - \bar{W}(\lambda - x), \ \eta(0) > 0 \\
&= \infty \qquad\qquad\qquad\qquad , \eta(0) = 0, \quad (3.11)
\end{aligned}$$

where for every $x \geq 0$,

$$\bar{W}(x) = \int_0^x W(y) dy.$$

Proof. We only prove (i), the proof of (ii) is easily obtained from (i) and hence is omitted. Let U^{α} be as defined in Corollary 1, then

$$
\begin{aligned}
E_x[e^{-\alpha\hat{T}_0}] &= 1-\alpha U^{\alpha}\mathbf{I}_{(-\infty,\lambda)}(x) \\
&= 1-\alpha\int_{-\infty}^{\lambda}\{W^{\alpha}(\lambda-x)e^{-(\lambda-y)\eta(\alpha)}-W^{\alpha}(y-x)\}dy \\
&= 1+\alpha\int_{x}^{\lambda}W^{\alpha}(y-x)dy-\alpha W^{\alpha}(\lambda-x)\int_{-\infty}^{\lambda}e^{-(\lambda-y)\eta(\alpha)}dy \\
&= Z^{(\alpha)}(\lambda-x)-\frac{\alpha}{\eta(\alpha)}W^{(\alpha)}(\lambda-x),
\end{aligned}
$$

where the first equation follows from (3.7), the second equation follows from (3.9), the third equation follows since $W^{(\alpha)}(x)=0,\ x<0$, and the last equation follows from the definition of $Z^{(\alpha)}$. ■

For any Borel set $B\subset R_+\times R$, we let $M(B)$ be the Poisson random measure counting the number of jumps of the process I in B with Lévy measure ν, where if $B=[0,t)\times A,\ A\subset R$, then $E[M(B)]=t\upsilon(A)$.

Proposition 3. Let $\overset{(1)}{u^{\alpha}}$ be as given in (3.8), and $x\le\lambda\le z$, then

$$
E_x[e^{-\alpha\hat{T}_0}, I_{\hat{T}_0}\in dz,\ \hat{T}_0<T_a^-]=\int_a^{\lambda}\upsilon(dz-y)\overset{(1)}{u^{\alpha}}(x,y)dy \qquad (3.12)
$$

Proof. Let T be as defined in Proposition 1. For $x<\lambda,\alpha\ge 0,\ C\subset[\lambda,\infty)$ and $D\subset(a,\lambda)$ we have

$$
\begin{aligned}
E_x[e^{-\alpha\hat{T}_0},\ I_{\hat{T}_0} &\in C, I_{\hat{T}_0-}\in D,\hat{T}_0<T_a^-] \\
&= E_x[\int_{[0,\infty)\times(0,\infty)}e^{-\alpha t}\mathbf{I}_{\{\bar{I}_{t-}<\lambda,\underline{I_{t-}}>a,I_{t-}\in D\}}\mathbf{I}_{\{y\in C-I_{t-}\}}M(dt,dy)] \\
&= E_x[\int_{[0,\infty)}e^{-\alpha t}\mathbf{I}_{\{\bar{I}_{t-}<\lambda,\underline{I_{t-}}>a\}}\mathbf{I}_{\{I_t\in D\}}\nu(C-I_t)dt] \\
&= E_x[\int_{[0,\infty)}e^{-\alpha t}\mathbf{I}_{\{t<T\}}\nu(C-I_t)\mathbf{I}_{\{I_t\in D\}}dt)] \\
&= E_x[\int_{[0,\infty)\times D}e^{-\alpha t}\mathbf{I}_{\{t<T\}}\nu(C-y)\mathbf{I}_{\{I_t\in dy\}}dt] \\
&= \int_D\nu(C-y)\overset{(1)}{u^{\alpha}}(x,y)dy,
\end{aligned}
$$

where the second equation follows from the *compensation formula* (Theorem 4.4. of [7]). Our assertion is proved by taking $D=[a,\lambda]$. ■

The following corollary gives a formula needed to compute the total discounted cost.

Corollary 2. Let u^α be as defined in (3.9). For $\alpha \geq 0$ and for $x \leq \lambda \leq z$,

$$E_x[e^{-\alpha \hat{T}_0}, I_{\hat{T}_0} \in dz\,] = \int_{-\infty}^{\lambda} v(dz - y) u^\alpha(x, y) dy. \quad (3.13)$$

Proof. The proof follows immediately from (3.9) and (3.12) by letting $a \to -\infty$. ■

We now turn our attention to the case where the input process is a spectrally positive Lévy process reflected at its infimum. In this case, the killed process has state space $[0, \lambda)$. Let $\overset{(2)}{U}{}^\alpha$ be the α- potential of this process.

Proposition 4. For any $x, y \in [0, \lambda)$,

$$\overset{(2)}{U}{}^\alpha(x, dy) = \frac{W^{(\alpha)}(\lambda - x) W^{(\alpha)}(dy)}{W_+^{(\alpha)'}(\lambda)} - W^{(\alpha)}(y - x) dy, \quad (3.14)$$

where for $x, y \in [0, \lambda)$, $W^{(\alpha)}(dy) = W^{(\alpha)}(0)\delta_0(dy) + W_+^{(\alpha)'}(y)dy$, and δ_0 is the delta measure in zero.

Proof. Note that for each $t \geq 0$,

$$\begin{aligned} I_t &= Y_t - \underline{Y_t} \quad (3.15) \\ &= \overline{\hat{Y}}_t - \hat{Y}_t, \end{aligned}$$

where the process $Y = \{Y_t, t \geq 0\}$ is a spectrally positive Lévy process. The result follows from part (ii) of Theorem 1 of [10], since the process $\hat{Y}$ is a spectrally negative Lévy process. ■

The following provides results parallel to (3.10) and (3.11), resectively.

Proposition 5. Assume that the input process is a spectrally positive Lévy process reflected at its infimum. Then

(i) For $\alpha \geq 0$ and $x \leq \lambda$ we have

$$E_x[e^{-\alpha \hat{T}_0}] = Z^{(\alpha)}(\lambda - x) - W^{(\alpha)}(\lambda - x) \frac{\alpha W^{(\alpha)}(\lambda)}{W_+^{(\alpha)'}(\lambda)}. \quad (3.16)$$

(ii) For $x \leq \lambda$ we have

$$E_x[\hat{T}_0] = W(\lambda - x) \frac{W(\lambda)}{W_+'(\lambda)} - \bar{W}(\lambda - x). \quad (3.17)$$

Proof. The proof of part (i) follows from (3.7) and (3.14), in a manner similar to the proof of (3.10). The proof of part (ii) follows from part (i) by direct differentiation. ■

To find a formula analogous to (3.13), when the input is a spectrally positive Lévy process reflected at its infimum, we first need few definitions. Define

$$\begin{aligned} l_\alpha(dz) &= W^{(\alpha)}(\lambda - x)\int_0^\lambda W^{(\alpha}(dy)\upsilon(dz-y) \\ &\quad -W_+^{(\alpha)'}(\lambda)\int_0^\lambda dyW^{(\alpha)}(y-x)\upsilon(dz-y)], z > \lambda. \quad (3.18) \\ L_\alpha(z) &= \int_{(z,\infty)} l_\alpha(du). \quad (3.19) \\ V_\alpha(\lambda) &= W_+^{(\alpha)'}(\lambda)Z^{(\alpha)}(\lambda - x) - \alpha W^{(\alpha)}(\lambda - x)W^{(\alpha)}(\lambda). \quad (3.20) \end{aligned}$$

The following proposition gives the required formula.

Proposition 6. (i) For $\alpha \geq 0$ and for $x \leq \lambda < z$,

$$E_x[e^{-\alpha\hat{T}_0}, I_{\hat{T}_0} \in dz] = \frac{l_\alpha(dz)}{W_+^{(\alpha)'}(\lambda)}. \quad (3.21)$$

(ii) For $\alpha \geq 0$

$$E_x[e^{-\alpha\hat{T}_0}, I_{\hat{T}_0} = \lambda] = \frac{V_\alpha(\lambda) - L_\alpha(\lambda)}{W_+^{(\alpha)'}(\lambda)}. \quad (3.22)$$

Proof. (i) Consider the spectrally positive Lévy process $Y = \{Y_t, t \geq 0\}$, given in the proof of Proposition 4. For any $a \in R$, we define $\mathcal{F}_a$ as the sigma algebra generated by $(Y_s\ , s \leq t)$, $\tau_a^+ = \inf\{t \geq 0 : Y_t\ \geq a\}$, $\tau_a^- = \inf\{t \geq 0 : Y_t\ \leq a\}$, $\sigma_a^+ = \inf\{t \geq 0 : \hat{Y}_t\ \geq a\}$, and $\sigma_a^- = \inf\{t \geq 0 : \hat{Y}_t\ \leq a\}$. From (3.15), for $x \geq 0$, $I_0\ = x$ if and only if $Y_0 = x$ if and only if $\hat{Y}_0 = -x$. Furthermore, $\hat{T}_0 = \tau_\lambda^+$ and $I_{\hat{T}_0} = Y_{\tau_\lambda^+}$ almost surely on $\{\tau_\lambda^+ < \tau_0^-\}$. Therefore

$$\begin{aligned} E_x[e^{-\alpha\hat{T}_0}, I_{\hat{T}_0} \in dz] &= E_x[e^{-\alpha\hat{T}_0}, I_{\hat{T}_0} \in dz, \tau_\lambda^+ < \tau_0^-] + E_x[e^{-\alpha\hat{T}_0}, I_{\hat{T}_0} \in dz, \tau_\lambda^+ \geq \tau_0^-] \\ &= E_x[e^{-\alpha\tau_\lambda^+}, Y_{\tau_\lambda^+} \in dz, \tau_\lambda^+ < \tau_0^-] \\ &\quad +E_x[e^{-\alpha\tau_0^-}, \tau_\lambda^+ \geq \tau_0^-] \times E_0[e^{-\alpha\hat{T}_0}, I_{\hat{T}_0} \in dz] \\ &= E_x[e^{-\alpha\tau_\lambda^+}, Y_{\tau_\lambda^+} \in dz, \tau_\lambda^+ < \tau_0^-] \\ &\quad +E_{-x}[e^{-\alpha\sigma_0^+}, \sigma_{-\lambda}^- \geq \sigma_0^+] \times E_0[e^{-\alpha\hat{T}_0}, I_{\hat{T}_0} \in dz] \\ &= E_x[e^{-\alpha\tau_\lambda^+}, Y_{\tau_\lambda^+} \in dz, \tau_\lambda^+ < \tau_0^-] \\ &\quad +E_{\lambda-x}[e^{-\alpha\sigma_\lambda^+}, \sigma_0^- > \sigma_\lambda^+] \times E_0[e^{-\alpha\hat{T}_0}, I_{\hat{T}_0} \in dz], \end{aligned}$$

where the second equation follows from the first equation by conditioning on $\mathcal{F}_{\tau_0^-}$ and then using the strong Markov property. The third and fourth equations follow from the definitions of $\hat{Y}$, τ_a^+,τ_a^- ,σ_a^+,σ_a^-.

Letting $a \to 0$ in (3.8) and (3.12), we find that the first term in the last equation above is equal to $\int_0^\lambda \nu(dz-y)[W^{(\alpha)}(\lambda-x)\frac{W^{(\alpha)}(y)}{W^{(\alpha)}(\lambda)} - W^{(\alpha)}(y-x)]dy$. The second term is equal to $\frac{W^{(\alpha)}(\lambda-x)}{W^{(\alpha)}(\lambda)}$ (see (8.8) of [7]) and the third term is equal to $\frac{h_\alpha(dz)}{W_+^{(\alpha)\prime}(\lambda)}$ (this follows from Theorem 4.1 of [11] by letting the $\beta, \gamma \to 0$).

Our assertion is satisfied by replacing each of the three terms in the last equation by the corresponding value indicated above and after some algebraic manipulations, which we omit.

(ii) The proof is immediate from (3.16) and (3.21). ∎

Now we turn our attention to computing $C_\alpha(x,\tau,M)$, $E_x[\exp(-\alpha\overset{*}{T}_0)]$, and $E_x[\overset{*}{T}_0]$, when $x \in [\lambda, V]$.

Let $\eta_\tau = \inf\{t \geq 0 : \overset{*}{I}_t \leq \tau\}$ and, for each $t \geq 0$,

$$\overset{*}{X}_t = \{\overset{*}{I}_t, t < \eta_\tau\}. \quad (3.23)$$

Note that, the state space of the process $\overset{*}{X}$ is the interval $(\tau, V]$, and let $\overset{*}{U}^\alpha$ be its α-potential. Starting at any $x \in [\lambda, V]$, $\eta_\tau = \overset{*}{T}_0$ almost everywhere, furthermore the sample paths of a spectrally positive Lévy process and a spectrally positive Lévy process reflected at its infimum behave the same way until they reach level τ, thus $\overset{*}{X}$ behaves the same way in both cases. It follows that, for each $x \in [\lambda, V]$,

$$C_\alpha(x,\tau,M) = \overset{*}{U}^\alpha g^*(x). \quad (3.24)$$

Denote the process $I-M$ by N, note that this process is a spectrally positive Lévy process with the Lévy exponent $\phi_M(\theta) = \phi(\theta) + \theta M$, $\theta \geq 0$. We denote its $\alpha-$scale and adjoint $\alpha-$scale functions by $W_M^{(\alpha)}$ and $Z_M^{(\alpha)}$, respectively.

Theorem 2. For $\alpha \geq 0$, $\overset{*}{U}^\alpha$ is absolutely continuous with respect to the Lebsegue measure on $(\tau, V]$, and a version of its density is given by

$$\overset{*}{u}^\alpha(x,y) = \frac{Z_M^{(\alpha)}(V-x)W_M^{(\alpha)}(y-\tau)}{Z_M^{(\alpha)}(V-\tau)} - W_M^{(\alpha)}(y-x)\ , \quad x,y \in (\tau, V]. \quad (3.25)$$

Proof. For each $t \geq 0$, we define $B_t = N_t - V$. For any $b \in R$, we define $\sigma_b^- = \inf\{t \geq 0 : B_t - \overline{B}_t < b\}$ and $\gamma_b^+ = \inf\{t \geq 0 : \hat{B}_t - \underline{\hat{B}_t} > b\}$. For any Borel set $A \subseteq (\tau, V]$ and $x \in (\tau, V]$ we have

$$\begin{aligned}
P_x\{\overset{*}{X}_t \in A\} &= P_x\{\overset{*}{I}_t \in A, t < \eta_\tau\} \\
&= P_x\{N_t - \sup_{s \leq t}((N_s - V) \vee 0) \in A, t < \eta_\tau\} \\
&= P_{x-V}\{B_t - \overline{B}_t \in A - V, t < \sigma_{\tau - V}^-\} \\
&= P_{V-x}\{\hat{B}_t - \underline{\hat{B}_t} \in V - A, t < \gamma_{V-\tau}^+\}
\end{aligned}$$

Using Theorem 1 (i) of [10], the result follows. ■

The following theorem gives Laplace transform of the distribution of the stopping time $\overset{*}{T}_0$ and $E_x[\overset{*}{T}_0]$ when $x \in [\lambda, V]$.

Theorem 3. (i) Let $x \in [\lambda, V]$ and $\alpha \in R_+$, then

$$E_x[e^{-\alpha \overset{*}{T}_0}] = \frac{Z_M^{(\alpha)}(V-x)}{Z_M^{(\alpha)}(V-\tau)}. \quad (3.26)$$

(ii) For $x \in [\lambda, V]$

$$E_x[\overset{*}{T}_0] = \overline{W}_M(V-\tau) - \overline{W}_M(V-x), \quad (3.27)$$

where, $\overline{W}_M(x) = \int_0^x W_M(y)dy$.

Proof. We only prove (i), the proof of (ii) follows easily from (i) and is omitted. For $x \in [\lambda, V]$, we have

$$\begin{aligned}
E_x[e^{-\alpha \overset{*}{T}_0}] &= 1 - \alpha \overset{*}{U}^{\alpha} \mathbf{I}_{(\tau,V]}(x) \\
&= 1 - \alpha \int_\tau^V \overset{*}{u}^{\alpha}(x, dy) \\
&= 1 - \alpha \int_\tau^V [\frac{Z_M^{(\alpha)}(V-x) W^\alpha(y-\tau)}{Z_M^{(\alpha)}(V-\tau)} - W_M^{(\alpha)}(y-x)]dy \\
&= 1 - \alpha[\frac{Z_M^{(\alpha)}(V-x)}{Z_M^{(\alpha)}(V-\tau)}\{\frac{Z^\alpha(V-\tau)-1}{\alpha}\} - \{\frac{Z_M^{(\alpha)}(V-x)-1}{\alpha}\}] \\
&= \frac{Z_M^{(\alpha)}(V-x)}{Z_M^{(\alpha)}(V-\tau)} - Z_M^{(\alpha)}(V-x) + Z_M^{(\alpha)}(V-x) \\
&= \frac{Z_M^{(\alpha)}(V-x)}{Z_M^{(\alpha)}(V-\tau)},
\end{aligned}$$

where the third equation follows from (3.25), the fourth equation follows from the definition of the function $Z_M^{(\alpha)}$ and the fifth equation follows the fourth equation after obvious manipulations. ■

Remark 1. When $V = \infty$, for $\alpha \geq 0$ we let $\eta_M(\alpha) = \sup\{\theta : \phi(\theta) - \theta M = \alpha\}$. Since $Z_M^{(\alpha)}(y) = O(e^{\eta_M(\alpha) y})$ as $y \to \infty$, then we have

$$\begin{aligned} E_x[\overset{*}{T}_0] &= \frac{(x-\tau)}{\eta_M(\alpha)'(0)} \\ &= \frac{(x-\tau)}{M - E(I_1)} \text{ , if } M > E(I_1) \\ &= \infty \qquad\qquad \text{ , if } M \leq E(I_1). \end{aligned}$$

This is consistent with the well known fact about the busy period of the M/G/1 queuing system.

The following gives $E_x[\exp(-\alpha \overset{*}{T}_0)]$, when $x < \lambda$, a result that is needed to compute the total discounted cost.

Theorem 4. Assume that the input process is a spectrally positive Lévy process. For $z > \lambda$, we define

$$h_\alpha(x, dz) = \int_{-\infty}^{\lambda} u^\alpha(x, y) v(dz - y) dy,$$

where $u^\alpha(x, y)$ is defined in (3.9).

Then, for $\alpha \geq 0, x < \lambda$

$$E_x[e^{-\alpha \overset{*}{T}_0}] = \frac{1}{Z_M^{(\alpha)}(V-\tau)} [\int_\lambda^V Z_M^{(\alpha)}(V-z) h_\alpha(x, dz) + \int_V^\infty h_\alpha(x, dz)] \tag{3.28}$$

Proof. We write

$$\begin{aligned} E_x[e^{-\alpha \overset{*}{T}_0}] &= E_x[e^{-\alpha \hat{T}_0 - \alpha(\overset{*}{T}_0 - \hat{T}_0)}] \\ &= E_x[E_x[e^{-\alpha \hat{T}_0 - \alpha(\overset{*}{T}_0 - \hat{T}_0)} \mid \sigma(\hat{T}_0, I_{\hat{T}_0})]] \\ &= E_x[e^{-\alpha \hat{T}_0} \quad E_{(I_{\hat{T}_0} \wedge V)}[e^{-\alpha \overset{*}{T}_0}]] \\ &= \frac{1}{Z_M^{(\alpha)}(V-\tau)} E_x[e^{-\alpha \hat{T}_0} \quad Z_M^{(\alpha)}(V - (I_{\hat{T}_0} \wedge V))] \\ &= \frac{1}{Z_M^{(\alpha)}(V-\tau)} [\int_\lambda^V Z_M^{(\alpha)}(V-z) h_\alpha(x, dz) + \int_V^\infty h_\alpha(x, dz)], \end{aligned}$$

where the third equation follows since, given $\hat{T}_0$ and $I_{\hat{T}_0}$, $\overset{*}{T}_0 - \hat{T}_0$ is equal to $\overset{*}{T}_0$ almost everywhere. The fourth equation follows from (3.26). The last equation follows from (3.13), the fact that $Z_M^{(\alpha)}(0) = 1$, and the definition of $h_\alpha(x, dz)$. ■

The following theorem gives a result analogous to (3.28) when the input process is a spectrally positive Lévy process reflected at its infimum.

Theorem 5. Assume that the input process is a spectrally positive Lévy process reflected at its infimum. For $z \geq \lambda$, let $l_\alpha(dz)$, $L_\alpha(z)$, and $V_\alpha(\lambda)$ be as defined in (3.18), (3.19) and (3.20), respectively. Define

$$g_\alpha(x, dz) = \begin{cases} = \frac{l_\alpha(dz)}{W_+^{(\alpha)'}(\lambda)}, \ z > \lambda \\ = \frac{V_\alpha(\lambda) - L_\alpha(\lambda)}{W_+^{(\alpha)'}(\lambda)} \delta_\lambda(dz). \end{cases}$$

Then, for $\alpha \geq 0, x < \lambda$

$$E_x[e^{-\alpha \overset{*}{T}_0}] = \frac{1}{Z_M^{(\alpha)}(\lambda - \tau)} [\int_\lambda^V Z_M^{(\alpha)}(\lambda - z) g_\alpha(x, dz) + \int_V^\infty g_\alpha(x, dz)]. \quad (3.29)$$

Proof. The proof follows in a manner similar to the proof of (3.28), using (3.21), (3.22) and (3.26). ■

4. The expected total discounted and long-run average costs

Consider a finite dam controlled by a $P_{\lambda,\tau}^M$ policy as described in Section 1. Assume that the input process, I, is spectrally positive Lévy, and define α to be the discount factor. For $x \in [\tau, V]$, we let $C_x^\alpha(\lambda, \tau)$, and $C(\lambda, \tau)$ be the expected total discounted cost and long-run average cost, respectively, given $I_0 = 0$. Furthermore, we define $C^\alpha(x)$ as the expected discounted cost during the interval $[0, \overset{*}{T}_0)$, given the initial water content is equal to x.

Modifying (3.1) of [1], it follows that for $x \in [\tau, V]$,

$$C_x^\alpha(\lambda, \tau) = C^\alpha(x) + \frac{E_x[\exp(-\alpha \overset{*}{T}_0)] C^\alpha(\tau)}{1 - E_\tau[\exp(-\alpha \overset{*}{T}_0)]}. \ (4.1)$$

From the definition of the $P_{\lambda,\tau}^M$ policy, it follows that for $\lambda < x < V$

$$C^{\alpha}(x) = M\{K_1 - RE_x \int_0^{\overset{*}{T}_0} e^{-\alpha t}dt\} + C_{\alpha}(x,\tau,M), \quad (4.2)$$

and for $x \in [\tau, \lambda]$

$$\begin{aligned} C_{\alpha}(x) \quad &= \quad M\{K_2 + K_1 \, E_x[e^{-\alpha \hat{T}_0}] - \frac{R}{\alpha}\{E_x[e^{-\alpha \hat{T}_0}] - E_x[e^{-\alpha \overset{*}{T}_0}]\} \\ &+ C_{\alpha}(x,\lambda,0) + E_x[e^{-\alpha \hat{T}_0} C_{\alpha}((I_{\hat{T}_0} \wedge V), \tau, M)], \qquad (4.3) \end{aligned}$$

where $C_{\alpha}(x,\lambda,0)$, and $C_{\alpha}(x,\tau,M)$ are given in (3.6) and (3.24), respectively. Using (3.9), (3.10), (3.13), (3.25), (3.26) and (3.28) we obtain $C^{\alpha}(x)$. Finally, the expected total discounted cost can be determined explicitly by substituting (4.2), (4.3), (3.26) and (3.28) into (4.1).

To determine the long-run average cost using a given $P^{M}_{\lambda,\tau}$ policy, we proceed as follows. Let $C(\lambda,\tau)$ denote the long-run average cost, and define $C^0(x)$ as the expected non-discounted cost during the interval $[0, \overset{*}{T}_0)$, given the initial water content is equal to x, $x \in [\tau, V]$. It follows that

$$C(\lambda,\tau) = \frac{C^0(\tau)}{E_{\tau}[\overset{*}{T}_0]}. \ (4.4)$$

From the strong Markov property we have

$$E_{\tau}[\overset{*}{T}_0] = E_{\tau}[\hat{T}_0] + E_{\tau}[E_{(I_{\hat{T}_0} \wedge V)}[\overset{*}{T}_0]]. \quad (4.5)$$

Furthermore

$$C^0(\tau) = M\{K - R(E_{\tau}[\overset{*}{T}_0] - E_{\tau}[\hat{T}_0])\} + C_0(\tau,\lambda,0) + E_{\tau}[C_0((I_{\hat{T}_{\lambda}} \wedge V), \tau, M)], \quad (4.6)$$

where $K = K_1 + K_2$. Letting $\alpha = 0$ in (3.13) and substituting the result, along with (3.11) and (3.27) into (4.5) we obtain $E_{\tau}[\overset{*}{T}_0]$. Using (3.6), (3.9), (3.11), (3.13), (3.24), (3.25) and (4.5) we obtain (4.6). Substituting (4.5) and (4.6) into (4.4) the long-run average cost is determined.

The corresponding results for the spectrally positive Lévy reflected at its infimum input follow similarly.

5. Special Cases

In this section we consider the cases where the input process is a spectrally positive Lévy of bounded variation, Brownian motion reflected at its infimum and Wiener process. For the first case, we extend the results of [6], we also simplify some of their results. For the second case, we obtain results similar to those of [4] and [8]. In the third case we obtain the results of [12].

Case 1. Assume that the input is a spectrally positive Lévy process of bounded variation with Lévy exponent described in (2.3), reflected at its infimum. Let $\mu = \int_0^\infty x\upsilon(dx)$ and assume that $\mu < \infty$. For every $x \in R_+$, we define the probability density function $f(x) = \frac{\upsilon([x,\infty))}{\mu}$. We have $\int_0^\infty (1-e^{-\theta x})\upsilon(dx) = \theta\mu \int_0^\infty\ e^{-\theta x} f(x)dx$. Define $\rho = \frac{\mu}{\varsigma}$, $F(x)$ as the distribution function corresponding to f . Assume that $\rho < 1$, it follows that, $\frac{1}{\phi(\theta)} = \frac{1}{\varsigma}\int_0^\infty e^{-\theta x}dx \sum\limits_{n=0}^{\infty} \rho^n F^{(n)}(x)$. Therefore, the 0−scale function is given is given as follows

$$W(x) = \frac{1}{\varsigma}\sum_{n=0}^{\infty} \rho^n F^{(n)}(x). \quad (5.1)$$

For $\alpha > 0, W^{(\alpha)}$ is computed using (2.5) and (5.1).

Define $\overset{*}{\varsigma} = \varsigma + M$, and $\overset{*}{\rho} = \frac{\mu}{\overset{*}{\varsigma}}$. Let $W_M^{(\alpha)}$ be as defined in the paragrah proceeding Theorem 2, and denote $W_M^{(0)}$ by W_M , using an argument similar to the one above we have

$$W_M(x) = \frac{1}{\overset{*}{\varsigma}}\sum_{n=0}^{\infty} \overset{*}{\rho}{}^n F^{(n)}(x). \quad (5.2)$$

It follows that, for a spectrally positive Lévy process of bounded variation, $W(0) = 0$. Thus, the α-potential $\overset{(2)}{U}{}^\alpha$ (given in (3.14) is absolutely continuous. From (4.4) (4.5), and (4.6), the long-run average cost is determined once for $x \in [\lambda, V]$, $E_\tau[\hat{T}_0]$, $E_x[\overset{*}{T}_0]$, $\overset{(2)}{U}{}^0$, $\overset{*}{u}{}^0$, and the didtribution of $I_{\hat{T}_0}$ are computed. Using (5.1) and (3.17) we compute $E_\tau[\hat{T}_0]$, and using (5.2) and (3.27) $E_x[\overset{*}{T}_0]$ is determined for $x \in [\lambda, V]$. Furthermore, $\overset{(2)}{U}{}^0$ is computed using (3.14) and (5.1). From (3.25) it follows that, for $x, y \in (\tau < V]$, $u^0(x, y) = W_M(y-\tau) - W_M(y-x)$, which is determined using (5.2). The distribution of $I_{\hat{T}_0}$ is given by letting $\alpha \to 0$ in (3.21) and (3.22), and using (3.18), (3.19), (3.20), and (5.1).

The corresponding results for the total discounted cost follow similarly.

Remark 2.

(a) Bae *et al* [6] obtain the long-run average cost, when the input process is a compound Poisson process with a negative drift. In this case, $\upsilon(dx) = \lambda G(dx)$, where $\lambda > 0$ and G is a distribution function of a positive random variable $[0,\infty)$, describing the size of each jump of the compound Poisson process. In this case, $f(x) = \frac{\bar{G}(x)}{m}$ and $\rho = \frac{\lambda m}{\varsigma}$, where $\bar{G} = 1-G$ and $m = \int_0^\infty \bar{G}(x)dx$, which is assumed to be finite. We note that their entities $w(x)$ and $E[L^a(\lambda,\tau)]$ given in p.521 and p.523 of [6], respectively, are nothing but our $C_0(x,\lambda,0)$ and $E_\tau[\hat{T}_0]$, respectively. Furthermore, for $x \in [\tau, V]$, their functions $u(x)$ and $E[T_\tau^M\ (x)]$ given in page 524 are our $C_0(x,\tau,M)$ and $E_x[\overset{*}{T}_0]$, respectively. The distribution of $L(\tau)$ (the overshoot) given on page 525 follows in an obvious manner from the distribution of $I_{\hat{T}_0}$. The formulas for computing $C_0(x,\lambda,0)$, $E_\tau[\hat{T}_0]$, $C_0(x,\tau,M)$, $E_x[\overset{*}{T}_0]$, and the distribution of $I_{\hat{T}_0}$, follow from the corresponding results obtained in Case 1. We note that our formulas for computing $w(x)$ and $E[L^a(\lambda,\tau)]$ are identical to those of [6], while our formulas for $u(x)$, $E[T_\tau^M\ (x)]$, and the distribution of $L(\tau)$ are simpler than theirs.

(b) Assume that the input process is a gamma process with negative drift. The Lévy measure is given by $\upsilon(dx) = a\frac{e^{-bx}}{x}dx$, $a, b > 0$. In this case, $E(I_1) = \varsigma - \frac{a}{b}$, which is assumed to be nonnegative and $\rho = \frac{a}{\varsigma b} < 1$. It follows that $f(x) = b\int_x^\infty \frac{e^{-by}}{y}dy$, the right hand side is denoted by $E_1(x)$ in p. 227 of [3], Direct integrations yield, $F(x) = (1-e^{-bx}) + xf(x)$.

(c) Assume that the input process is an inverse Gaussian process with a negative drift, and with Lévy measure is given by $\upsilon(dx) = \frac{1}{\sigma\sqrt{2\pi x^3}}e^{-xc^2/2\sigma^2}$, $\sigma, c > 0$. It follows that $E(I_1) = \varsigma - \frac{1}{c}$, which is assumed to be greater than zero. In this case $\rho = \frac{1}{c\varsigma} < 1$, $f(x) = c\int_x^\infty \upsilon(dy)$, and $F(x) = \operatorname{erf}(c\sqrt{y/2\sigma^2}) + xf(x)$.

(d) Since, for all $n \geq 0$, $F^{(n)}(x) \leq [F(x)]^n$, then for all x, $W(x) \leq \frac{1}{\varsigma - \mu F(x)}$, if $\rho < 1$.

Case 2. Assume that the input process is a Brownian motion with drift term $\mu \in R$, variance term σ^2, reflected at its infimum. From (3.6), (3.24), and (4.1)-(4.6), the total discounted and long-run average costs are determined once $E_x[e^{-\alpha\hat{T}_0}]$, $E_\lambda[e^{-\alpha\overset{*}{T}_0}]$, $E_\tau[\hat{T}_0]$, $E_\lambda[\overset{*}{T}_0]$, $\overset{(2)}{U^\alpha}$, and $\overset{*}{u}{}^\alpha$ are computed.

In this case, $I_{\hat{T}_0} = \lambda$ almost everywhere, the Lévy measure $\nu = 0$, and from (2.1) we have, for $\theta \geq 0$, $\phi(\theta) = -\mu\theta + \frac{\theta^2\sigma^2}{2}$. It follows that, for $\alpha \geq 0$, $\eta(\alpha) = \frac{\sqrt{2\alpha\sigma^2+\mu^2}+\mu}{\sigma^2}$. Let $\delta = \sqrt{2\alpha\sigma^2+\mu^2}$, we have, $W^{(\alpha)}(x) = \frac{2}{\delta}e^{\mu x/\sigma^2}\sinh(\frac{x\delta}{\sigma^2})$ and $Z^{(\alpha)}(x) = e^{\mu x/\sigma^2}\left(\cosh(\frac{x\delta}{\sigma^2}) - \frac{\mu}{\delta}\sinh(\frac{x\delta}{\sigma^2})\right)$. We note that

$W^{\alpha}(x)$ is differentiable, and $W^{(\alpha)'}(x) = \frac{\mu}{\sigma^2}W^{(\alpha)}(x) + \frac{2}{\sigma^2}e^{\mu x/\sigma^2}\cosh(\frac{x\delta}{\sigma^2})$; hence $\frac{W^{(\alpha)}(\lambda)}{W^{(\alpha)'}(\lambda)} = \left(\frac{\sigma^2}{\mu+\delta\coth(\frac{\lambda\delta}{\sigma^2})}\right)$. Substituting the values of $Z^{(\alpha)}(\lambda - x), W^{(\alpha)}(\lambda - x)$ and $\frac{W^{(\alpha)}(\lambda)}{W^{(\alpha)'}(\lambda)}$ in (3.16), we have, for $\alpha \geq 0, x \leq \lambda$

$$E_x[e^{-\alpha\hat{T}_0}] = e^{\mu(\lambda-x)}\left[\cosh\left(\frac{(\lambda-x)\delta}{\sigma^2}\right) - \frac{1}{\delta}\sinh\left(\frac{(\lambda-x)\delta}{\sigma^2}\right)\left(\mu + \frac{2\alpha\sigma^2}{\mu+\delta\coth(\frac{\lambda\delta}{\sigma^2})}\right)\right], \quad (5.3)$$

a simpler and more explicit formula than (4.6) of [4].

We note that, $W^{(\alpha)}(0) = 0$, $\overset{(2)}{U}{}^{\alpha}$ is absolutely continuous with respect to the Lebesgue measure on $[0, \lambda)$. Substituting the values of $W^{(\alpha)}(\lambda - x)$, $W^{\alpha'}(\lambda)$, $W^{(\alpha)}(y - x)$, and $W^{\alpha'}(y)$ in (3.14) we get a version of the the density of $\overset{(2)}{U}{}^{\alpha}$.

Let $\overset{*}{\mu} = \mu - M, \overset{*}{\delta} = \sqrt{2\alpha\sigma^2 + \overset{*}{\mu}{}^2}$, it follows that $W_M^{(\alpha)}(x) = \frac{2}{\delta}e^{\overset{*}{\mu}x/\sigma^2}\sinh(\frac{x\overset{*}{\delta}}{\sigma^2})$, and $Z_M^{(\alpha)}(x) = e^{\overset{*}{\mu}x/\sigma^2}\left(\cosh(\frac{x\overset{*}{\delta}}{\sigma^2}) - \frac{\mu}{\delta}\sinh(\frac{x\overset{*}{\delta}}{\sigma^2})\right)$. Hence, $\overset{*}{u}{}^{\alpha}$ is computed using (3.25). Since $I_{\hat{T}_0} = \lambda$ almost everywhere,

$$E_x[e^{-\alpha\overset{*}{T}_0}] = E_x[e^{-\alpha\hat{T}_0}]E_\lambda[e^{-\alpha\overset{*}{T}_0}], \quad (5.4)$$

where $E_x[e^{-\alpha\hat{T}_0}]$ is given in (5.3) and $E_\lambda[e^{-\alpha\overset{*}{T}_0}] = \frac{Z_M^{(\alpha)}(V-\lambda)}{Z_M^{(\alpha)}(V-\tau)}$, which follows from (3.26).

To compute $E_\tau[\hat{T}_0]$, $E_\lambda[\overset{*}{T}_0]$, we first assume that $\mu\#0$. In this case, for $x \geq 0$, $W(x) = \frac{e^{2\mu x/\sigma^2}-1}{\mu}, W'(x) = \frac{2e^{2\mu x/\sigma^2}}{\sigma^2}$ and $\bar{W}(x) = \frac{\sigma^2}{2\mu^2}(e^{2\mu x/\sigma^2} - 1) - \frac{x}{\mu}$. Substituting the values of $W(\lambda - x)$, $\bar{W}(\lambda - x), W(\lambda)$ and $W'(\lambda)$ in (3.17) we have,

$$E_\tau[\hat{T}_0] = \frac{\lambda - \tau}{\mu} + \frac{\sigma^2}{2\mu^2}\left[e^{-2\mu\lambda/\sigma^2} - e^{-2\mu\tau/\sigma^2}\right]. \quad (5.5)$$

Since, $I_{\hat{T}_0} = \lambda$ almost everywhere, from (4.5) we have

$$E_\tau[\overset{*}{T}_0] = E_\tau[\hat{T}_0] + E_\lambda[\overset{*}{T}_0]. \quad (5.6)$$

We note that $\bar{W}_M$ is computed from $\bar{W}$ above by replacing the term μ by $\overset{*}{\mu}$ defined in the preceding paragraph. Let $\lambda^* = V - \lambda$, and $\tau^* = V - \tau$, substituting the values $\bar{W}_M(V - \tau)$ and $\bar{W}_M(V - \lambda)$ in (3.27) we have

$$E_\lambda[\overset{*}{T}_0] = \frac{\lambda^* - \tau^*}{\mu^*} + \frac{\sigma^2}{2\overset{*}{\mu}^2}\left[e^{2\overset{*}{\mu}\tau^*/\sigma^2} - e^{2\mu\lambda^*/\sigma^2}\right]. \quad (5.7)$$

If $\mu = 0$, then $W(x) = \frac{2x}{\sigma^2}$ and $\overline{W}(x) = \frac{x^2}{\sigma^2}$. From (3.17) it follows that

$$E_\tau[\hat{T}_0] = \frac{\lambda^2 - x^2}{\sigma^2}. \quad (5.8)$$

It is easily shown that, $W_M(x) = \frac{1}{M}(\ 1 - e^{-2xM/\sigma^2}\ )$, using (3.27) we have

$$E_\lambda[\overset{*}{T}_0] = \frac{\tau^* - \lambda^*}{M} + \frac{\sigma^2}{2M^2}\left[e^{-2M\tau^*/\sigma^2} - e^{-2M\lambda^*/\sigma^2}\right]. \quad (5.9)$$

We note that our (5.5) is consistent with (4.9) of [4], while (5.8) is identical to the corresponding equation given in p. 298 of the same reference.

Case 3. Assume that the input process is a Brownian motion with drift term $\mu > 0$ and variance parameter σ^2. It follows that $\eta(0) = \frac{2\mu}{\sigma^2}$. Substituting the values of $W^{(\alpha)}(x), Z^{(\alpha)}(x)$, given in Case 2, in (3.10) we have, for $x \le \lambda$, $E_x[e^{-\alpha\hat{T}_0}] = \exp((\delta - \mu)(x - \lambda))$. Substituting $\frac{1}{\mu}(e^{2\mu x/\sigma^2} - 1)$ and $\frac{\sigma^2}{2\mu^2}(e^{2\mu x/\sigma^2} - 1) - \frac{x}{\mu}$ for $W(x)$ and $\overline{W}(x)$, respectively, in (3.11), we have, for $x \le \lambda$, $E_x[\hat{T}_0] = \frac{\lambda - x}{\mu}$. These results are consistent with the results of Zuckerman [12], p.423. The values of $E_\lambda[e^{-\alpha\overset{*}{T}_0}]$ and $E_\lambda[\overset{*}{T}_0]$ are the same whether the input process is spectrally positive Lévy or spectrally positive Lévy reflected at its infimum. The computations of the total discounted and long-run average costs can be obtained using (3.9), (3.13), (3.25), (4.1)-(4.6) and the values of $E_x[e^{-\alpha\hat{T}_0}]$, $E_\lambda[e^{-\alpha\overset{*}{T}_0}]$, $E_x[\hat{T}_0]$, $E_\lambda[\overset{*}{T}_0]$, in manners similar to those discussed in Case 2.

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