

Critical Properties of S^4 System Restudied via Generalized Migdal-Kadanoff Bond-moving Renormalization

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Abstract

We study the critical properties of the spin-continuous S^4 system on the typical translational invariant triangular lattices by combining the recently-developed generalized Migdal-Kadanoff bond-moving recursion procedures with the cumulative expansion technique. In three different cases of nearest-neighbor, next nearest neighbor and external field we obtain the critical points and further calculate the critical exponents according to the scaling theory. In all case it is found that there exists three fixed points. The correlation length critical exponents obtained near the Wilson-Fisher fixed points are found getting smaller and smaller with the increasing of the system complexity. Others are found similar to the results of the classical Gaussian model and different from those of the Ising system.

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I. INTRODUCTION

The S^4 model, which is another extension of the classical Ising model, is a kind of typical spin-continuous systems which allows the spin to take any real value between $(-\infty, +\infty)$ instead of a discrete one. The study of it is of great value for better understanding the properties of the ferromagnetic systems from either a theoretical or a practical viewpoint. Due to this point it has been widely studied in the past few years [1–3]. For an example, critical properties of the S^4 model on some lattice systems with translational symmetry has been found to have close dependent on the space dimensionality d : when d is more than 4, only the Gaussian fixed point is obtained while for low-dimensional systems with d less equal to 4, not only the Wilson-Fisher fixed point but also the Gaussian one can be obtained [1]. Similar dimensionality dependent behaviors are also found for the S^4 model on the fractal lattices [2].

However, critical behaviors of the S^4 system is far from having been well studied. This may in some case due to the self-complexity of the model caused by its four-body interactions. Other restrictions may come from the particular lattice system on which the model is constructed. Recently, a type of generalization of the remarkable Migdal-Kadanoff bond-moving renormalization procedure is developed which enables us to handle complex systems in a relatively easy way [4]. For the great convenience it has brought in the coarse-graining process, in this paper, we extend this renormalization procedure to study the critical properties of the S^4 system on translational invariant triangular lattice by combining it with the cumulative expansion method [2, 3, 5].

The paper is organized as follows: in Sec. II, a brief review of the generalized bond-moving recursion procedures is presented; in Sec. III, the procedures are used to study the critical properties of the spin-continuous S^4 model constructed on the translational invariant triangular lattice; Sec. IV serves as a summary of our conclusion in which further implicit applications are also discussed.

II. RENORMALIZATION PROCEDURE AND CUMULATIVE EXPANSION

In Ref.[4], a new type of generalization of the remarkable Migdal-Kadanoff bond-moving renormalization procedure is developed bringing with much convenience for the study of

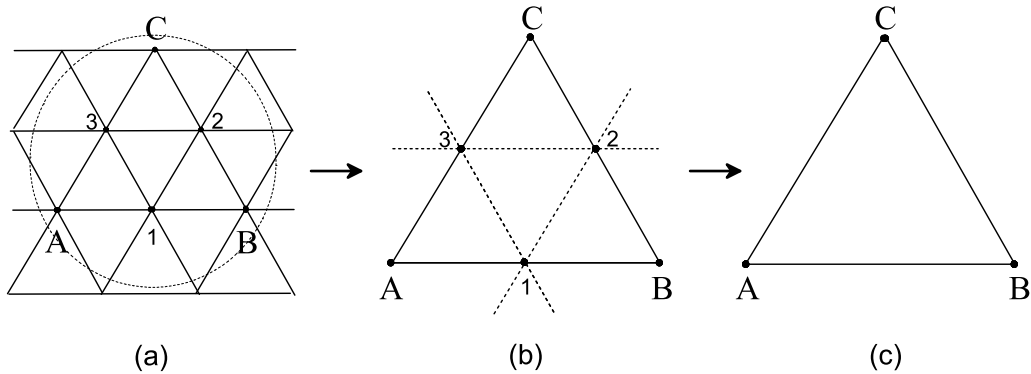


FIG. 1: Generalized bond-moving procedure recurring on the triangular lattice where the peripheral bonds connecting the three to be eliminated sites 1, 2 and 3 around the selected triplet $\triangle ABC$ are drawn at half length.

spin-continues systems. By introducing in some kind of symmetrical half-length bond operations, the renormalization process is greatly simplified as has been witnessed in the study of Gaussian system. For the example of systems constructed on the triangular lattice as illustrated in Fig.1, this was realized by supposing each bond connecting the to be eliminated sites denoted as 1, 2 and 3 in $\triangle ABC$ to schlep symmetrically two half-length bonds. Thus the number of lattice sites that a basic unit for recursion contains can be reduced to only six which provides a good foundation for the following integral calculation.

The technique of cumulative expansion is always used as an auxiliary method in calculating high order integrations [2, 3, 5]. For a system with effective Hamiltonian: $H_{\text{eff}} = H_0 + V$, a partial trace (PT) is defined as

$$\begin{aligned}
 (PT) &= \int_{-\infty}^{+\infty} \left(\prod_i^m ds_i \right) e^{H_{\text{eff}}} \\
 &= \int_{-\infty}^{+\infty} \left(\prod_i^m ds_i \right) e^{H_0} \frac{\int_{-\infty}^{+\infty} (\prod_i^m ds_i) e^{H_0+V}}{\int_{-\infty}^{+\infty} (\prod_i^m ds_i) e^{H_0}} \\
 &= A \langle e^V \rangle_0,
 \end{aligned} \tag{1}$$

where

$$\langle \dots \rangle_0 = \frac{\int_{-\infty}^{+\infty} (\prod_i^m ds_i) (\dots) e^{H_0}}{\int_{-\infty}^{+\infty} (\prod_i^m ds_i) e^{H_0}} \tag{2}$$

is namely the cumulative expansion average and

$$A = \int_{-\infty}^{+\infty} \left(\prod_i^m ds_i \right) e^{H_0}. \quad (3)$$

The integrations in the above equations trace over all the to be decimated sites in the renormalization process.

On the supposition of V is a small variable, we can expand e^V by a Maclaurin series as

$$e^V = 1 + V + \frac{1}{2!}V^2 + \frac{1}{3!}V^3 + \dots. \quad (4)$$

The partial trace is then obtained to be

$$(PT) = A \left(1 + \langle V \rangle_0 + \frac{1}{2!} \langle V^2 \rangle_0 + \frac{1}{3!} \langle V^3 \rangle_0 + \dots \right). \quad (5)$$

Noticing that the partition function should be kept unchanged after the renormalization group (RG) transformation. That is, supposing

$$\int_{-\infty}^{+\infty} \left(\prod_i^m ds_i \right) e^{H_{\text{eff}}} = C e^{H'_{\text{eff}}}, \quad (6)$$

then H'_{eff} represents an effective Hamiltonian after RG transformation (C is a RG constant).

Thus we can obtain from the above equations

$$H'_{\text{eff}} = \ln A + \langle V \rangle_0 + \frac{1}{2} (\langle V^2 \rangle_0 - \langle V \rangle_0^2) + \dots, \quad (7)$$

where the approximation relation $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ is used.

Now we have found a convenient way to calculate the effective Hamiltonian H'_{eff} from the partial trace defined herein before. Supposing

$$\int_{-\infty}^{+\infty} \left(\prod_i^m ds_i \right) e^{H_0} = e^{H'_0}, \quad (8)$$

the first part in the right-hand site of Eq.(7) can be regarded as the zero order approximation of H'_{eff} . Similar understandings are still true as well for other right-hand parts. In the following sections, this will be applied to study the critical behavior of the S^4 system constructed on the triangular lattice combining with the generalized bond-moving renormalization procedures.

III. CRITICAL BEHAVIOR OF THE S^4 SYSTEM

The S^4 model is another spin-continuous extension of the Ising model [6]. By introducing a probability distribution function

$$W(s_1, s_2, \dots, s_N) = e^{-\frac{b}{2} \sum_i s_i^2 - \sum_i u s_i^4}, \quad (9)$$

for the classical Ising spin system, the effective Hamiltonian of the S^4 model in an external field h can be obtained as

$$H_{\text{eff}} = \sum_{\langle ij \rangle} K s_i s_j - \frac{b}{2} \sum_i s_i^2 - u \sum_i s_i^4 + h \sum_i s_i, \quad (10)$$

where $K = J/k_B T$ is the reduced interaction with $K > 0$ denotes the ferromagnetic system; k_B is the Boltzmann constant and T the thermodynamic temperature; h is the external field. The $\langle ij \rangle$ in the first summation generally represents a certain nearest-neighbor spin pair. Seen from Eq.(10), the S^4 model Hamiltonian contains synchronously the four-spins and two-spins interactions denoted by u ($u > 0$) and Gaussian constant b respectively. This will undoubtedly made it more difficult to be studied, however these can all be settled following the generalized bond-moving renormalization procedures.

A. nearest-neighbor interaction

Firstly we give our study on the case of only considering nearest-neighbor interactions. The procedure of the generalized bond-moving RG transformation for obtaining a certain bond K' is shown in Fig.2 as an example. From which we can see that the number of to be decimated sites reduces to be only one. Thus the effective Hamiltonian of the basic unit for RG transformation in the case of $h = 0$ can be given as

$$H_{\text{eff}} = 2K (s_a s_1 + s_1 s_b) - b (2s_1^2 + s_a^2 + s_b^2) - 2u (2s_1^4 + s_a^4 + s_b^4), \quad (11)$$

where in the RG procedures two types of interactions K_e and K together with two types of self-energy $(-b_e s^2/2)$ and $(-b s^2/2)$ are assigned respectively for differentiation of the to be and not to be eliminated bonds Gefen et al did on the fractal [7–9]. For the particular case of triangular lattices the numerical value of K_e and K is actually identical as well as that of b_e and b (or u_e and u). In the recursion procedures the two half length bonds are considered acting effectively as a whole one and be moved regularly as other bonds.

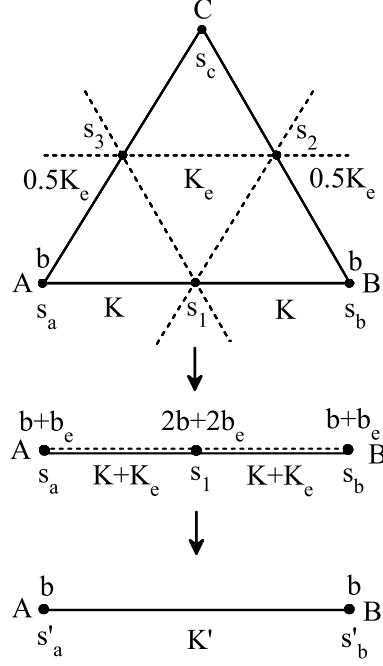


FIG. 2: Bond-moving and decimation procedures for the renormalized bond K' between sites A and B on the triangular lattices.

Supposing

$$V = -4us_1^4, \quad (12)$$

and

$$H_0 = 2K (s_a s_1 + s_1 s_b) - b (2s_1^2 + s_a^2 + s_b^2) - 2u (s_a^4 + s_b^4), \quad (13)$$

the zero-order approximation of the effective Hamiltonian after RG transformation can be obtained from Eq.(8) as

$$H'_0 = K_{01} s_a s_b + K_{02} (s_a^2 + s_b^2) + K_{03} (s_a^4 + s_b^4), \quad (14)$$

with

$$K_{01} = \frac{K^2}{b}, \quad (15a)$$

$$K_{02} = \frac{K^2}{2b} - b, \quad (15b)$$

$$K_{03} = -2u. \quad (15c)$$

Continuously we can obtain the first-order one by Eq.(2) as

$$\begin{aligned}\langle V \rangle_0 &= \frac{\int_{-\infty}^{+\infty} (\prod_i^m ds_i) V e^{H_0}}{\int_{-\infty}^{+\infty} (\prod_i^m ds_i) e^{H_0}} \\ &= K_{11} s_a s_b + K_{12} (s_a^2 + s_b^2) + K_{13} (s_a^4 + s_b^4),\end{aligned}\quad (16)$$

where

$$K_{11} = -\frac{3K^2 u}{b^3}, \quad (17a)$$

$$K_{12} = -\frac{3K^2 u}{2b^3}, \quad (17b)$$

$$K_{13} = -\frac{K^4 u}{4b^4}. \quad (17c)$$

In the same way, the second-order term can also be approximately obtained as

$$\frac{1}{2} (\langle V^2 \rangle_0 - \langle V \rangle_0^2) = K_{21} s_a s_b + K_{22} (s_a^2 + s_b^2) + K_{23} (s_a^4 + s_b^4), \quad (18)$$

with

$$K_{21} = \frac{105K^2 u^2}{4b^5}, \quad (19a)$$

$$K_{22} = \frac{105K^2 u^2}{8b^5}, \quad (19b)$$

$$K_{23} = \frac{87K^4 u^2}{16b^6}. \quad (19c)$$

Combining Eqs.(15), (17) and (28), we obtain the effective Hamiltonian after RG transformation

$$\begin{aligned}H'_{\text{eff}} &= (K_{01} + K_{11} + K_{21}) s_a s_b + (K_{02} + K_{12} + K_{22}) (s_a^2 + s_b^2) \\ &\quad + (K_{03} + K_{13} + K_{23}) (s_a^4 + s_b^4),\end{aligned}\quad (20)$$

In order to get the recursion relations of the RG transformation, we rescale the spins by $s'_i = \xi_i s_i$ ($i = a, b$) and then Eq. (21) can be rewritten as

$$H'_{\text{eff}} = K' s'_a s'_b - \frac{b}{2} (s'^2_a + s'^2_b) - u' (s'^4_a + s'^4_b), \quad (21)$$

where

$$\xi_i = \sqrt{\frac{-2(K_{11} + K_{12} + K_{13})}{b}}, \quad (22a)$$

$$K' = \frac{(K_{01} + K_{02} + K_{03})}{\xi_i^2}, \quad (22b)$$

$$u' = -\frac{(K_{21} + K_{22} + K_{23})}{\xi_i^4}. \quad (22c)$$

Setting $K' = K = K^*$ and $u' = u = u^*$, the fixed points of the system can be found are

$$A : K^* = 0, u^* = 0; \quad (23a)$$

$$B : K^* = b, u^* = 0; \quad (23b)$$

$$C : K^* = 0.436b, u^* = 0.370b^2; \quad (23c)$$

in which A is found to be a nonphysical steady fixed point while B is a Gaussian fixed point and C a Wilson-Fisher one.

Actually, in the three fixed point, the Wilson-Fisher one is most crucial for the critical properties of the S^4 system. By expanding K' and u' in the neighborhood of the Wilson-Fisher fixed point C and preserving only the linear items, the transformation matrix is obtained to be

$$R_L(K, u) = \left(\begin{array}{cc} \frac{\partial K'}{\partial K} & \frac{\partial K'}{\partial u} \\ \frac{\partial u'}{\partial K} & \frac{\partial u'}{\partial u} \end{array} \right)_C = \left(\begin{array}{cc} 2.855 & 2.895 \\ 1.356 & 2.447 \end{array} \right), \quad (24)$$

with two eigenvalues $\lambda_1 = 4.643 > 1$ and $\lambda_2 = 0.659 < 1$ revealing that C is a barrier point. Thus a scale power can be obtained according to the scaling theory $p = \frac{\ln \lambda_1}{d \ln L} = 1.107$ where $d = 2$ is the dimensionality of the triangular lattice and $L = 2$ is the scaling factor. The correlation length exponent can be obtained from the relations $\nu = \frac{1}{pd}$ to be $\nu = 0.451$ smaller than the Gaussian case which contains only two-body interactions [4, 10] while in good conformity with previous S^4 system studies on other lattices [2, 3].

B. external field

In order to obtain all the critical exponents, we study in the following the case of S^4 system in an external field h where the effective Hamiltonian can be written as Eq.(10). For simplicity, we give out here only the primary results while leaving the process of derivation in the appendixes. Considering only nearest-neighbor interactions, three fixed points can also be obtained by following similar RG transformation procedures. They are

$$A : K^* = 0, u^* = 0, h^* = 0; \quad (25a)$$

$$B : K^* = b, u^* = 0, h^* = 0; \quad (25b)$$

$$C : K^* = 0.436b, u^* = 0.352b^2, h^* = 0; \quad (25c)$$

the transformation matrix in the neighborhood of the Wilson-Fisher fixed point C is now

$$R_L(K, u) = \begin{pmatrix} \frac{\partial K'}{\partial K} & \frac{\partial K'}{\partial u} & \frac{\partial K'}{\partial h} \\ \frac{\partial u'}{\partial K} & \frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial h} \\ \frac{\partial h'}{\partial K} & \frac{\partial h'}{\partial u} & \frac{\partial h'}{\partial h} \end{pmatrix}_C = \begin{pmatrix} 2.872 & 3.032 & 0 \\ 1.307 & 2.450 & 0 \\ 0 & 0 & 2.875 \end{pmatrix}, \quad (26)$$

with three eigenvalues $\lambda_1 = 4.663 > 1$, $\lambda_2 = 0.659 < 1$ and $\lambda_3 = 2.875 > 1$. Where λ_1 and λ_2 is related to the temperature while λ_3 is related to the external field. Then we can obtain the two scale powers according to the scaling theory

$$p = \frac{\ln \lambda_1}{d \ln L} = 1.110, \quad q = \frac{\ln \lambda_3}{d \ln L} = 0.762; \quad (27)$$

The critical exponents at this point can be obtained following $\alpha = \frac{2p-1}{p}$, $\beta = \frac{1-q}{p}$, $\gamma = \frac{2q-1}{p}$, $\delta = \frac{q}{1-q}$, $\eta = 2 + d(1 - 2q)$, $\nu = \frac{1}{pd}$ to be $\alpha = 1.099$, $\beta = 0.215$, $\gamma = 0.471$, $\delta = 3.197$, $\eta = 0.953$, $\nu = 0.450$ describing in the neighborhood of the critical point the variations in heat, magnetization, magnetic field, magnetic susceptibility, correlation function and correlation length, respectively.

C. next-nearest neighbor interaction

For the enriching of the S^4 results, we present in this paper also the case of considering next-nearest neighbor interactions. All the approximation of the effective Hamiltonian after RG transformation can be obtained also by combining the generalized bond-moving operations and the cumulative expansion techniques. The only change need to be made is to select a cluster of containing several sites (as is shown in Fig.3) as a basic unit for recursion. The various coefficients of H'_{eff} after RG derivation is also left in the appendixes for simplicity, we give out here only the primary results.

In the case of considering next-nearest neighbor interactions, we can also obtain by RG transformation three fixed points

$$A : K^* = 0, u^* = 0; \quad (28a)$$

$$B : K^* = 0.829b, u^* = 0; \quad (28b)$$

$$C : K^* = 0.257b, u^* = 5.405b^2; \quad (28c)$$

in which A is found to be a nonphysical steady fixed point while B is a Gaussian fixed point and C a Wilson-Fisher one. After some algebra we can obtain the correlation length critical

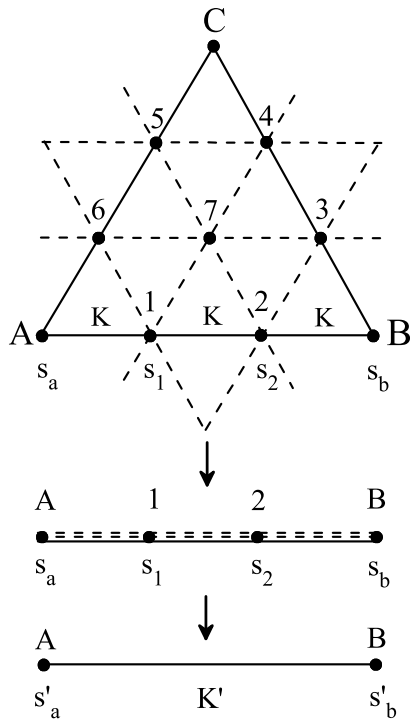


FIG. 3: Bond-moving and decimation procedures for the renormalized bond K' in the case of considering next nearest-neighbor interactions.

exponent at the Wilson-Fisher fixed point to be $\nu = 0.341$ also in conformity with previous studies. However we find it becomes relatively smaller than previous results. This reveals the effects of next nearest neighbor interactions.

IV. SUMMARY AND DISCUSSIONS

In summary, we have presented in this paper a detailed study on the critical behavior of the S^4 system constructed on the triangular lattices by combining the generalized bond-moving RG transformation method and the cumulative expansion techniques. In all cases it is found to have three fixed points. The critical exponents obtained near the Wilson-Fisher fixed points are found relatively smaller than that of the Gaussian model and different from those of the Ising model. This reveals the particular effect of the four-body interactions on the critical properties of the S^4 system. In further by comparing all the results of correlation

length exponent obtained in all cases we find that it becomes smaller and smaller with the increasing of the complexity of the system. This reveals the decisive influence of the system complexity on the critical behavior.

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Appendix A: RG coefficients of H'_{eff} in external field

For simplicity we give out here only the various RG coefficients of H'_{eff} from zero to second order approximations instead of showing all the details of derivation that is very similar to those have appeared in this paper hereinbefore. In the case of considering the influence of an external field h , these coefficients are

$$K_{01} = \frac{K^2}{b}, \quad (\text{A1})$$

$$K_{02} = \frac{K^2 - 2b^2}{2b}, \quad (\text{A2})$$

$$K_{03} = -2u, \quad (\text{A3})$$

$$K_{04} = \frac{(K - 2b)h}{b}, \quad (\text{A4})$$

$$K_{11} = -\frac{3K^2u}{b^4}(b + h^2), \quad (\text{A5})$$

$$K_{12} = -\frac{3K^2u}{2b^4}(b + h^2), \quad (\text{A6})$$

$$K_{13} = -\frac{K^4u}{4b^4}, \quad (\text{A7})$$

$$K_{14} = -\frac{Kuh}{b^4}(3b + h^2), \quad (\text{A8})$$

$$K_{21} = \frac{3K^2u^2}{4b^5}\left(35 + \frac{93h^2}{b} + \frac{9h^4}{b^2} + \frac{h^6}{b^3}\right), \quad (\text{A9})$$

$$K_{22} = \frac{3K^2u^2}{8b^5}\left(35 + \frac{93h^2}{b} + \frac{9h^4}{b^2} + \frac{h^6}{b^3}\right), \quad (\text{A10})$$

$$K_{23} = \frac{K^4u^2}{16b^6}\left(87 + \frac{174h^2}{b} + \frac{17h^4}{b^2}\right), \quad (\text{A11})$$

$$K_{24} = \frac{hKu^2}{4b^5}\left(105 + \frac{105h^2}{b} + \frac{21h^4}{b^2} + \frac{h^6}{b^3}\right). \quad (\text{A12})$$

where K_{i4} ($i = 0, 1, 2$) represents the various RG coefficients of H'_{eff} which is related with the external field.

Appendix B: RG coefficients of next-nearest H'_{eff}

The RG coefficients for various order approximation of the effective Hamiltonian in the case of considering next-nearest neighbor interactions are

$$K_{01} = \frac{48K^2b + 51K^3}{16(4b^2 - K^2)}, \quad (\text{B1})$$

$$K_{02} = -\frac{96b^3 - 75K^2b - 12K^2}{16(4b^2 - K^2)}, \quad (\text{B2})$$

$$K_{03} = -3u, \quad (\text{B3})$$

$$K_{11} = \frac{3K^2u}{4b^3(4b^2 - K^2)^2}, \quad (\text{B4})$$

$$K_{12} = -\frac{K^2u(272b^4 + 128kb^3 + 136K^2b^2 - 17K^4)}{64b^3(4b^2 - K^2)^2}, \quad (\text{B5})$$

$$K_{13} = -\frac{K^4u(4112b^4 + 2176Kb^3 + 2824K^2b^2 + 1088K^3b + 771K^4)}{4096b^4(4b^2 - K^2)^2}, \quad (\text{B6})$$

$$K_{21} = -\frac{2K^2u^2(656b^5 + 1802b^4K + 452b^3K^2 + 374b^2K^3 + 56bK^4 + 17K^5)}{9(4b^2 - K^2)^5}, \quad (\text{B7})$$

$$K_{22} = -\frac{K^2u^2(2584b^5 + 1408b^4K + 1258b^3K^2 + 176b^2K^3 + 119bK^4 + 8K^5)}{9(4b^2 - K^2)^5}, \quad (\text{B8})$$

$$K_{23} = \frac{K^4u^2}{24576(4b^3 - bK^2)^6} (15790080b^{12} + 9469952b^{11}K + 6037504b^{10}K^2 + 5431296b^9K^3 + 4420864b^8K^4 + 1114112b^7K^5 + 185088b^6K^6 - 208896b^5K^7 - 152592b^4K^8 + 13056b^3K^9 + 20808b^2K^{10} - 867K^{12}). \quad (\text{B9})$$

where the next nearest-neighbor interaction is set one quarter in strength as the nearest-neighbor interaction.

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