THE MAXIMUM LIKELIHOOD DRIFT ESTIMATOR FOR MIXED FRACTIONAL BROWNIAN MOTION

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Abstract. The paper is concerned with the maximum likelihood estimator (MLE) of the unknown drift parameter $\theta \in \mathbb{R}$ in the continuous-time regression model

$$X_t = \theta t + B_t + B_t^H, \quad t \in [0, T]$$

where B_t is a Brownian motion and B_t^H is an independent fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. We derive the exact formula for the MLE in terms of the solution of an integral equation and find the asymptotic distribution of the estimation error. In particular, it turns out that the Brownian part does not contribute to the asymptotic variance of the MLE.

Another contribution of this paper is a formula for the Radon–Nikodym derivative of the probability, induced by the mixed fractional Brownian motion $\xi_t = B_t + B_t^H$, $H > 3/4$ with respect to the Wiener measure.

1. Introduction and the main result

Consider the continuous-time regression model

$$X_t = \theta t + \sigma B_t + B_t^H, \quad t \in [0, T], \tag{1.1}$$

where B_t is a Brownian motion and B_t^H is an independent fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$, *i.e.*, zero mean Gaussian process with the covariance function

$$\mathbb{E} B_t^H B_s^H = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big), \quad s, t \in [0, T].$$

As is well known, for $H \in (\frac{1}{2}, 1)$ the process B_t^H exhibits the long-range dependence property

$$\sum_{j=1}^{\infty} \mathbb{E} B_1^H \big(B_{j+1}^H - B_j^H\big) = \infty,$$

and hence $\xi_t := \sigma B_t + B_t^H$, called in [4] the *mixed* fractional Brownian motion (fBm), can be thought of as observation noise with both "white" and heavily correlated components. The mixed fBm has a number of peculiar probabilistic properties,

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studied in *e.g.* [4], [2], [17], which have some relevance to mathematical finance (see *e.g.* [3]).

The constant $\sigma > 0$, controlling the intensity of the Brownian part, and the Hurst parameter H can be reconstructed precisely from the trajectory $X^T := \{X_t, t \in [0,T]\}$ (see, *e.g.*, [1]) and hence are assumed to be known.

Given the sample path X^T, it is required to estimate the unknown drift parameter $\theta \in \mathbb{R}$. The parameter estimation problems in models with mixed fBm have been considered in the recent monographs [8] and [12], where the construction of the maximum likelihood estimator (MLE) of θ appears as an open problem (see Remark (iii) page 181 in [12] and the discussion on page 354 in [8]). Our main result aims at filling this gap:

Theorem 1.1. *The MLE of θ is given by*

$$\hat{\theta}_T = \frac{\int_0^T g(t,T)dX_t}{\int_0^T g(t,T)dt}, \tag{1.2}$$

where[1] *the function $g(t,T)$, $t \in [0,T]$ is the unique $L^2[0,T]$ solution of the integral equation*

$$\sigma^2 g(t,T) + H(2H-1)\int_0^T g(s,T)|s-t|^{2H-2}ds = \sigma, \quad \text{for a.a. } t \in [0,T]. \tag{1.3}$$

This estimator is strongly consistent and the corresponding estimation error is normal

$$\hat{\theta}_T - \theta \sim N\left(0, \frac{\sigma}{\int_0^T g(t,T)dt}\right), \tag{1.4}$$

with the asymptotic behavior of the variance

$$\lim_{T\to\infty} T^{2-2H}\mathbb{E}_\theta(\hat{\theta}_T - \theta)^2 = \lambda_H := \frac{2H\Gamma(H+\frac{1}{2})\Gamma(3-2H)}{\Gamma(\frac{3}{2}-H)}, \tag{1.5}$$

where $\Gamma(x)$ is the standard Gamma function.

Remark 1.2.

1. The asymptotic variance in (1.5) is independent of σ and coincides with the asymptotic variance of the MLE in the problem with $\sigma = 0$, *i.e.*, estimating the drift of fBm without additional Brownian component (see Section 5.1 in [6]). This means that the Brownian part is asymptotically negligible.
2. Actually, it will appear (see Remark 2.3 below) that for some constant κ_H the following limits hold:

$$\lim_{\sigma\to 0} \frac{g(t,T)}{\sigma} = k_H(T,t) := \kappa_H^{-1} t^{\frac{1}{2}-H}(T-t)^{\frac{1}{2}-H}, \tag{1.6}$$

[1] the stochastic integral in the numerator is defined through the usual limit procedure, recalled in Subsection 2.2

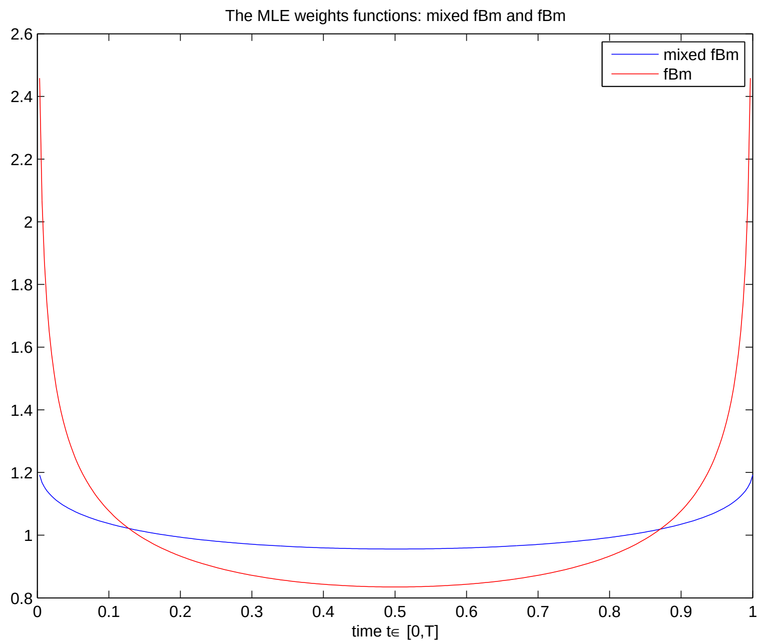

FIGURE 1. The MLE weight function for mixed fBM versus fBm ($\sigma = 1$, $T = 1$, $H = 3/4$)

$$\lim_{\sigma \to 0} \int_0^T \frac{g(t,T)}{\sigma} \, dt = \int_0^T k_H(T,t) \, dt = \lambda_H^{-1} T^{2-2H}. \tag{1.7}$$

Hence, the limiting form of (1.2) for σ tending to 0 is

$$\hat{\theta}_T = \frac{\lambda_H}{T^{2-2H}} \int_0^T k_H(T,t) dX_t, \tag{1.8}$$

which is nothing else but the expression obtained in [6] for the MLE in the model with $\sigma = 0$. It is easy to check that this estimator is applicable to the data X, generated by the model with any $\sigma > 0$ and its asymptotic variance coincides with (1.5). In other words, the estimator (1.8) has the same asymptotic accuracy as the genuine MLE.

Remark 1.3.

(1) The proof of Theorem 1.1 suggests an approximation procedure for the function $g(t,T)$ (see (2.5) and (2.6)). Its typical form, depicted in Figure 1 versus the weight function from the estimator (1.8), indicates significant difference in the non-asymptotic regime.

(2) The integral equation (1.3) is known as the second type Fredholm equation with weakly singular kernel (see [11]) or as the Wiener-Hopf equation on the finite interval. Its solution can be reduced to a particular instance of

the Riemann boundary value problem, which unfortunately doesn't seem to be helpful in our case. It is well known, however, that (1.3) has a unique continuous solution, which enjoys some regularity properties (see, *e.g.*, [16]).

Another interesting outcome of our approach is a formula for the Radon–Nikodym derivative of the probability, induced by the mixed fractional Brownian motion with respect to the Wiener measure. It is shown in [4] (see also [2]) that for $H > 3/4$ these measures are mutually absolutely continuous, however no expression for the corresponding derivative is given. The derivative is calculated in terms of reproducing kernels in [17], but the author points out that it might be hard to obtain more explicit expression (see remark (iii) on page 63). The following proposition gives a representation formula in terms of the solution of (1.3):

Proposition 1.4. *For $H > 3/4$, the probability μ^ξ, induced by $\xi_t := B_t + B_t^H$, $t \in [0,T]$ is absolutely continuous with respect to the Wiener measure μ^W and*

$$\frac{d\mu^\xi}{d\mu^W}(\xi) = \exp\left\{-\int_0^T \varphi_s(\xi)d\xi_s - \frac{1}{2}\int_0^T \varphi_s^2(\xi)ds\right\}$$

with $\varphi_s(\xi) = \int_0^s \frac{\dot g(r,s)}{g(s,s)}d\xi_r$ and $\dot g(r,t) = \frac{\partial}{\partial t}g(r,t)$ where $g = \big(g(s,t),\, 0 \le s \le t \le T\big)$ satisfies the equation

$$g(s,t) + H(2H-1)\int_0^t g(r,t)|r-s|^{2H-2}dr = 1, \quad s \in [0,t]. \tag{1.9}$$

2. Proof of Theorem 1.1

2.1. **The likelihood function and the MLE.** Let $\widetilde B = (\widetilde B_t)_{t\in[0,T]}$ and $B = (B_t^H)_{t\in[0,T]}$ be processes defined on a measurable space $(\Omega, \mathcal{F})$ and $\mathbb{P}_\theta$ be a probability, under which $\widetilde B$ and B^H are independent, B^H is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $\widetilde B$ is a Brownian motion with drift $\frac{\theta}{\sigma}$, *i.e.*,

$$\sigma \widetilde B_t = \theta t + \sigma B_t, \quad t \in [0,T].$$

Under $\mathbb{P}_\theta$, the process $X = \sigma\widetilde B + B^H$ is the mixed fBm with drift θ as defined in (1.1). By Girsanov's theorem and independence of $\widetilde B$ and B^H

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0} = \exp\left(\frac{\theta}{\sigma}\widetilde B_T - \frac{1}{2}\frac{\theta^2}{\sigma^2}T\right).$$

In order to eliminate some confusion, existing in the literature (see, *e.g.*, [14]), let us stress that this derivative is *not* the likelihood for the problem at hand, since ,*e.g.*, it is not measurable with respect to the observed σ-algebra $\mathcal{F}_T^X = \sigma\{X_t, t\in[0,T]\}$. The proper likelihood function is obtained by conditioning on $\mathcal{F}_T^X$.

More precisely, let μ_θ be the probability induced by X on the space of continuous functions with the usual supremum topology under probability $\mathbb{P}_\theta$. Then for a measurable set A,

$$\mu_\theta(A) = \mathbb{P}_\theta(X \in A) = \mathbb{E}_0 \frac{d\mathbb{P}_\theta}{d\mathbb{P}_0} \mathbf{1}_{\{X \in A\}} = \mathbb{E}_0 \mathbb{E}_0 \Big(\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0} \big| \mathcal{F}_T^X \Big) \mathbf{1}_{\{X \in A\}} = \int_A \Phi(x) \mu_0(dx),$$

where $\Phi(x)$ is a measurable functional, such that

$$\Phi(X) = \mathbb{E}_0 \Big(\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0} \big| \mathcal{F}_T^X \Big), \quad \mathbb{P}_0 - a.s.$$

The latter means that $\mu_\theta \ll \mu_0$ for any $\theta \in \mathbb{R}$ and, since $\widetilde{B} = B$ under $\mathbb{P}_0$, the corresponding likelihood function is given by

$$L_T(X;\theta) = \mathbb{E}_0 \Big(\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0} \big| \mathcal{F}_T^X \Big) = \mathbb{E}_0 \left(\exp \left(\frac{\theta}{\sigma} B_T - \frac{1}{2} \frac{\theta^2}{\sigma^2} T \right) \Big| \mathcal{F}_T^X \right) = \exp \left(\frac{\theta}{\sigma} M_T + \frac{1}{2} \frac{\theta^2}{\sigma^2} \big(V_T - T \big) \right).$$

The latter equality holds with $M_T := \mathbb{E}_0(B_T | \mathcal{F}_T^X)$ and $V_T = \mathbb{E}_0 \big(B_T - M_T \big)^2$, since the process (B, X) is Gaussian and hence the conditional distribution of B_T given $\mathcal{F}_T^X$ is Gaussian as well.

Let $(\mathcal{F}_t)$ and $(\mathcal{F}_t^X)$ be the natural filtrations of (B, B^H) and X respectively and set

$$M_t = \mathbb{E}_0 \big(B_t | \mathcal{F}_t^X \big), \quad t \in [0, T]. \tag{2.1}$$

Since B is an $(\mathcal{F}_t)$-martingale and $\mathcal{F}_t^X \subset \mathcal{F}_t$, the process M is an $(\mathcal{F}_t^X)$-martingale with respect to $\mathbb{P}_0$. Moreover, since $V_t = \mathbb{E}_0(B_t^2 | \mathcal{F}_t^X) - M_t^2$ and $B_t^2 - t$ is an $(\mathcal{F}_t)$-martingale, for $s \le t$,

$$\mathbb{E}_0 \big(M_t^2 - (t - V_t) | \mathcal{F}_s^X \big) = \mathbb{E}_0 \big(\mathbb{E}_0(B_t^2 | \mathcal{F}_t^X) - t | \mathcal{F}_s^X \big) = \mathbb{E}_0 (B_t^2 - t | \mathcal{F}_s^X) = \mathbb{E}_0(B_s^2 | \mathcal{F}_s^X) - s = M_s^2 - (s - V_s),$$

i.e., the quadratic variation process of the martingale M is $\langle M \rangle_t = t - V_t$, and the likelihood function reads

$$L_T(X;\theta) = \exp \left(\frac{\theta}{\sigma} M_T - \frac{1}{2} \frac{\theta^2}{\sigma^2} \langle M \rangle_T \right).$$

The MLE of θ, being the maximizer of the above expression, is given by

$$\hat{\theta}_T := \sigma \frac{M_T}{\langle M \rangle_T}.$$

This estimator is unbiased:

$$\mathbb{E}_\theta \sigma \frac{M_T}{\langle M\rangle_T} = \sigma \mathbb{E}_0 L_T(X;\theta)\frac{M_T}{\langle M\rangle_T} =$$
$$\sigma^2 \frac{1}{\langle M\rangle_T}\exp\left(-\frac{1}{2}\frac{\theta^2}{\sigma^2}\langle M\rangle_T\right)\frac{d}{d\theta}\mathbb{E}_0 \exp\left(\frac{\theta}{\sigma}M_T\right) =$$
$$\sigma^2 \frac{1}{\langle M\rangle_T}\exp\left(-\frac{1}{2}\frac{\theta^2}{\sigma^2}\langle M\rangle_T\right)\frac{d}{d\theta}\exp\left(\frac{1}{2}\frac{\theta^2}{\sigma^2}\langle M\rangle_T\right) = \theta,$$

with the variance

$$\begin{aligned}
&\mathbb{E}_\theta\big(\hat\theta_T-\theta\big)^2 = \mathbb{E}_\theta \hat\theta_T^2 - \theta^2 = \sigma^2 \mathbb{E}_\theta \frac{M_T^2}{\langle M\rangle_T^2} - \theta^2 = \\
&\sigma^2 \exp\left(-\frac{1}{2}\frac{\theta^2}{\sigma^2}\langle M\rangle_T\right)\mathbb{E}_0 \exp\left(\frac{\theta}{\sigma}M_T\right)\frac{M_T^2}{\langle M\rangle_T^2} - \theta^2 = \\
&\frac{\sigma^4}{\langle M\rangle_T^2}\exp\left(-\frac{1}{2}\frac{\theta^2}{\sigma^2}\langle M\rangle_T\right)\frac{d^2}{d\theta^2}\mathbb{E}_0 \exp\left(\frac{\theta}{\sigma}M_T\right) - \theta^2 = \\
&\frac{\sigma^4}{\langle M\rangle_T^2}\left(\frac{\theta^2}{\sigma^4}\langle M\rangle_T^2 + \frac{\langle M\rangle_T}{\sigma^2}\right) - \theta^2 = \frac{\sigma^2}{\langle M\rangle_T}.
\end{aligned}\tag{2.2}$$

To recap, the MLE error is a zero mean Gaussian random variable with variance $\sigma^2/\langle M\rangle_T$. Next we shall derive an explicit characterization of the martingale M in terms of the solution of the integral equation (1.3) and will find the appropriate asymptotic as $T\to\infty$.

2.2. **The martingale representation.** Let us recall briefly some relevant properties of the integrals with respect to fractional Brownian motion. Following the notations of [10], define the spaces

$$\begin{aligned}
&L^2[0,T] := \Big\{f:[0,T]\mapsto\mathbb{R} \text{ such that } \int_0^T f^2(u)du<\infty\Big\},\\
&|\Lambda|_T^{H-\frac12} := \Big\{f:[0,T]\mapsto\mathbb{R} \text{ such that } \int_0^T\int_0^T |f(u)||f(v)||u-v|^{2H-H}dudv<\infty\Big\},\\
&\Lambda_T^{H-\frac12} := \Big\{f:[0,T]\mapsto\mathbb{R} \text{ such that } \int_0^T \Big(s^{\frac12-H}\big(\mathbf{I}_{T-}^{H-\frac12}u^{H-\frac12}f(u)\big)(s)\Big)^2 ds<\infty\Big\},
\end{aligned}$$

where $\mathbf{I}_{T-}^{H-\frac12}$ is the Riemann-Liouville fractional integral operator (see [15]). For $H\in(\frac12,1)$ the inclusions $L^2[0,T]\subset|\Lambda|_T^{H-\frac12}\subset\Lambda_T^{H-\frac12}$ hold (see Remark 4.2 in [10]).

For the simple function of the form,

$$f(u)=\sum_{k=1}^n f_k \mathbf{1}_{\{[u_k,u_{k+1})\}}(u),\quad f_k\in\mathbb{R},\quad 0=u_1<u_2<\ldots<u_k=T,$$

the stochastic integral with respect to B^H is defined by

$$\int_0^T f(t)dB_t^H := \sum_{k=1}^n f_k\big(B^H_{u_{k+1}} - B^H_{u_k}\big).$$

Since the simple functions are dense in $\Lambda_T^{H-\frac{1}{2}}$ (see Theorem 4.1 in [10]) , the definition of $\int_0^T f(t)dB_t^H$ is extended to $f\in\Lambda_T^{H-\frac{1}{2}}$ through the limit

$$\int_0^T f(t)dB_t^H := \lim_n \int_0^T f_n(t)dB_t^H,$$

where f_n is any sequence of simple functions, such that $\lim_n \|f-f_n\|_{\Lambda_T^{H-\frac{1}{2}}}=0$.

It turns out however (see Section 5 of [10]), that the image of $\Lambda_T^{H-\frac{1}{2}}$ under the map $f\mapsto \int_0^T f(t)dB_t^H$ is a strict subset of $\overline{\mathrm{sp}}_{[0,T]}(B^H)$, the closure in $L^2(\Omega,\mathcal{F},\mathbb{P}_0)$ of all possible linear combinations of the increments of B^H. In other words, some linear functionals of B^H cannot be realized as stochastic integrals of the above type. Nevertheless we have the following:

Lemma 2.1. *Assume $H\in(\frac{1}{2},1)$ and let η be a Gaussian random variable, such that (η, X_t), $t\in[0,T]$ is a Gaussian random process. Then there exists a function $g(\cdot,T)\in L^2[0,T]$, such that*

$$\mathbb{E}_0(\eta|\mathcal{F}_T^X) = \mathbb{E}_0\eta + \int_0^T g(s,T)dX_s, \quad \mathbb{P}_0 - a.s. \tag{2.3}$$

Proof. Following the arguments of the proof of Lemma 10.1 in [7], let (t_i), $i=0,...,2^n$ be the dyadic partition of $[0,T]$, *i.e.*, $t_i = i2^{-n}$, $i=0,...,2^n$ and $\mathcal{F}^X_{T,n} = \sigma\{X_{t_i} - X_{t_{i-1}}, i=1,...,2^n\}$. Then $\mathcal{F}^X_{T,n}\nearrow \mathcal{F}^X_T$ and by the martingale convergence

$$\lim_n \mathbb{E}_0(\eta|\mathcal{F}^X_{T,n}) = \mathbb{E}_0(\eta|\mathcal{F}^X_T), \quad \mathbb{P}_0 - a.s. \tag{2.4}$$

as well as in $L^2(\Omega,\mathcal{F},\mathbb{P}_0)$, since $\mathbb{E}_0(\eta|\mathcal{F}^X_{T,n})$ are uniformly integrable. By the Normal Correlation theorem,

$$\mathbb{E}_0(\eta|\mathcal{F}^X_{T,n}) = \mathbb{E}_0\eta + \sum_{i=1}^{2^n} g^n_{i-1}\big(X_{t_i} - X_{t_{i-1}}\big), \tag{2.5}$$

with constants g^n_{i-1}, $i=1,...,2^n$. Define

$$g_n(t,T) := \sum_{i=1}^{2^n} g^n_{i-1}\mathbf{1}_{\{[t_{i-1},t_i)\}}(t), \tag{2.6}$$

then

$$\mathbb{E}_0\big(\eta|\mathcal{F}^X_{T,n}\big) = \mathbb{E}_0\eta + \sigma\int_0^T g_n(t,T)dB_t + \int_0^T g_n(t,T)dB_t^H,$$

and

$$\mathbb{E}_0\Big(\mathbb{E}_0\big(\eta|\mathcal{F}^X_{T,n}\big)-\mathbb{E}_0\big(\eta|\mathcal{F}^X_{T,m}\big)\Big)^2=\sigma^2\int_0^T\big(g_n(t,T)-g_m(t,T)\big)^2dt+$$
$$c_H\int_0^T\int_0^T\big(g_n(t,T)-g_m(t,T)\big)\big(g_n(s,T)-g_m(s,T)\big)|s-t|^{2H-2}dsdt,$$

where $c_H:=H(2H-1)$. Since the kernel in the last integral is positive definite

$$\lim_n\sup_{m\ge n}\sigma^2\int_0^T\big(g_n(t,T)-g_m(t,T)\big)^2dt\le\lim_n\sup_{m\ge n}\mathbb{E}_0\Big(\mathbb{E}_0\big(\eta|\mathcal{F}^X_{T,n}\big)-\mathbb{E}_0\big(\eta|\mathcal{F}^X_{T,m}\big)\Big)^2=0,$$

where the latter equality holds by (2.4), since $L^2(\Omega,\mathcal{F},\mathbb{P}_0)$ is complete. Since $L^2[0,T]$ is a complete space, there exists a function $g(t,T)\in L^2[0,T]$, such that $\lim_n\|g-g_n\|_2=0$. Then

$$\mathbb{E}_0\bigg(\mathbb{E}_0(\eta|\mathcal{F}^X_T)-\mathbb{E}_0\eta-\sigma\int_0^T g(t,T)dB_t-\int_0^T g(t,T)dB^H_t\bigg)^2\le$$
$$3\mathbb{E}_0\Big(\mathbb{E}_0(\eta|\mathcal{F}^X_T)-\mathbb{E}_0(\eta|\mathcal{F}^X_{T,n})\Big)^2+3\sigma^2\int_0^T\big(g_n(t,T)-g(t,T)\big)^2dt+$$
$$3c_H\int_0^T\int_0^T\big(g_n(t,T)-g(t,T)\big)\big(g_n(s,T)-g(s,T)\big)|s-t|^{2H-2}dsdt\xrightarrow{n\to\infty}0,$$

where the latter convergence holds, since $L^2[0,T]\subset|\Lambda|_T^{H-\frac12}$. □

Applying the above lemma, we obtain the claimed formulas (1.2) and (1.4):

Lemma 2.2. *Let* (M_t) *be the* $(\mathcal{F}^X_t)-$*martingale defined by* (2.1). *The following representations hold:*

$$M_T=\int_0^T g(t,T)dX_t, \tag{2.7}$$

and

$$\langle M\rangle_T=\sigma\int_0^T g(t,T)dt, \tag{2.8}$$

where $g(t,T)$ *is the unique solution of* (1.3).

Proof. By Lemma 2.1, there exists $g(\cdot,T)\in L^2[0,T]$, such that

$$M_T=\mathbb{E}_0(B_T|\mathcal{F}^X_T)=\int_0^T g(t,T)dX_t,\quad \mathbb{P}_0-a.s.$$

holds. For an arbitrary function $h \in L^2[0,T]$,

$$\mathbb{E}_0\left(B_T - \int_0^T g(r,T)dX_r\right)\int_0^T h(s)dX_s =$$

$$\mathbb{E}_0\left(\int_0^T dB_t - \sigma\int_0^T g(t,T)dB_t - \int_0^T g(t,T)dB_t^H\right)\left(\sigma\int_0^T h(t)dB_t + \int_0^T h(t)dB_t^H\right) =$$

$$\int_0^T h(s)\left(\sigma - \sigma^2 g(s,T) - c_H\int_0^T g(r,T)|s-r|^{2H-2}dr\right)ds.$$

By the orthogonality property of the conditional expectation and by arbitrariness of h, it follows that $g(t,T)$ satisfies (1.3) for almost all $t \in [0,T]$. This solution is unique (see, *e.g.*, [16]). Further, since M is a Gaussian martingale,

$$\langle M\rangle_T = \mathbb{E}_0 M_T^2 = \mathbb{E}_0\left(\int_0^T g(s,T)dX_s\right)^2 =$$

$$\int_0^T g(t,T)\left(\sigma^2 g(t,T) + c_H\int_0^T g(s,T)|s-t|^{2H-2}ds\right)dt = \sigma\int_0^T g(t,T)dt.$$

□

2.3. **The large sample asymptotic.** Finally we shall derive the asymptotic announced in (1.5). Let $\mu := T^{2H-1}$ and define $g_\mu(u) := T^{2H-1}g(uT,T)$, $u \in [0,1]$. Then (1.3) reads

$$\frac{1}{\mu}\sigma^2 g_\mu(u) + c_H\int_0^1 g_\mu(v)|u-v|^{2H-2}dv = \sigma, \quad u \in [0,1], \tag{2.9}$$

and, moreover,

$$\langle M\rangle_T = \sigma\int_0^T g(s,T)ds = \sigma T^{2-2H}\int_0^1 g_\mu(u)du. \tag{2.10}$$

Define the operator K

$$Kf(u) = c_H\int_0^1 f(v)|u-v|^{2H-2}dv, \quad f \in |\Lambda|_T^{H-\frac{1}{2}} \tag{2.11}$$

and the scalar products

$$\langle f,h\rangle := \int_0^1 f(s)h(s)ds, \quad f,h \in L^2[0,1]$$

and

$$\langle f,h\rangle_K := c_H\int_0^1\int_0^1 f(v)h(u)|u-v|^{2H-2}dvdu, \quad h,f \in |\Lambda|_T^{H-\frac{1}{2}}.$$

In terms of these notations, the equation (2.9) becomes

$$\frac{\sigma^2}{\mu}g_\mu + Kg_\mu = \sigma.$$

We shall also consider the first type auxiliary integral equation

$$Kg = \sigma, \tag{2.12}$$

which admits the unique solution (see Lemma 3 in [6] and the references therein):

$$g(u) = \frac{\sigma}{c_H} \frac{\beta(2-2H, H-\frac{1}{2})}{\Gamma^2(H-\frac{1}{2})\Gamma(2-2H)} u^{\frac{1}{2}-H}(1-u)^{\frac{1}{2}-H}, \quad u \in (0,1),$$

where $\beta(\alpha,\gamma) = \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}$.

The function $\delta_\mu := g_\mu - g$ satisfies

$$\frac{\sigma^2}{\mu}\delta_\mu + K\delta_\mu = -\frac{\sigma^2}{\mu} g.$$

Since $g \in L^2[0,1] \subset |\Lambda|_1^{H-\frac{1}{2}}$, multiplying by δ_μ and integrating we obtain

$$\frac{\sigma^2}{\mu}\|\delta_\mu\|_2^2 + \|\delta_\mu\|_K^2 = \frac{\sigma^2}{\mu}\big|\langle g, \delta_\mu\rangle\big|,$$

and, in particular, $\|\delta_\mu\|_2^2 \le \big|\langle g, \delta_\mu\rangle\big|$. On the other hand, by the Cauchy-Schwarz inequality $\big|\langle g, \delta_\mu\rangle\big| \le \|g\|_2\|\delta_\mu\|_2$ and hence $\|\delta_\mu\|_2 \le \|g\|_2$. Note that δ_μ also satisfies

$$\frac{\sigma^2}{\mu} g_\mu + K\delta_\mu = 0.$$

Multiplying both sides of this equation by g and integrating, we get

$$\frac{\sigma^2}{\mu}\langle g_\mu, g\rangle + \langle K\delta_\mu, g\rangle = 0.$$

But

$$\big|\langle g_\mu, g\rangle\big| \le \big|\langle \delta_\mu, g\rangle\big| + \|g\|_2^2 \le \|\delta_\mu\|_2\|g\|_2 + \|g\|_2^2 \le 2\|g\|_2^2 < \infty$$

and hence

$$\sigma|\langle \delta_\mu, 1\rangle| = |\langle \delta_\mu, Kg\rangle| = |\langle K\delta_\mu, g\rangle| = \frac{\sigma^2}{\mu}\big|\langle g_\mu, g\rangle\big| \le \frac{\sigma^2}{\mu} 2\|g\|_2^2 \xrightarrow{\mu\to\infty} 0,$$

and

$$\lim_{\mu\to\infty} \int_0^1 g_\mu(u)du = \int_0^1 g(u)du.$$

Finally, by the formulas (2.2) and (2.10)

$$T^{2-2H}\mathbb{E}_\theta\big(\hat\theta_T - \theta\big)^2 = T^{2-2H}\frac{\sigma^2}{\langle M\rangle_T} = \frac{\sigma^2}{\sigma\int_0^1 g_\mu(u)du} \xrightarrow{T\to\infty} \frac{\sigma}{\int_0^1 g(u)du}.$$

The asymptotic (1.5) now follows, since

$$\int_0^1 g(r)dr = \frac{1}{C_2(H)}\frac{1}{\Gamma(2-2H)}\int_0^1 r^{\frac{1}{2}-H}(1-r)^{\frac{1}{2}-H}dr =$$
$$\frac{1}{H(2H-1)}\frac{\beta(2-2H,H-\frac{1}{2})}{\Gamma^2(H-\frac{1}{2})}\frac{1}{\Gamma(2-2H)}\beta\Big(\frac{3}{2}-H,\frac{3}{2}-H\Big) =$$
$$\frac{1}{H(2H-1)}\frac{\Gamma(2-2H)\Gamma(H-\frac{1}{2})}{\Gamma(\frac{3}{2}-H)\Gamma^2(H-\frac{1}{2})}\frac{1}{\Gamma(2-2H)}\frac{\Gamma^2(\frac{3}{2}-H)}{\Gamma(3-2H)} =$$
$$\frac{\Gamma(\frac{3}{2}-H)}{H(2H-1)\Gamma(H-\frac{1}{2})\Gamma(3-2H)} = \frac{\Gamma(\frac{3}{2}-H)}{2H\Gamma(H+\frac{1}{2})\Gamma(3-2H)},$$

where we used the property $\Gamma(x+1) = x\Gamma(x)$, $x > 0$.

Of course, the martingale property of the process $M_T^H := \int_0^T g(t,T)d\xi_t$, $T \geq 0$ with $\xi_t := \sigma B_t + B_t^H$ and the representation of the error:

$$\hat{\theta}_T - \theta = \frac{M_T^H}{\langle M^H\rangle_T},$$

where $\langle M^H\rangle_T \to \infty$ when $T \to \infty$ implies the strong consistency of $\hat{\theta}_T$ due to the strong law of large numbers for martingales.

Remark 2.3. By means of a study similar to the previous, one can check that the limits (1.6) and (1.7) hold.

Remark 2.4. It is worth mentioning that, following the terminology of [9],[6] and [5], the martingale M^H merits to be called the *fundamental martingale* associated to the mixed fractional Brownian motion ξ. It will play also a key role in the statistical analysis of models more general than (1.1) such as the mixed fractional Ornstein-Uhlenbeck process. The progress in this direction will be reported elsewhere.

3. Proof of Proposition 1.4

In the following lemmas, we shall first prove a number of useful properties of the solution of (1.9), assuming hereafter that $H > 3/4$.

Lemma 3.1. *The solution $g(s,t)$ of (1.9) is continuously differentiable at $t > 0$ for any $s \in (0,t)$. The derivative $\dot{g}(s,t) := \frac{\partial}{\partial t}g(s,t)$ satisfies the equation*

$$\dot{g}(s,t) + c_H\int_0^t \dot{g}(r,t)|r-s|^{2H-2}dr = -c_H g(t,t)|t-s|^{2H-2}, \quad s \in (0,t),\ t > 0, \tag{3.1}$$

and $\int_0^t \dot{g}^2(s,t)ds < \infty$.

Proof. The function $g_t(u) := g(ut,t)$, $u \in [0,1]$, $t > 0$ satisfies the integral equation

$$g_t(u) + c_H t^{2H-1}\int_0^1 g_t(v)|v-u|^{2H-2}dv = 1, \quad u \in [0,1].$$

This equation has a unique continuous solution for any $t > 0$ (see [16]) and, in terminology of [13], any point $\lambda := t^{2H-1}$ is regular. Since for $H > 3/4$ the kernel (2.11) belongs to $L_2[0,1]$, it follows from ,*e.g.*, Theorem on page 154 in [13], that the solution $g_t(u)$ is analytic at $t > 0$. By [16] the solution $g_t(u)$ is continuously differentiable at $u \in (0,1)$ and hence the function $g(s,t) = g_t(s/t)$ is continuously differentiable at $t > 0$ for any $s \in (0,t)$. The equation (3.1), obtained by taking the derivative of both sides of (1.9), has a unique solution in $L_2[0,t]$ for $H > 3/4$. □

Lemma 3.2. *The solution $g(s,t)$ of (1.9) is such that $g(t,t) \neq 0$ for all $t > 0$.*

Proof. Letting $g'(s,t) := \frac{\partial}{\partial s} g(s,t)$ and taking the derivative of (1.9), we obtain

$$g'(s,t) = -c_H \frac{\partial}{\partial s} \int_0^t g(r,t)|r-s|^{2H-2} dr = -c_H \frac{\partial}{\partial s} \left(\int_{-s}^{t-s} g(u+s,t)|u|^{2H-2} du \right) =$$

$$- c_H \int_{-s}^{t-s} g'(u+s,t)|u|^{2H-2} du + c_H g(t,t)|t-s|^{2H-2} - c_H g(0,t)|s|^{2H-2} =$$

$$- c_H \int_0^t g'(r,t)|r-s|^{2H-2} dr + c_H g(t,t) \Big(|t-s|^{2H-2} - |s|^{2H-2} \Big),$$

where we used the obvious symmetry $g(t-s,t) = g(s,t)$ and $g(t,t) = g(0,t)$ in particular. Now suppose $g(t,t) = 0$ for some $t > 0$. Then

$$g'(s,t) + c_H \int_0^t g'(r,t)|r-s|^{2H-2} dr = 0, \quad s \in [0,t].$$

This equation has the unique solution $g'(s,t) \equiv 0$, *i.e.*, $g(s,t)$ is a constant function. But since $g(t,t) = 0$, it follows that $g(s,t) = 0$ for all $s \in [0,t]$, which contradicts (1.9). □

Lemma 3.3. *The solution $g(s,t)$ of (1.9) is such that for all $t > 0$,*

$$\int_0^t g(s,t) ds = \int_0^t g^2(s,s) ds. \tag{3.2}$$

Proof. We shall use Krein's method for solving integral equations with difference kernels on a finite interval (see §13.13-1 in [11]). Let $y(s,t)$ satisfy the equation

$$y(s,t) + \int_0^t y(r,t) k(r-s) dr = f(s), \quad s \in [0,t],$$

where $k(u) = c_H |u|^{2H-2}$ and f is a continuous function and consider the auxiliary equation with parameter $\tau \in [0,t]$:

$$g(s,\tau) + \int_0^\tau g(r,\tau) k(r-s) dr = 1, \quad s \in [0,\tau].$$

Then

$$y(s,t) = F(t) g(s,t) - \int_s^t g(s,\tau) F'(\tau) d\tau, \tag{3.3}$$

where

$$F(\tau)=\frac{1}{g^2(\tau,\tau)}\frac{d}{d\tau}\int_0^\tau g(s,\tau)f(s)ds. \tag{3.4}$$

In particular, by uniqueness of the solutions, for $f\equiv 1$ we get $y(s,t)=g(s,t)$, $s\in[0,t]$ and setting $s:=t$ in (3.3), we see that $F(t)=1$. Now by (3.4)

$$g^2(t,t)=\frac{d}{dt}\int_0^t g(s,t)ds,$$

and (3.2) follows by integration. □

Now we are ready to prove Proposition 1.4. By Lemma 2.2 and Lemma 3.3,

$$\langle M\rangle_t=\int_0^t g(s,t)ds=\int_0^t g^2(s,s)ds,\quad t\in[0,T].$$

Hence by the Levy theorem and Corollary 3.2

$$W_t=\int_0^t\frac{1}{g(s,s)}dM_s,\quad t\in[0,T],$$

is a Brownian motion. On the other hand,

$$M_t=\int_0^t g(s,t)d\xi_s=\int_0^t g(s,s)d\xi_s+\int_0^t\Big(g(r,t)-g(r,r)\Big)d\xi_r=$$
$$\int_0^t g(s,s)d\xi_s+\int_0^t\int_r^t\dot g(r,s)dsd\xi_r=\int_0^t g(s,s)d\xi_s+\int_0^t\int_0^s\dot g(r,s)d\xi_r ds,$$

and hence

$$W_t=\int_0^t\frac{1}{g(s,s)}dM_s=\xi_t+\int_0^t\int_0^s\frac{\dot g(r,s)}{g(s,s)}d\xi_r ds=:\xi_t+\int_0^t\varphi_s(\xi)ds.$$

The desired claim follows from Theorem 7.7 in [7], once we check

$$\int_0^T\mathbb{E}\varphi_t^2(B)dt<\infty\quad\text{and}\quad\int_0^T\mathbb{E}\varphi_t^2(\xi)dt<\infty. \tag{3.5}$$

Since $\varphi_t(\cdot)$ is additive and $\xi_t=B_t+B_t^H$, where B and B^H are independent, it is enough to check only the latter condition. By (3.1) the function $R(s,t):=\frac{\dot g(s,t)}{g(t,t)}$ satisfies

$$R(s,t)+c_H\int_0^t R(r,t)|r-s|^{2H-2}dr=-c_H|t-s|^{2H-2},\quad s\in(0,t),\ t>0. \tag{3.6}$$

Hence for $H > 3/4$, using the latter equation we get

$$\mathbb{E}\varphi_t^2(\xi) = \mathbb{E}\left(\int_0^t R(r,t)d\xi_r\right)^2 = \int_0^t R^2(s,t)ds + c_H \int_0^t \int_0^t R(s,t)R(r,t)|r-s|^{2H-2}drds =$$

$$\int_0^t R(s,t)\left(R(s,t) + c_H \int_0^t R(r,t)|r-s|^{2H-2}dr\right)ds = -c_H \int_0^t R(s,t)|t-s|^{2H-2}ds \le$$

$$c_H \left(\int_0^t R^2(s,t)ds\right)^{1/2} \left(\int_0^t |t-s|^{4H-4}ds\right)^{1/2} = \frac{c_H}{\sqrt{4H-3}} \left(\int_0^t R^2(s,t)ds\right)^{1/2} t^{2H-3/2}.$$

Since the kernel is positive definite, multiplying (3.6) by $R(s,t)$ and integrating gives

$$\int_0^t R^2(s,t)ds \le -c_H \int_0^t R(s,t)|t-s|^{2H-2}ds \le \frac{c_H}{\sqrt{4H-3}} \left(\int_0^t R^2(s,t)ds\right)^{1/2} t^{2H-3/2},$$

and consequently

$$\left(\int_0^t R^2(s,t)ds\right)^{1/2} \le \frac{c_H}{\sqrt{4H-3}} t^{2H-3/2}.$$

Plugging this bound back gives $\mathbb{E}\varphi_t^2(\xi) \le \frac{c_H^2}{4H-3} t^{4H-3}$ and in turn

$$\int_0^T \mathbb{E}\varphi_t^2(\xi)dt \le \frac{c_H^2}{4H-3}\int_0^T t^{4H-3}dt = \frac{c_H^2}{(4H-3)(4H-2)} T^{4H-2},$$

which verifies (3.5) and completes the proof. □

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