Distribution of the largest eigenvalue for real Wishart and Gaussian random matrices and a simple approximation for the Tracy-Widom distribution

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Abstract

We derive the exact distribution of the largest eigenvalue for finite dimensions real Wishart matrices and for the Gaussian Orthogonal Ensemble (GOE). We compare the exact distribution with the Tracy-Widom distribution which arises in many fields as the limiting distribution of the largest eigenvalue of large random matrices. In this regard we show that the Tracy-Widom distribution can be approximated by a properly scaled and shifted Gamma distribution, with great accuracy for the values of common interest in statistical applications.

Keywords: Random Matrix Theory, characteristic roots, largest eigenvalue, Tracy-Widom distribution, Wishart matrices, Gaussian Orthogonal Ensemble.

1. Introduction

The distribution of the largest eigenvalue of Wishart and Gaussian random matrices plays an important role in many fields of multivariate analysis, including principal component analysis, analysis of large data sets, communication theory and mathematical physics [1, 2].

The exact distribution of the largest eigenvalue for finite dimension uncorrelated central complex Wishart matrices is given in [3]. The extension to non-central uncorrelated complex Wishart is derived in [4], while the case of correlated central complex Wishart matrices with arbitrary one-sided correlation can be obtained by following the approach in [5, 6, 7].

In this paper we derive a simple expression for the exact distribution of the largest eigenvalue for real Wishart matrices and for the GOE, which can be used for arbitrary dimensions.

Also, we compare the exact distribution with the Tracy-Widom distribution, which arises in many fields as the limiting distribution of the largest eigenvalue of large random matrices, and whose applications include principal component analysis, analysis of large data sets, communication theory and mathematical physics [8, 9, 10, 11, 12, 13]. In this regard we show that the Tracy-Widom distribution can be approximated by a properly scaled and shifted Gamma distribution, with great accuracy for the values of common interest in statistical applications.

2. Exact distribution of the eigenvalues for finite dimensions Wishart and Gaussian symmetric matrices

We derive the exact distribution of the largest eigenvalue for real Wishart matrices and for random symmetric Gaussian matrices. For completeness we also summarize the analogous distributions for the complex case (Wishart and Gaussian Unitary Ensemble).

2.1. Real random matrices: uncorrelated Wishart and the Gaussian Orthogonal Ensemble (GOE)

Assume a Gaussian real $p \times m$ matrix $\mathbf{X}$ with independent, identically distributed (i.i.d.) columns, each with zero mean and covariance $\mathbf{\Sigma} = \mathbf{I}$. Denoting $n_{\min} = \min\{m, p\}$, $n_{\max} = \max\{m, p\}$, the distribution of the (real) ordered eigenvalues $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_{n_{\min}} \geq 0$ of the real Wishart matrix $\mathbf{W} = \mathbf{X}\mathbf{X}^T$ is given by [14, 1]

$$f_{\boldsymbol{\lambda}}(x_1, \ldots, x_{n_{\min}}) = K \prod_{i=1}^{n_{\min}} e^{-x_i/2} x_i^{(n_{\max}-n_{\min}-1)/2} \cdot \prod_{i<j}^{n_{\min}} (x_i - x_j) \tag{1}$$

where K is a normalizing constant given by

$$K = \frac{\pi^{n_{\min}^2/2}}{2^{n_{\min}n_{\max}/2}\Gamma_{n_{\min}}(n_{\max}/2)\Gamma_{n_{\min}}(n_{\min}/2)} \tag{2}$$

with $\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma(a - (i-1)/2)$.

Denoting $\mathbf{x} = [x_1, x_2, \ldots, x_{n_{\min}}]$, the probability distribution function (p.d.f.) in (1) can be written alternatively, in terms of the Vandermonde matrix $\mathbf{V}(\mathbf{x})$ with elements $\{x_j^{i-1}\}$, as

$$f_{\boldsymbol{\lambda}}(\mathbf{x}) = K \, |\mathbf{V}(\mathbf{x})| \prod_{i=1}^{n_{\min}} e^{-x_i/2} x_i^{\alpha} \tag{3}$$

where $\alpha \triangleq (n_{\max} - n_{\min} - 1)/2$ and $|\cdot|$ stands for determinant.

Similarly, for the Gaussian Orthogonal Ensemble the interest is in the distribution of the (real) eigenvalues for real $n \times n$ symmetric matrices whose entries are i.i.d. Gaussian $\mathcal{N}(0, 1/2)$ on the upper-triangle, and i.i.d. $\mathcal{N}(0, 1)$ on the diagonal [13]. Their joint p.d.f. is $f_{\boldsymbol{\lambda}}(x_1, \ldots, x_{n_{\min}}) \propto \prod_{i=1}^{n} e^{-x_i^2/2} \cdot \prod_{i<j}^{n} (x_i - x_j)$ [15, 13]. Then, by following the same approach as for Wishart, the joint p.d.f. of the ordered eigenvalues for these matrices can be written

$$f_{\boldsymbol{\lambda}}(\mathbf{x}) = K_{GOE} \, |\mathbf{V}(\mathbf{x})| \prod_{i=1}^{n} e^{-x_i^2/2} \tag{4}$$

where $K_{GOE} = (2^{n/2} \prod_{i=1}^{n} \Gamma[i/2])^{-1}$ is a normalizing constant. Note that the eigenvalues here are distributed over all the reals.

Theorem 1. *The cumulative distribution function (CDF) of the largest eigenvalue of the real Wishart matrix* $\mathbf{W}$ *is*

$$F_{\lambda_1}(x_1) = \Pr\{\lambda_1 \leq x_1\} = K\sqrt{|\mathbf{A}(x_1)|} \quad (5)$$

where for even n_{min} *the elements of the* $n_{\mathit{min}} \times n_{\mathit{min}}$ *skew-symmetric matrix* $\mathbf{A}(x_1)$ *are*

$$a_{i,j}(x_1) = 2^{2\alpha+i+j}\left\{\gamma\left(\alpha+i, \frac{x_1}{2}\right)\gamma\left(\alpha+j, \frac{x_1}{2}\right) + -2\int_0^{x_1/2} t^{\alpha+i-1}e^{-t}\gamma(\alpha+j,t)\,dt\right\} \quad (6)$$

for $i, j = 1, \ldots, n_{\mathit{min}}$. *Note that* $a_{i,j}(x_1) = -a_{j,i}(x_1)$ *and* $a_{i,i}(x_1) = 0$.

When n_{min} *is odd, the elements of the* $(n_{\mathit{min}}+1)\times(n_{\mathit{min}}+1)$ *skew-symmetric matrix* $\mathbf{A}(x_1)$ *are as in* (6)*, with the additional elements*

$$\begin{aligned} a_{i,n_{\mathit{min}}+1}(x_1) &= 2^{\alpha+i}\gamma\left(\alpha+i, \frac{x_1}{2}\right) && i = 1, \ldots, n_{\mathit{min}} \\ a_{n_{\mathit{min}}+1,j}(x_1) &= -a_{j,n_{\mathit{min}}+1}(x_1) && j = 1, \ldots, n_{\mathit{min}} \\ a_{n_{\mathit{min}}+1,n_{\mathit{min}}+1}(x_1) &= 0 \end{aligned}$$

where $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$ *is the lower incomplete gamma function.*

Proof 1. *We start by writing*

$$F_{\lambda_1}(x_1) = \Pr\{\lambda_1 \leq x_1\} = \underset{\mathcal{D}(x_1)}{\int\cdots\int} f_{\boldsymbol{\lambda}}(w_1, \ldots, w_{n_{\mathit{min}}})d\mathbf{w} \quad (7)$$

with $\mathcal{D}(x_1) = \{w_1, \ldots, w_{n_{\mathit{min}}} : x_1 \geq w_1 \geq w_2 \ldots \geq w_{n_{\mathit{min}}} \geq 0\}$.

Then we have

$$\underset{\mathcal{D}(x_1)}{\int\cdots\int} f_{\boldsymbol{\lambda}}(w_1, \ldots, w_{n_{\mathit{min}}})d\mathbf{w} = K\underset{\mathcal{D}(x_1)}{\int\cdots\int} |\mathbf{V}(\mathbf{w})| \prod_{i=1}^{n_{\mathit{min}}} \xi(w_i)\,d\mathbf{w} \quad (8)$$

where $\xi(x) = e^{-x/2}x^{\alpha}$. *To evaluate this integral we recall that for a generic* $m \times m$ *matrix* $\mathbf{\Phi}(\mathbf{w})$ *with elements* $\{\Phi_i(w_j)\}$ *the following identity holds [16, 17]*

$$\underset{b\geq w_1 \geq w_2 \ldots \geq w_m \geq a}{\int\cdots\int} |\mathbf{\Phi}(\mathbf{w})|\,d\mathbf{w} = Pf(\mathbf{A}) \quad (9)$$

where $Pf(\mathbf{A}) = \sqrt{|\mathbf{A}|}$ *is the Pfaffian, and the skew-symmetric matrix* $\mathbf{A}$ *is* $m \times m$ *for* m *even, and* $(m+1)\times(m+1)$ *for* m *odd, with*

$$a_{i,j} = \int_a^b \int_a^b sgn(y-x)\Phi_i(x)\Phi_j(y)dxdy \qquad i, j = 1, \ldots, m. \quad (10)$$

For m *odd the additional elements are* $a_{i,m+1} = -a_{m+1,i} = \int_a^b \Phi_i(x)dx$, $i = 1, \ldots, m$, *and* $a_{m+1,m+1} = 0$.

Using (9) *in* (8) *with* $a = 0, b = x_1, \Phi_i(x) = x^{i-1}e^{-x/2}x^{\alpha}$ *with some simple manipulations gives Theorem 1.*

Theorem 2. *The CDF of the largest eigenvalue for the Gaussian Orthogonal Ensemble (GOE) matrices is*

$$F_{\lambda_1}(x_1) = \Pr\{\lambda_1 \leq x_1\} = K_{GOE}\sqrt{|\mathbf{A}(x_1)|} \tag{11}$$

where for even n the elements of the $n \times n$ skew-symmetric matrix $\mathbf{A}(x_1)$ are

$$a_{i,j}(x_1) = \psi(i, x_1)\psi(j, x_1) - 2\int_{-\infty}^{x_1} t^{i-1} e^{-t^2/2}\psi(j, x)dt \tag{12}$$

and

$$\psi(i, x) \triangleq 2^{\frac{i}{2}-1}\left(\mathit{sgn}(x)^i \gamma\left(\frac{i}{2}, \frac{x^2}{2}\right) - (-1)^i \Gamma\left(\frac{i}{2}\right)\right)$$

for $i, j = 1, \ldots, n$.

When n is odd, the elements of the $(n+1)\times(n+1)$ skew-symmetric matrix $\mathbf{A}(x_1)$ are as in (12)*, with the additional elements*

$$\begin{aligned} a_{i,n+1}(x_1) &= \psi(i, x_1) \qquad i = 1, \ldots, n \\ a_{n+1,j}(x_1) &= -a_{j,n+1}(x_1) \qquad j = 1, \ldots, n \\ a_{n+1,n+1}(x_1) &= 0. \end{aligned}$$

Proof 2. *Starting from* (4) *the proof is similar to that for Theorem 1.*

2.2. Complex random matrices: uncorrelated Wishart and the Gaussian Unitary Ensemble (GUE)

Assume now a Gaussian complex $p \times m$ matrix $\mathbf{X}$ with i.i.d. columns, each with zero mean and covariance $\mathbf{\Sigma}$. The distribution of the (real) ordered eigenvalues of the complex Wishart matrix $\mathbf{W} = \mathbf{X}\mathbf{X}^H$ is known since many years from [14] in terms of hypergeometric functions of matrix arguments. Unfortunately, the expressions given in [14] are not easy to use, due to the difficulties in evaluating zonal polynomials. The first expression of practical usage for the joint distribution of the eigenvalues of a complex Wishart matrix with correlation has been given in [5] by expressing the hypergeometric function of matrix arguments as product of determinants of matrices. More recently, that approach has been expanded to cover the case where $\mathbf{\Sigma}$ has eigenvalues of arbitrary multiplicity, and to find several statistics regarding the marginal eigenvalues distribution [6, 7, 18]. By using these approaches, the exact statistics of an arbitrary subset of the ordered eigenvalues can be evaluated easily for finite dimension complex quadratic forms and Wishart (uncorrelated and correlated) matrices.

Regarding the largest eigenvalue statistic, below we report a known result for the particular case of uncorrelated complex Wishart matrices (i.e., for $\mathbf{\Sigma} = \mathbf{I}$).

Theorem 3. *The CDF of the largest eigenvalue of the uncorrelated complex Wishart matrix $\mathbf{W}$ is [3]*

$$F_{\lambda_1}(x_1) = \Pr\{\lambda_1 \leq x_1\} = K_C\,|\mathbf{A}(x_1)| \tag{13}$$

where the elements of the $n_{min} \times n_{min}$ matrix $\mathbf{A}(x_1)$ are

$$a_{i,j}(x_1) = \int_0^{x_1} t^{n_{max}-n_{min}+i+j-2} e^{-t} dt = \gamma\left(n_{max} - n_{min} + i + j - 1, x_1\right) \tag{14}$$

and K_C is a normalizing constant given by

$$K_C = \frac{\pi^{n_{min}(n_{min}-1)}}{\tilde{\Gamma}_{n_{min}}(n_{max})\tilde{\Gamma}_{n_{min}}(n_{min})} \tag{15}$$

with $\tilde{\Gamma}_m(n) = \pi^{m(m-1)/2} \prod_{i=1}^{m}(n-i)!$.

The following is a similar result for complex hermitian random matrices with i.i.d. $\mathcal{CN}(0, 1/2)$ entries on the upper-triangle, and $\mathcal{N}(0, 1/2)$ on the diagonal. These matrices constitute the so called Gaussian Unitary Ensemble (GUE) [13].

Theorem 4. *The CDF of the largest eigenvalue for the GUE is*

$$F_{\lambda_1}(x_1) = \Pr\{\lambda_1 \le x_1\} = K_{GUE} \left|\mathbf{A}(x_1)\right| \tag{16}$$

where the elements of the $n \times n$ *matrix* $\mathbf{A}(x_1)$ *are*

$$a_{i,j}(x_1) = \int_{-\infty}^{x_1} t^{i+j-2} e^{-t^2} dt = \frac{1}{2}\left[\gamma\left(\frac{i+j-1}{2}, x_1^2\right) sgn(x_1)^{i+j-1} + (-1)^{i+j}\right] \tag{17}$$

and $K_{GUE} = 2^{n(n-1)/2}(\pi^{n/2} \prod_{i=1}^{n} \Gamma[i])^{-1}$ *is a normalizing constant.*

Proof 3. *For the GUE the joint distribution of the ordered eigenvalues can be written as [13]*

$$f_{\boldsymbol{\lambda}}(\mathbf{x}) = K_{GUE} \left|\mathbf{V}(\mathbf{x})\right|^2 \prod_{i=1}^{n} e^{-x_i^2} \tag{18}$$

Then, by using [6, Th. 7] with $a = -\infty, b = x_1, \Psi_i(x_j) = \Phi_i(x_j) = x_j^{i-1}, \xi(x) = e^{-x^2}$ *we get immediately the result.*

These thorems can be used for finite dimensional random matrices of limited dimensions in the uncorrelated case. For the extension of Theorem 3 to correlated complex Wishart see [18, 6, 7] and references therein.

We remark that the previous theorems can be used to obtain explicit expressions for the distribution of the largest eigenvalue. In fact, $\gamma(a, x)$ can be written as combinations of exponentials and powers of x when a is an integer, and as the combination of exponentials, powers of x, and erf(x) when a is a multiple of $1/2$. Thus, we can write explicit expressions for the CDF and p.d.f. of λ_1 for all previous theorems.

For example, by expanding (5) we derived the following expressions for the CDF of the largest eigenvalue for real Wishart matrices.

For $n_{\text{min}} = n_{\text{max}} = 2$:

$$F_{\lambda_1}(x_1) = \sqrt{\frac{x_1 \pi}{2}} e^{-x_1/2} \text{erf} \sqrt{\frac{x_1}{2}} + e^{-x_1} - 1 \tag{19}$$

For $n_{\text{min}} = n_{\text{max}} = 3$:

$$F_{\lambda_1}(x_1) = e^{-3x/2}\left(e^{x/2}\left(e^x - x - 1\right)\operatorname{erf}\sqrt{\frac{x}{2}} - \sqrt{\frac{2x}{\pi}}\left(e^x(x-1)+1\right)\right) \tag{20}$$

For $n_{\text{min}} = n_{\text{max}} = 4$:

$$\begin{aligned} F_{\lambda_1}(x_1) &= \frac{e^{-2x}}{\sqrt{32}}\Big(\sqrt{2}\left(4e^{2x} - e^x\left(x^3+2x^2+2x+8\right)+2(x+2)\right) + \\ &\quad -\sqrt{\pi x}e^{x/2}\left(e^x\left(x^2-4x+6\right)-2(x+3)\right)\operatorname{erf}\sqrt{\frac{x}{2}}\Big) \end{aligned} \tag{21}$$

Similar expressions can be derived for the p.d.f., for complex Wishart, for GOE and for GUE. These expressions becomes cumbersome for large matrices.

3. Limiting behavior for large random matrices: the Tracy-Widom distribution

The pioneering works [8, 9] and [11] have shown the importance of the Tracy-Widom distribution, which arises in many fields as the limiting distribution of the largest eigenvalue of large random matrices. This distribution, originally derived in the study of the Gaussian unitary ensemble, has been shown to be related to many areas concerned with large random matrices. Applications include principal component analysis, analysis of large data sets, combinatorics, communication theory, representation theory, probability, statistics and mathematical physics [10, 11, 19, 12, 13, 20].

For example, it has been shown in [11, 12] for principal component analysis (PCA) that if $\mathbf{X}$ is an $n \times p$ matrix whose entries are i.i.d. standard Gaussian and λ_1 is the largest eigenvalue of $\mathbf{X}\mathbf{X}^H$, then for $n, p \to \infty$

$$\frac{\lambda_1 - \mu_{np}}{\sigma_{np}} \xrightarrow{\mathcal{D}} \mathcal{TW}_\beta \tag{22}$$

where $\mathcal{TW}_\beta$ denotes a random variable (r.v.) with Tracy-Widom distribution of order β, for $\beta = 1, 2, 4$. In the previous expression $\beta = 1$ when the entries of $\mathbf{X}$ are standard real Gaussian, and $\beta = 2$ when the entries are standard complex Gaussian. We recall that a random variable Z is said to have a standard complex Gaussian distribution (denoted $\mathcal{CN}(0,1)$) if $Z = (Z_1 + iZ_2)$, where Z_1 and Z_2 are i.i.d. real Gaussian $\mathcal{N}(0, 1/2)$. The scaling and centering parameters in (22) are [11, 12]

$$\mu_{np} = \left(\sqrt{n+a_1} + \sqrt{p+a_2}\right)^2 \tag{23}$$

$$\sigma_{np} = \sqrt{\mu_{np}}\left(\frac{1}{\sqrt{n+a_1}} + \frac{1}{\sqrt{p+a_2}}\right)^{1/3} \tag{24}$$

where the best adjustment parameters a_1, a_2 are known to be $a_1 = a_2 = -1/2$ for real Wishart ($\beta = 1$) and $a_1 = a_2 = 0$ for complex Wishart ($\beta = 2$). A similar behavior can be proved for more general conditions when the entries of $\mathbf{X}$ are not Gaussian [19, 21]. Due to the simplicity of this result, the Tracy-Widom distribution is of extreme usefulness for problems involving PCA with large dimensional matrices.

The Tracy-Widom CDFs are given by [8, 9, 13]

$$F_1(x) = \exp\left\{-\frac{1}{2}\int_x^\infty q(y) + (y-x)q^2(y)dy\right\} \tag{25}$$

$$F_2(x) = \exp\left\{-\int_x^\infty (y-x)q^2(y)dy\right\} \tag{26}$$

$$F_4\left(\frac{x}{\sqrt{2}}\right) = \cosh\left\{\frac{1}{2}\int_x^\infty q(y)dy\right\}\sqrt{F_2(x)} \tag{27}$$

where $q(y)$ is the unique solution to the Painvalé II differential equation

$$q''(y) = yq'(y) + 2q^3(y) \tag{28}$$

satisfying the condition

$$q(y) \sim \ Ai(y) \qquad y \to \infty \tag{29}$$

and $Ai(y)$ denotes the Airy function.

The function $F_4(x)$ can be derived from the other two. In fact, from (25)(26) and (27) we can write

$$F_4\left(x\right) = \frac{1}{2}\left(F_1(x\sqrt{2}) + \frac{F_2(x\sqrt{2})}{F_1(x\sqrt{2})}\right) \tag{30}$$

and

$$f_4(x) = \frac{1}{\sqrt{2}}\left[f_1(x\sqrt{2}) + \frac{f_2(x\sqrt{2})F_1(x\sqrt{2}) - F_2(x\sqrt{2})f_1(x\sqrt{2})}{F_1^2(x\sqrt{2})}\right] \tag{31}$$

where $f_\beta(x) = dF_\beta(x)/dx$ is the p.d.f.. So in the following we will mainly focus on $F_1(x)$ and $F_2(x)$.

These distributions can be evaluated numerically by solving the Painlevé II differential equation (28) or the corresponding Fredholm determinant [8, 11, 12, 22, 13, 23].

In this paper we propose a very simple approximation for the Tracy-Widom distribution, to avoid the need for numerical solution of differential equations of Fredholm determinants. The approximation is shown to be extremely accurate for values of the CDF or of the complementary complementary cumulative distribution function (CCDF) of practical uses.

4. A simple approximation of the Tracy-Widom distribution based on the gamma distribution

It is known that the exact distribution of the largest eigenvalue of a complex Wishart matrix is a mixture of gamma distributions, i.e., its p.d.f. can be expressed as the

Table 1: Parameters for Approximating $\mathcal{TW}_\beta$ with $\Gamma[k,\theta]-\alpha$.

	$\mathcal{TW}_1$	$\mathcal{TW}_2$	$\mathcal{TW}_4$
k	46.446	79.6595	146.021
θ	0.186054	0.101037	0.0595445
α	9.84801	9.81961	11.0016

weighted sum of terms $x^a e^{-bx}$ (see [3] for the uncorrelated case and [6, 18] for one-sided correlated Wishart matrices). For finite dimensions matrices, it can be shown that the exact distribution is very well approximated by a (single) gamma distribution, with proper parameters chosen to match the first moments of the true distribution.

Based on these observations, we propose the approximation

$$\mathcal{TW}_\beta \simeq \mathcal{G} - \alpha \tag{32}$$

where α is a constant, and $\mathcal{G} \sim \Gamma(k,\theta)$ denotes a Gamma r.v. with shape parameter k and scale parameter θ. Thus the CDF and p.d.f. of $\mathcal{TW}_\beta$ are approximated as:

$$F_\beta(x) \simeq \frac{1}{\Gamma[k]}\gamma\left(k, \frac{x+\alpha}{\theta}\right) \ , \qquad\qquad x > -\alpha \tag{33}$$

$$f_\beta(x) \simeq \frac{1}{\Gamma[k]\theta^k}\left(x+\alpha\right)^{k-1} e^{-\frac{x+\alpha}{\theta}} \ , \qquad\qquad x > -\alpha \tag{34}$$

where $\Gamma[.]$ is the Gamma function, and $\gamma\left(k,x\right) = \int_0^x t^{k-1}e^{-t}dt$ is the lower incomplete Gamma function.

The parameters k, θ, α should be suitably chosen according to some criterion. For example, we have chosen to set k, θ, α for matching the first three moments of the distributions $\mathcal{TW}_\beta$. To this aim we recall that for the Gamma r.v. the meas is $\mathbb{E}\left\{\mathcal{G}\right\} = k\theta$, the variance is var $\left\{\mathcal{G}\right\} = k\theta^2$ and the skewness is Skew $\left\{\mathcal{G}\right\} = \frac{2}{\sqrt{k}}$. If $\mu_\beta, \sigma_\beta^2, S_\beta$ are the mean, variance and skewness of the Tracy-Widom (see e.g. [12]), matching the first three moments gives:

$$k \;=\; \frac{4}{S_\beta^2} \tag{35}$$

$$\theta \;=\; \sigma_\beta \frac{S_\beta}{2} \tag{36}$$

$$\alpha \;=\; k\theta - \mu_\beta \tag{37}$$

The parameters for the approximation (33) (34) obtained from these equations are reported in Table 1.

The comparison with pre-calculated p.d.f. values from [22] is shown in Fig. 1.Since in linear scale the exact and approximated distributions are practically indistinguishable, in Fig. 2 we report the CDF and CCDF in logarithmic scale for Tracy-Widom 2 (similar for the others). It can be seen that the approximation is in general very good for all values of the CDF of practical interest. In particular there is an excellent agreement between the exact and approximate distributions for the right tail. The left tail is less accurate but still of small relative error for values of the CDF of practical statistical uses. Note that, differently from the true distribution which goes to zero only asymptotically, the left tail is exactly zero for $x < -\alpha$.

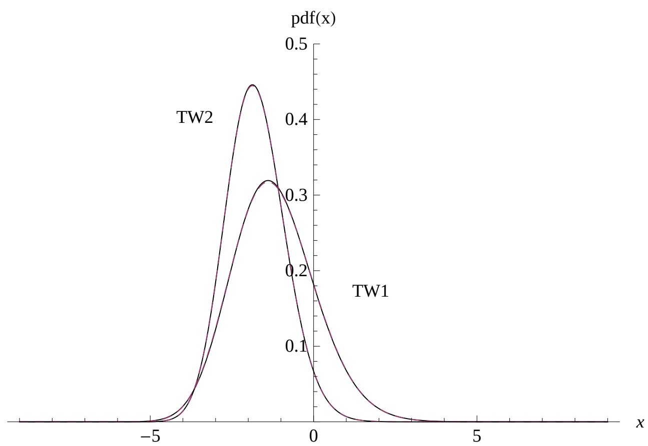

Figure 1: Comparison between the exact (continuous line) and approximated (dashed) PDF for the Tracy-Widom 1 and Tracy-Widom 2. The exact and approximated curves are practically indistinguishable on this scale.

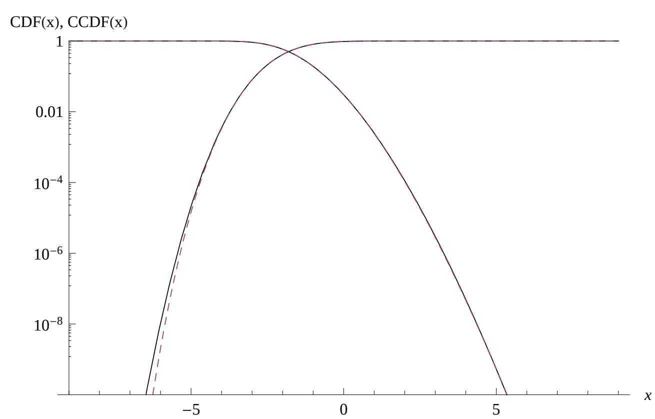

Figure 2: Comparison between the exact (continuous line) and approximated (dashed) CDF, CCDF, Tracy-Widom $\mathcal{TW}_2$, log scale. The two CCDF are practically indistinguishable.

Some specific values are given in Table 2 and 3 where it can be noted that, for values of common use, the relative error is small.

Table 2: Precision of the approximation: CDF of $\mathcal{TW}_1$ vs. $\Gamma[k,\theta]-\alpha$ for some percentiles.

x	Target CDF	CDF [22]	CDF approximation	CDF rel. error (%)
-4.64	0.001	0.0011	0.0009	-17.40
-3.90	0.010	0.0099	0.0095	-4.02
-3.18	0.050	0.0500	0.0501	0.16
-2.78	0.100	0.1004	0.1010	0.65
-1.91	0.300	0.3001	0.3011	0.32
-1.27	0.500	0.4995	0.4995	-0.01
-0.59	0.700	0.7006	0.6998	-0.12
0.45	0.900	0.9000	0.8996	-0.04
0.98	0.950	0.9500	0.9500	-0.00
2.02	0.990	0.9899	0.9901	0.02
3.24	0.999	0.9989	0.9990	0.01

5. Numerical results

The calculation of the exact distribution of the largest eigenvalue is easy by using Theorems 1-4 for not too large random matrices. For example, we show in Fig. 3 and Fig. 4 the distribution of the largest eigenvalue for real and complex Wishart matrices, with different $n_{\text{min}}, n_{\text{max}}$. In the figure we report the exact distributions given by (5), (13) and the centered and scaled Tracy-Widom distribution (22) (here we can use the exact Tracy-Widom or the approximations (33) which are non distinguishable in this scale).

In Fig. 5 and Fig. 6 we report the exact distribution for GOE (eq. (11)) and for GUE (eq. (16)), for $n = 2, 5, 10, 15, 20, 25$. In the same figures we report the approximation based on the Tracy-Widom distribution, which in these cases is [8, 9, 13]:

$$\frac{\lambda_1 - \mu_n'}{\sigma_n'} \xrightarrow{\mathcal{D}} \mathcal{TW}_\beta \tag{38}$$

with $\mu_n' = 2\sigma_0\sqrt{n-a_1}$ and $\sigma_n' = \sigma_0(n-a_2)^{-1/6}$, where $\sigma_0^2 = 1/2$ is the variance of the off-diagonal elements in the ensembles in our normalization. In the previous expression we must use $\beta = 1$ and $\beta = 2$ for the GOE and GUE, respectively. While [13] indicates $a_1 = a_2 = 0$, we have observed that the approximations are better for small n if we choose the adjusting parameters $a_1 = a_2 = 1/2$ for the GOE and $a_1 = 0, a_2 = 1$ for the GUE.

We note that, for large dimension problems, the asymptotic distributions predicted by the Tracy-Widom laws converge soon to the exact. In particular, for GOE and GUE the properly scaled and centered Tracy-Widom laws are already very close to the exact for very small matrices ($n = 2$). Also, we remark that the simple approximations (33), (34) can be used instead of the pre-calculated tables for the Tracy-Widom distribution for values of practical interest in statistic.

Table 3: Precision of the approximation: CDF of $\mathcal{TW}_2$ vs. $\Gamma[k,\theta]-\alpha$ for some percentiles.

x	Target CDF	CDF [22]	CDF approximation	CDF rel. error (%)
-4.29	0.001	0.0010	0.0009	-8.89
-3.72	0.010	0.0102	0.0100	-1.77
-3.19	0.050	0.0505	0.0506	0.16
-2.90	0.100	0.1003	0.1006	0.35
-2.26	0.300	0.3025	0.3029	0.14
-1.80	0.500	0.5022	0.5021	-0.02
-1.32	0.700	0.7018	0.7014	-0.06
-0.59	0.900	0.9012	0.9011	-0.02
-0.23	0.950	0.9503	0.9503	0.00
0.48	0.990	0.9901	0.9901	0.01
1.31	0.999	0.9990	0.9990	0.00

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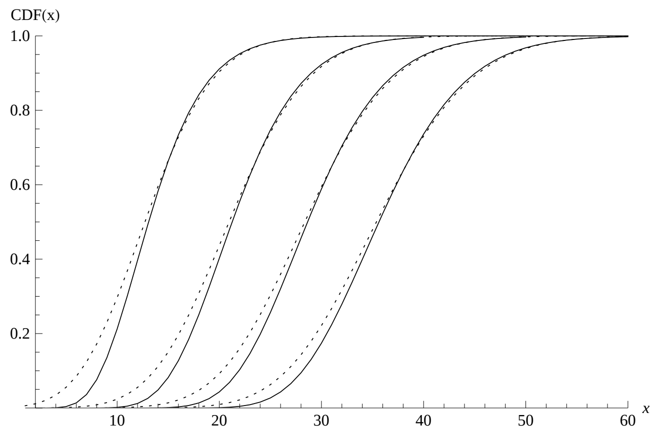

Figure 3: CDF of the largest eigenvalue, real Wishart matrix, $n_{\min} = 5, n_{\max} = 5, 10, 15, 20$. Comparison between the exact distribution (5) (continuous line) and the scaled and centered $\mathcal{TW}_1$ as in (22) (dotted).

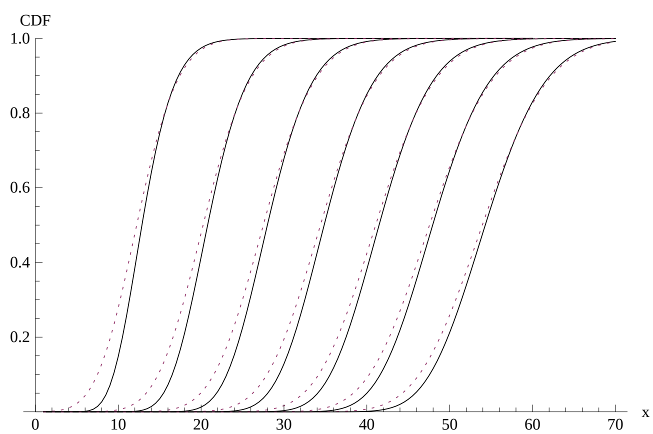

Figure 4: CDF of the largest eigenvalue, complex Wishart matrix, $n_{\min} = 5$, $n_{\max} = 5, 10, 15, 20, 25, 30$. Comparison between the exact distribution (13) (continuous line) and the scaled and centered $\mathcal{TW}_2$ as in (22) (dotted).

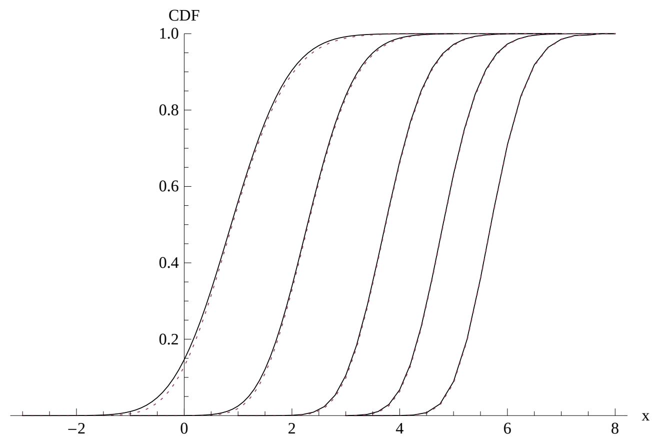

Figure 5: CDF of the largest eigenvalue, GOE. From left to right: $n = 2, 5, 10, 15, 20$. Comparison between the exact distribution (11) (continuous lines) and the scaled and centered $\mathcal{TW}_1$ as in (38) (dotted lines).

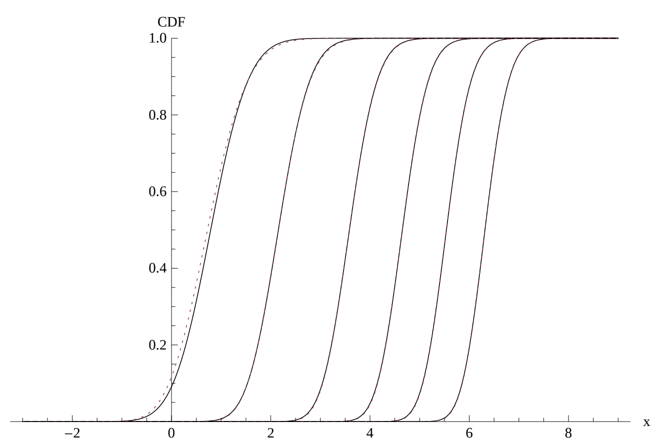

Figure 6: CDF of the largest eigenvalue, GUE. From left to right: $n = 2, 5, 10, 15, 20, 25$. Comparison between the exact distribution (16) (continuous lines) and the scaled and centered $\mathcal{TW}_2$ as in (38) (dotted lines).