

# Spectral Risk Measures, With Adaptions For Stochastic Optimization

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## Abstract

Stochastic optimization problems often involve the expectation in its objective. When risk is incorporated in the problem description as well, then risk measures have to be involved in addition to quantify the acceptable risk, often in the objective. For this purpose it is important to have an adjusted, adapted and efficient evaluation scheme for the risk measure available. In this article different representations of an important class of risk measures, the spectral risk measures, are elaborated. The results allow concise problem formulations, they are particularly adapted for stochastic optimization problems. Efficient evaluation algorithms can be built on these new results, which finally make optimization problems involving spectral risk measures eligible for stochastic optimization.

## 1 Introduction

Spectral risk measures have been introduced and studied first in [?] and [?], they represent a risk measure in the general axiomatic environment introduced by Artzner et al. in the seminal paper [?]. An important study of spectral risk measures, although under the name *distortion functional*, was provided in [?]. The concept of spectral risk measures and distortion functionals is essentially the same – they differ just in an opposite sign, resulting in a concave rather than a convex description. But this additional name *distortion functional* exhibits the interpretation that some outstanding observations are overvalued, whereas others are undervalued, creating a somehow distorted overall picture for the entire observations.

Spectral risk measures constitute an elementary and important class of risk measures, as every risk measure can be described as a supremum of spectral risk measures. They are moreover defined in an explicit way, hence there is an explicit evaluation scheme available.

The most important spectral risk measure, which made its way to the top, is the Average Value-at-Risk. It is the essential advantage of the Average Value-at-Risk that different representations are known, which makes this risk measure eligible in varying situations: by conjugate duality there is an expression in the form of a supremum, but in applications, particularly in optimization, an equivalent expression as an infimum is extremely convenient: developed in the paper *Optimization of Conditional Value-at-Risk* [?] in this journal, and brought to beauty in *Some Remarks on the*

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*Value-at-Risk and the Conditional Value-at-Risk* [?]. In this article we extend both formulations to spectral risk measures. A description as a supremum is a first result of this article. Besides that – and this is of importance for applications and the main result – a description involving an infimum is elaborated as well. In comparison with the Average Value-at-Risk there is a price to pay to allow a representation as an infimum, as the decision space substantially increases.

The alternative representation of spectral risk measures is of particular importance for stochastic programming, as the new formulation allows a direct approach to solve stochastic optimization problems by employing spectral risk measures in the objective. Applications in portfolio optimization (asset allocation) are immediate.

The article is organized as follows. The spectral risk measure is introduced in Section 2. Its description as a supremum is contained in Section 3, and as an infimum in Section 4. The implications for stochastic optimization (asset allocation, e.g.) are outlined and explained in Section 5.

## 2 Spectral Risk Measures

Risk measures evaluate on random variables defined on a probability space  $(\Omega, \Sigma, P)$ . In the following definition we follow the axiomatic environment which has been proposed and formulated in [?]. Often, risk measures are restricted to non-atomic probability spaces. To make the results eligible for stochastic optimization and numerical approximations we emphasize that we do not require the probability space to be atom-less.

**Definition 1.** A positively homogeneous *risk measure* is a mapping  $\mathcal{R}: L^p \rightarrow \mathbb{R} \cup \{\infty\}$  with the following properties:

- (M) MONOTONICITY:  $\mathcal{R}(Y_1) \leq \mathcal{R}(Y_2)$  whenever  $Y_1 \leq Y_2$  almost surely;
- (C) CONVEXITY:  $\mathcal{R}((1 - \lambda)Y_0 + \lambda Y_1) \leq (1 - \lambda)\mathcal{R}(Y_0) + \lambda\mathcal{R}(Y_1)$  for  $0 \leq \lambda \leq 1$ ;
- (T) TRANSLATION EQUIVARIANCE<sup>1</sup>:  $\mathcal{R}(Y + c) = \mathcal{R}(Y) + c$  if  $c \in \mathbb{R}$ ;
- (H) POSITIVE HOMOGENEITY:  $\mathcal{R}(\lambda Y) = \lambda\mathcal{R}(Y)$  whenever  $\lambda > 0$ .

In the convex and monotone setting exposed in Definition 1 the random variable  $Y$  can be naturally associated with loss, it is hence common, for example, in an actuarial context in the insurance industry. In a banking or investment environment the interpretation of a reward is more natural, in this context the mapping  $\rho: Y \mapsto \mathcal{R}(-Y)$  is often considered – and called *coherent risk measure* – instead (note, that essentially the monotonicity condition (M) and translation property (T) reverse for  $\rho$ ).

The term *acceptability functional* is often employed in energy or decision theory to quantify and classify acceptable strategies. In this context the concave mapping  $\mathcal{A}: Y \mapsto -\mathcal{R}(-Y)$ , the acceptability functional, is employed instead (here, (C) modifies to concavity).

An important class of positively homogeneous risk measures is provided by spectral risk measures.

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<sup>1</sup>In an economic or monetary environment this is often called CASH INVARIANCE instead.

**Definition 2** (Spectral Risk Measure). For  $\sigma \in L^1(0, 1)$  a non-decreasing probability density function (i.e.  $\sigma \geq 0$  and  $\int_0^1 \sigma(u) du = 1$ ) a mapping

$$\begin{aligned} \mathcal{R}_\sigma : L^1 &\rightarrow \mathbb{R} \cup \{\infty\} \\ Y &\mapsto \int_0^1 F_Y^{-1}(\alpha) \sigma(\alpha) d\alpha \end{aligned} \quad (1)$$

is called *spectral risk measure* with *spectrum* (or *spectral density*)  $\sigma$ .

$F_Y^{-1}(\alpha)$  is the left-continuous, lower semi-continuous *quantile* or *Value-at-Risk* at level  $\alpha$  (sometimes also *lower inverse cdf*),

$$F_Y^{-1}(\alpha) := \text{V@R}_\alpha(Y) := \inf \{y : P(Y \leq y) \geq \alpha\}.$$

It is evident that for general  $\sigma \in L^1$  and  $Y \in L^1$  the risk measure  $\mathcal{R}_\sigma$  might evaluate to  $+\infty$  or  $-\infty$ . To overcome we will occasionally require  $\sigma \in L^\infty(0, 1)$  or  $Y \in L^\infty$  and state this explicitly in the text.

*Remark 3.* For  $Y \geq 0$  a.s. the spectral risk measures can be expressed by use of  $Y$ 's cdf ( $F_Y(y) := P(Y \leq y)$ ) directly, it has the representation

$$\mathcal{R}_\sigma(Y) = \int_0^\infty \tau_\sigma(F_Y(q)) dq, \quad (2)$$

where  $\tau_\sigma = \int_\alpha^1 \sigma(p) dp$ . This formulation can be found in insurance (the loss is usually positive), but as well when associating  $Y$  with cost.

The Average Value-at-Risk is the most important spectral risk measure.

**Definition 4** (Average Value-at-Risk). The spectral risk measure with spectrum

$$\sigma_\alpha(\cdot) := \frac{1}{1-\alpha} \mathbb{1}_{(\alpha, 1]}(\cdot) \quad (3)$$

is

$$\text{AV@R}_\alpha(Y) = \frac{1}{1-\alpha} \int_\alpha^1 \text{V@R}_p(Y) dp,$$

the *Average Value-at-Risk* at level  $\alpha$  ( $0 \leq \alpha < 1$ ). The Average Value-at-Risk at level  $\alpha = 1$  is

$$\text{AV@R}_1(Y) := \lim_{\alpha \nearrow 1} \text{AV@R}_\alpha(Y) = \text{ess sup } Y.$$

Due to the defining equation (1) of the spectral risk measure the same real number is assigned to all random variables  $Y$  sharing the same law, irrespective of the underlying probability space. This gives rise to the notion of version independence:

**Definition 5.** A risk measure  $\mathcal{R}$  is *version independent*<sup>2</sup>, if  $\mathcal{R}(Y_1) = \mathcal{R}(Y_2)$  whenever  $Y_1$  and  $Y_2$  share the same law, that is  $P(Y_1 \leq y) = P(Y_2 \leq y)$  for all  $y \in \mathbb{R}$ .

<sup>2</sup>sometimes also *law invariant* or *distribution based*.

The following representation underlines the central role of the Average Value-at-Risk for version independent risk measures. Moreover, it is the basis and justification for investigating spectral risk measures in much more detail.

**Theorem 6** (Kusuoka's representation). *Any version independent, positively homogeneous and lower semi-continuous risk measure  $\mathcal{R}$  on  $L^\infty$  of an atom-less probability space has the representation*

$$\mathcal{R}(Y) = \sup_{\mu \in \mathcal{M}} \int_0^1 \text{AV@R}_\alpha(Y) \mu(d\alpha), \quad (4)$$

where  $\mathcal{M}$  is a set of probability measures on  $[0, 1]$ .

*Proof.* Cf. [?, ?, ?]. □

In the context of spectral risk measures it is essential to observe that any spectral risk measure has an immediate representation as in (4), the measure  $\mu_\sigma$  corresponding to the density  $\sigma$  is

$$\mu_\sigma(A) := \sigma(0) \delta_0(A) + \int_A 1 - \alpha \, d\sigma(\alpha) \quad (A \subset [0, 1], \text{ measurable}).$$

$\mu_\sigma$  is a positive measure since  $\sigma$  is non-decreasing, and integration by parts reveals that it is a probability measure. Kusuoka's representation is immediate by Riemann-Stieltjes integration by parts for the set  $\mathcal{M} = \{\mu_\sigma\}$ , as

$$\begin{aligned} \int_0^1 \text{AV@R}_\alpha(Y) \mu_\sigma(d\alpha) &= \sigma(0) \text{AV@R}_0(Y) + \int_0^1 \frac{1}{1-\alpha} \int_\alpha^1 \text{V@R}_p(Y) \, dp \, (1-\alpha) \, d\sigma(\alpha) \\ &= \int_0^1 \text{V@R}_p(Y) \sigma(p) \, dp = \mathcal{R}_\sigma(Y). \end{aligned}$$

Conversely, the risk measure  $\int_0^1 \text{AV@R}_\alpha(Y) \mu(d\alpha)$  in Kusuoka's representation (6) often can be expressed as a spectral risk measure with spectral density  $\sigma_\mu$ , this is accomplished by the function

$$\sigma_\mu(\alpha) := \int_0^\alpha \frac{1}{1-p} \mu(dp). \quad (5)$$

Provided that  $\sigma_\mu$  is well defined (notice that possibly  $\mu(\{1\}) > 0$  is a problem to compute  $\sigma(1)$ ) it is positive and a density, as  $\int_0^1 \sigma(\alpha) \, d\alpha = \int_0^1 \frac{1}{1-p} \int_p^1 d\alpha \mu(dp) = 1$ .

As a matter of fact every risk measure  $\mathcal{R}$  allows a representation in terms of spectral risk measures as

$$\mathcal{R}(Y) = \sup_{\sigma \in \mathcal{S}} \mathcal{R}_\sigma(Y),$$

where  $\mathcal{S}$  is a set of continuous and (except for the expectation, for which  $\mathcal{S} = \{\mathbf{1}\}$ ) strictly increasing (thus invertible) density functions. A rigorous discussion is rather straight forward, although beyond the scope of this article.

### 3 Supremum-Representation of Spectral Risk Measures

The duality theory based on the Legendre transform is a usual way to develop alternative representations of convex functions. The resulting representation provides an expression of the spectral risk measure as a supremum.

**Definition 7** (Dual function). The *dual function* (convex conjugate) of an risk measure  $\mathcal{R}$  is defined as

$$\mathcal{R}^*(Z) = \sup_{Y \in L^\infty} \mathbb{E}YZ - \mathcal{R}(Y) \quad (Z \in L^1).$$

A random variable  $Z$  is called *feasible* for  $\mathcal{R}$  if

$$\mathcal{R}^*(Z) < \infty.$$

It follows from the Fenchel-Moreau duality theorem (cf. [?]) that <sup>3</sup>

$$\mathcal{R}(Y) = \sup_{Z \in L^1} \mathbb{E}YZ - \mathcal{R}^*(Z),$$

provided that  $\mathcal{R}$  is lower semi-continuous (cf. [?] and [?] for characterizations on lower semi-continuity). From positive homogeneity it is immediate that  $Z$  is feasible iff  $\mathcal{R}^*(Z) = 0$ .

To formulate the following representation result in a concise way we employ the notion of convex ordering. We basically follow [?].

**Definition 8** (Convex ordering). Let  $\tau, \sigma : [0, 1] \rightarrow \mathbb{R}$  be integrable functions.

(i)  $\sigma$  majorizes  $\tau$  (denoted  $\sigma \succcurlyeq \tau$  or  $\tau \preccurlyeq \sigma$ ) iff

$$\int_\alpha^1 \tau(p) dp \leq \int_\alpha^1 \sigma(p) dp \quad (\alpha \in [0, 1]) \quad \text{and} \quad \int_0^1 \tau(p) dp = \int_0^1 \sigma(p) dp.$$

(ii) The spectrum  $\sigma$  majorizes the random variable  $Z$  ( $Z \preccurlyeq \sigma$ ) iff

$$(1 - \alpha) \text{AV@R}_\alpha(Z) \leq \int_\alpha^1 \sigma(p) dp \quad (\alpha \in [0, 1]) \quad \text{and} \quad \mathbb{E}Z = \int_0^1 \sigma(p) dp.$$

*Remark 9.* It should be noted that

$$Z \preccurlyeq \sigma \text{ if and only if } p \mapsto \text{V@R}_p(Z) \preccurlyeq \sigma.$$

Moreover  $Z \preccurlyeq \sigma$  is related to a *convex order constraint* condition (cf. [?] or [?]). The dominance in convex (concave) order was used in studying risk measures for example in [?, ?].

The following Theorem 10 is a characterization by employing the convex conjugate relationship.

**Theorem 10** (Representation as a Supremum – Dual Representation of Spectral Risk Measures). *Let  $\mathcal{R}_\sigma(Y)$  be a spectral risk measure. Then*

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<sup>3</sup>Here,  $L^\infty$  is paired with (its pre-dual)  $L^1$  for convenience. Different pairings are possible as well, and adapted to particular problems.

(i) the representation

$$\begin{aligned}\mathcal{R}_\sigma(Y) &= \sup \{ \mathbb{E}YZ : Z \preceq \sigma \} \\ &= \sup \left\{ \mathbb{E}YZ : \mathbb{E}Z = 1, (1-\alpha) \text{AV@R}_\alpha(Z) \leq \int_\alpha^1 \sigma(p) dp, 0 \leq \alpha < 1 \right\}\end{aligned}\tag{6}$$

holds true,

(ii) and

$$\mathcal{R}_\sigma(Y) = \sup \{ \mathbb{E}Y \cdot \sigma(U) : U \text{ is uniformly distributed} \},\tag{7}$$

where the infimum is attained if  $Y$  and  $U$  are coupled in a co-monotone way.<sup>4</sup>

*Remark 11.* For the measure  $\mu_\sigma$  associated with  $\sigma$  it holds that  $\int_\alpha^1 \sigma(p) dp = \int_0^1 \min \left\{ \frac{1-\alpha}{1-p}, 1 \right\} \mu_\sigma(dp)$ , hence (6) can be stated equivalently as

$$\mathcal{R}_\sigma(Y) = \sup \left\{ \mathbb{E}YZ : \mathbb{E}Z = 1, \text{AV@R}_\alpha(Z) \leq \int_0^1 \min \left\{ \frac{1}{1-\alpha}, \frac{1}{1-p} \right\} \mu_\sigma(dp), 0 \leq \alpha < 1 \right\}.$$

*Remark 12.* The second statement of Theorem 10 implicitly and tacitly assumes that the probability space is rich enough to carry a uniform random variable. This is certainly the case if the probability space does not contain atoms. But even if the probability space has atoms, then this is not a restriction neither, as any probability space with atoms can be augmented to allow a uniformly distributed random variable.

The second statement of Theorem 10 is already contained in [?]. The theorem moreover is related to the paper [?].

*Proof of Theorem 10.* Recall the Legendre-Fenchel transformation for convex functions (cf. [?])

$$\begin{aligned}\mathcal{R}_\sigma(Y) &= \sup_{Z \in L^1} \mathbb{E}YZ - \mathcal{R}_\sigma^*(Z), \\ \mathcal{R}_\sigma^*(Z) &= \sup_{Y \in L^\infty} \mathbb{E}YZ - \mathcal{R}_\sigma(Y).\end{aligned}\tag{8}$$

As  $\mathcal{R}_\sigma$  is version independent the random variable  $Y$  minimizing (8) is coupled in a co-monotone way with  $Z$  (cf. [?] and [?, Proposition 1.8] for the respective rearrangement inequality, sometimes (cf. [?]) referred to as *Hardy and Littlewood's inequality* or *Hardy-Littlewood-Pólya inequality*). It follows that

$$\begin{aligned}\mathcal{R}_\sigma^*(Z) &= \sup_Y \mathbb{E}YZ - \mathcal{R}_\sigma(Y) \\ &= \sup \int_0^1 F_Y^{-1}(\alpha) F_Z^{-1}(\alpha) d\alpha - \int_0^1 F_Y^{-1}(\alpha) \sigma(\alpha) d\alpha,\end{aligned}$$

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<sup>4</sup> $U$  is uniformly distributed if  $P(U \leq u) = u$  for all  $u \in [0, 1]$ .

the infimum being among all cumulative distribution functions  $F_Y(y) = P(Y \leq y)$  of  $Y$ . Define  $G(\alpha) := \int_\alpha^1 F_Z^{-1}(p) dp$  and  $S(\alpha) := \int_\alpha^1 \sigma(p) dp$ , whence

$$\begin{aligned} \mathcal{R}_\sigma^*(Z) &= \sup_{F_Y} \int_0^1 F_Y^{-1}(\alpha) d(S(\alpha) - G(\alpha)) \\ &= \sup_{F_Y} [F_Y^{-1}(\alpha) (S(\alpha) - G(\alpha))]_{\alpha=0}^1 - \int_0^1 S(\alpha) - G(\alpha) dF_Y^{-1}(\alpha) \\ &= \sup_{F_Y} F_Y^{-1}(0) (G(0) - S(0)) + \int_0^1 G(\alpha) - S(\alpha) dF_Y^{-1}(\alpha) \end{aligned} \quad (9)$$

by integration by parts of the Riemann-Stieltjes integral and as  $-\|Y\|_\infty \leq F_Y^{-1} \leq \|Y\|_\infty$ . Consider the constant random variables  $Y \equiv c$  ( $c \in \mathbb{R}$ ), then  $F_Y^{-1} \equiv c$  and, by (9),

$$\mathcal{R}_\sigma^*(Z) \geq \sup_{c \in \mathbb{R}} c (G(0) - S(0)).$$

Note now that  $S(0) = \int_0^1 \sigma(p) dp = 1$ , whence

$$\mathcal{R}_\sigma^*(Z) \geq \sup_{c \in \mathbb{R}} c (G(0) - 1) = \begin{cases} 0 & \text{if } G(0) = 1 \\ \infty & \text{else} \end{cases} = \begin{cases} 0 & \text{if } \mathbb{E}Z = 1 \\ \infty & \text{else,} \end{cases}$$

because

$$G(0) = \int_0^1 F_Z^{-1}(p) dp = \mathbb{E}Z. \quad (10)$$

Assuming  $\mathbb{E}Z = 1$  it follows from (9) that

$$\mathcal{R}_\sigma^*(Z) = \sup_{F_Y} \int_0^1 G(\alpha) - S(\alpha) dF_Y^{-1}(\alpha).$$

Then choose an arbitrary measurable set  $B$  and consider the random variable  $Y_c := c \cdot \mathbb{1}_B$  ( $c > 0$ ). Note that  $F_{Y_c}^{-1} = \mathbb{1}_{[\alpha_0, 1]}$ , where  $\alpha_0 = P(B)$ . With this choice

$$\begin{aligned} \mathcal{R}_\sigma^*(Z) &\geq \sup_{F_{Y_c}} \int_0^1 G(\alpha) - S(\alpha) dF_{Y_c}^{-1}(\alpha) \geq \sup_{c \geq 0} c (G(\alpha_0) - S(\alpha_0)) = \\ &= \begin{cases} 0 & \text{if } G(\alpha_0) \leq S(\alpha_0) \\ \infty & \text{else} \end{cases} \end{aligned}$$

As  $B$  was chosen arbitrarily it follows that  $G(\alpha) \leq S(\alpha)$  has to hold for any  $0 \leq \alpha \leq 1$  for  $Z$  to be feasible.

Conversely, if (10) and  $G(\alpha) \leq S(\alpha)$  for all  $0 \leq \alpha \leq 1$ , then

$$\sup_{F_Y} \int_0^1 G(\alpha) - S(\alpha) dF_Y^{-1}(\alpha) \leq 0,$$

because  $\alpha \mapsto F_Y^{-1}(\alpha)$  is a non-decreasing function. Note now that

$$\begin{aligned} \int_{\alpha}^1 \sigma(p) dp &= S(\alpha) \geq G(\alpha) \\ &= \int_{\alpha}^1 F_Z^{-1}(p) dp = (1 - \alpha) \text{AV@R}_{\alpha}(Z), \end{aligned}$$

from which finally follows that

$$\mathcal{R}_{\sigma}^*(Z) = \begin{cases} 0 & \text{if } \mathbb{E}Z = 1 \text{ and } (1 - \alpha) \text{AV@R}_{\alpha}(Z) \leq \int_{\alpha}^1 \sigma(p) dp \text{ (} 0 \leq \alpha \leq 1 \text{)} \\ \infty & \text{else.} \end{cases}$$

As for the second assertion of the theorem consider  $Z = \sigma(U)$  for a uniformly distributed random variable  $U$ , then  $P(Z \leq \sigma(\alpha)) = P(\sigma(U) \leq \sigma(\alpha)) \geq P(U \leq \alpha) = \alpha$ , that is  $\text{V@R}_{\alpha}(Z) \geq \sigma(\alpha)$ . But as  $1 = \int_0^1 \sigma(\alpha) d\alpha \leq \int_0^1 \text{V@R}_{\alpha}(\sigma(U)) d\alpha = \mathbb{E}\sigma(U) = 1$  it follows that

$$\sigma(\alpha) = \text{V@R}_{\alpha}(Z)$$

almost everywhere. Observe now that any  $Z$  with  $\text{V@R}_{\alpha}(Z) \leq \sigma(\alpha)$  is feasible, because

$$\int_{\alpha}^1 \sigma(p) dp \geq \int_{\alpha}^1 \text{V@R}_p(Z) dp = (1 - \alpha) \text{AV@R}_{\alpha}(Z)$$

and  $\mathbb{E}Z = \mathbb{E}\sigma(U) = \int_0^1 \sigma(\alpha) d\alpha = 1$ . Now let  $U$  be coupled in an co-monotone way with  $Y$ , then  $\mathbb{E}YZ = \int_0^1 F_Y^{-1}(\alpha) F_Z^{-1}(\alpha) d\alpha = \int_0^1 F_Y^{-1}(\alpha) \text{V@R}_{\alpha}(\sigma(U)) d\alpha = \int_0^1 F_Y^{-1}(\alpha) \sigma(\alpha) d\alpha$  such that

$$\mathcal{R}_{\sigma}(Y) = \sup \{ \mathbb{E}Y\sigma(U) : U \text{ uniformly distributed} \},$$

which is finally the second assertion.  $\square$

*Remark 13.* We emphasize that the conditions  $(1 - \alpha) \text{AV@R}_{\alpha}(Z) \leq \int_{\alpha}^1 \sigma(p) dp$  and  $\mathbb{E}Z = 1$  together imply that  $Z \geq 0$  almost everywhere. Indeed, suppose that  $P(Z < 0) =: p > 0$ . Then  $1 = \mathbb{E}Z = \int_{\{Z < 0\}} Z dP + \int_{\{Z \geq 0\}} Z dP = \int_{\{Z < 0\}} Z dP + (1 - p) \text{AV@R}_p(Z)$ . As  $\int_{\{Z < 0\}} Z dP < 0$  it follows that  $(1 - p) \text{AV@R}_p(Z) > 1$ . But this contradicts the fact that  $(1 - p) \text{AV@R}_p(Z) \leq \int_p^1 \sigma(p') dp' \leq 1$ , hence  $Z \geq 0$  almost surely.

The characterization derived in the previous theorem naturally applies to the Average Value-at-Risk. This can be simplified further for the Average Value-at-Risk, as the following corollary exhibits. Whereas the first statement is just folklore, the second statement is listed for completeness and because it allows interesting interpretations.

**Corollary 14.** *The Average Value-at-Risk at level  $\alpha$  obeys the dual representations*

$$\begin{aligned} \text{AV@R}_{\alpha}(Y) &= \sup \{ \mathbb{E}YZ : \mathbb{E}Z = 1, 0 \leq Z, (1 - \alpha)Z \leq \mathbb{1} \} \\ &= \sup \left\{ \mathbb{E}YZ : \mathbb{E}Z = 1, \text{AV@R}_p(Z) \leq \frac{1}{1 - \alpha} \text{ for all } p > \alpha \right\}. \end{aligned}$$



*Proof.* The Average Value-at-Risk at level  $\alpha$  is provided by the Dirac measure  $\mu_\alpha(A) := \delta_\alpha(A) = \begin{cases} 1 & \text{if } \alpha \in A \\ 0 & \text{otherwise} \end{cases}$ , and the respective spectral density function is  $\sigma_\alpha$  (cf. (3)). It follows from  $\int_p^1 \sigma_\alpha(p') dp' = \min\left\{1, \frac{1-p}{1-\alpha}\right\}$  and Theorem 10 that

$$\text{AV@R}_\alpha(Y) = \inf \left\{ \mathbb{E}YZ : \mathbb{E}Z = 1, \text{AV@R}_p(Z) \leq \min \left\{ \frac{1}{1-p}, \frac{1}{1-\alpha} \right\} \right\}.$$

Observe next that for  $Z \geq 0$

$$\frac{1}{1-p} = \frac{1}{1-p} \mathbb{E}Z \geq \frac{1}{1-p} \int_p^1 F_Z^{-1}(p') dp' = \text{AV@R}_p(Z),$$

hence

$$\text{AV@R}_\alpha(Y) = \inf \left\{ \mathbb{E}YZ : \mathbb{E}Z = 1, \text{AV@R}_p(Z) \leq \frac{1}{1-\alpha} \right\}.$$

For  $p \leq \alpha$ , in addition,  $\text{AV@R}_p(Z) \leq \frac{1}{1-p} \leq \frac{1}{1-\alpha}$ .

This proves the second assertion.

As for the first observe that  $\frac{1}{1-\alpha} \geq \text{AV@R}_p(Z) \rightarrow \text{ess sup } Z$ , hence  $(1-\alpha)Z \leq \mathbf{1}$ ; conversely, if  $0 \leq Z$  and  $(1-\alpha)Z \leq \mathbf{1}$ , then

$$\frac{1}{1-\alpha} \geq \text{ess sup } Z \geq \text{AV@R}_p(Z),$$

which is the first assertion.  $\square$

**Corollary 15.** *The conditions  $\mathbb{E}Z = 1$  and  $\text{AV@R}_\alpha(Z) \geq 1$  (for all  $0 \leq \alpha \leq 1$ ) are equivalent to  $Z \equiv \mathbf{1}$ .*

*Proof.* The proof is immediate for the particular choice  $\alpha = 0$  in Corollary 14.  $\square$

## 4 Infimum-Representation of Spectral Risk Measures

The latter Theorem 10 exposes the spectral risk measure as a supremum. This expression (6), nor the initial defining equation (1), nor (2) are of good use for stochastic optimization (cf. the next section). For this it is desirable to have an expression as an infimum available: the following theorem, the main result of this article, provides a helpful alternative. The representation extends the well-known formula for the Average Value-at-Risk provided in [?], finally stated in the present form in [?].

**Theorem 16** (Representation as an Infimum). *For any  $Y \in L^\infty$  the spectral risk measure with spectrum  $\sigma$  has the representation*

$$\mathcal{R}_\sigma(Y) = \inf_f \mathbb{E}f(Y) + \int_0^1 f^*(\sigma(p)) dp, \quad (11)$$

where the infimum is among all arbitrary, measurable functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  and  $f^*$  is  $f$ 's convex conjugate function<sup>5</sup>.

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<sup>5</sup>The convex conjugate function of  $f$  is  $f^*(y) := \sup_x x \cdot y - f(x)$ . The convex conjugate may evaluate to  $+\infty$ .

The statement of the Inf-Representation Theorem 16 can be formulated equivalently by the following forms.

**Corollary 17.** *For any  $Y \in L^\infty$  the spectral risk measure with spectrum  $\sigma$  allows the representations*

$$\begin{aligned}\mathcal{R}_\sigma(Y) &= \inf_{f \text{ convex}} \mathbb{E} f(Y) + \int_0^1 f^*(\sigma(p)) \, dp \\ &= \inf \left\{ \mathbb{E} f(Y) : \int_0^1 f^*(\sigma(p)) \, dp \leq 0 \right\},\end{aligned}$$

where the infimum is among arbitrary, measurable functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ .

*Proof of Corollary 17.* It is well-known that the bi-conjugate function  $f^{**} := (f^*)^*$  is a convex and lower semi-continuous function satisfying  $f^{**} \leq f$  and  $f^{***} = f^*$  (cf. the analogous Fenchel-Moreau Theorem and equation (8)). The infimum in (11) hence – without any loss of generality – can be restricted to *convex* functions, that is

$$\mathcal{R}_\sigma(Y) = \inf_{f \text{ convex}} \mathbb{E} f(Y) + \int_0^1 f^*(\sigma(p)) \, dp.$$

As for the second assertion notice first that clearly

$$\begin{aligned}\mathcal{R}_\sigma(Y) &\leq \inf \left\{ \mathbb{E} f(Y) + \int_0^1 f^*(\sigma(p)) \, dp : \int_0^1 f^*(\sigma(p)) \, dp \leq 0 \right\} \\ &\leq \inf \left\{ \mathbb{E} f(Y) : \int_0^1 f^*(\sigma(p)) \, dp \leq 0 \right\}.\end{aligned}$$

Consider  $f_\alpha(x) := f(x) - \alpha$  (where  $\alpha$  a constant and  $f$  arbitrary). It holds that  $f_\alpha^*(y) = f^*(y) + \alpha$ , as exposed by the auxiliary Lemma 20 in the Appendix. Hence  $\int_0^1 f_\alpha^*(\sigma(p)) \, dp = \int_0^1 f^*(\sigma(p)) \, dp + \alpha$  and

$$\mathbb{E} f_\alpha(Y) + \int_0^1 f_\alpha^*(\sigma(p)) \, dp = \mathbb{E} f(Y) + \int_0^1 f^*(\sigma(p)) \, dp. \quad (12)$$

Choose  $\alpha := \int_0^1 f^*(\sigma(p)) \, dp$  such that  $\int_0^1 f_\alpha^*(\sigma(p)) \, dp = 0$ .  $f_\alpha$  hence is feasible for (11) with the same objective as  $f$  by (12), from which the assertion follows.  $\square$

*Remark 18.* Having a look at representation (11) it is not immediate that the axioms of Definition 1 are satisfied – except for monotonicity, (M). The transformations listed in Lemma 20 in the Appendix can be used in a straight forward manner to deduce the properties directly from (11).

*Remark 19.* Notice that  $\sigma$  has its range in the interval  $\{\sigma(x) : x \in [0, 1]\} = [0, \sigma(1)]$ , and from convexity of  $f^*$  it follows that the set  $\{f^* < \infty\}$  is convex. Hence  $f^*(y) < \infty$  necessarily has to hold for all  $y \in (0, \sigma(1))$  to ensure that  $\int_0^1 f^*(\sigma(u)) \, du < \infty$ . For  $f$  convex this means in turn that

$$\lim_{x \rightarrow -\infty} f'(x) \leq 0 \text{ and } \lim_{x \rightarrow \infty} f'(x) \geq \sigma(1),$$

limiting thus the class of interesting functions in Corollary 17 to convex functions satisfying  $f'(\mathbb{R}) \supset (0, \sigma(1))$ .

*Proof of Theorem 16.* From the definition of the convex conjugate  $f^*$  it is immediate that

$$f^*(\sigma) \geq y \cdot \sigma - f(y)$$

for all numbers  $y$  and  $\sigma$ , hence

$$f(Y) + f^*(\sigma(U)) \geq Y \cdot \sigma(U),$$

where  $U$  is any uniformly distributed random variable, i.e.  $U$  satisfies  $P(U \leq u) = u$ . Taking expectations it follows that

$$\mathbb{E}f(Y) + \mathbb{E}f^*(\sigma(U)) \geq \mathbb{E}Y \cdot \sigma(U).$$

As  $U$  is uniformly distributed it holds that

$$\mathbb{E}f^*(\sigma(U)) = \int_0^1 f^*(\sigma(u)) du,$$

such that

$$\mathbb{E}f(Y) + \int_0^1 f^*(\sigma(u)) du \geq \mathbb{E}Y \cdot \sigma(U),$$

irrespective of the uniform random variable  $U$ . Hence, by (7) in Theorem 10,

$$\mathbb{E}f(Y) + \int_0^1 f^*(\sigma(u)) du \geq \sup_{U \text{ uniform}} \mathbb{E}Y \cdot \sigma(U) = \mathcal{R}_\sigma(Y),$$

establishing the inequality

$$\mathcal{R}_\sigma(Y) \leq \mathbb{E}f(Y) + \int_0^1 f^*(\sigma(u)) du.$$

As for the converse inequality consider the function

$$f_0(y) := \int_0^1 F_Y^{-1}(\alpha) + \frac{1}{1-\alpha} (y - F_Y^{-1}(\alpha))_+ \mu_\sigma(d\alpha).$$

$f_0(y)$  is well-defined for all  $y$  because  $Y \in L^\infty$ ;  $f_0(y)$  is moreover increasing and convex, because  $y \mapsto (y - q)_+$  is increasing and convex, and because  $\mu_\sigma$  is positive.

Recall the formula

$$\text{AV@R}_\alpha(Y) = \inf_{q \in \mathbb{R}} q + \frac{1}{1-\alpha} \mathbb{E}(Y - q)_+$$

and the fact that the infimum is attained at  $q = F_Y^{-1}(\alpha)$  (cf. [?] for the general formula), providing thus the explicit form

$$\text{AV@R}_\alpha(Y) = F_Y^{-1}(\alpha) + \frac{1}{1-\alpha} \mathbb{E}(Y - F_Y^{-1}(\alpha))_+.$$

Note now that, by Fubini's Theorem,

$$\begin{aligned}
\mathcal{R}_\sigma(Y) &= \int_0^1 \text{AV@R}_\alpha(Y) \mu_\sigma(d\alpha) \\
&= \int_0^1 F_Y^{-1}(\alpha) + \frac{1}{1-\alpha} \mathbb{E}(Y - F_Y^{-1}(\alpha))_+ \mu_\sigma(d\alpha) \\
&= \mathbb{E} \int_0^1 F_Y^{-1}(\alpha) + \frac{1}{1-\alpha} (Y - F_Y^{-1}(\alpha))_+ \mu_\sigma(d\alpha) \\
&= \mathbb{E} f_0(Y).
\end{aligned} \tag{13}$$

To establish the assertion (11) it needs to be shown that  $\int_0^1 f_0^*(\sigma(u)) du \leq 0$ . For this observe first that  $f_0$  is almost everywhere differentiable (because it is convex), with derivative

$$\begin{aligned}
f_0'(y) &= \int_{\{\alpha: F_Y^{-1}(\alpha) \leq y\}} \frac{1}{1-\alpha} \mu_\sigma(d\alpha) \\
&= \int_0^{F_Y(y)} \frac{1}{1-\alpha} \mu_\sigma(d\alpha) = \sigma(F_Y(y))
\end{aligned}$$

(almost everywhere) by relation (5). Moreover  $f_0^*(\sigma(u)) = \sup_y \sigma(u)y - f_0(y)$ , the supremum being attained at every  $y$  satisfying  $\sigma(u) = f_0'(y) = \sigma(F_Y(y))$ , hence at  $y = F_Y^{-1}(u)$ , and it follows that

$$f_0^*(\sigma(u)) = \sigma(u) \cdot F_Y^{-1}(u) - f_0(F_Y^{-1}(u)).$$

Now

$$\begin{aligned}
\int_0^1 f_0^*(\sigma(u)) du &= \int_0^1 \sigma(u) \cdot F_Y^{-1}(u) du - \int_0^1 f_0(F_Y^{-1}(u)) du \\
&= \mathcal{R}_\sigma(Y) - \mathbb{E} f_0(Y).
\end{aligned}$$

But it was established already in (13) that  $\mathcal{R}_\sigma(Y) = \mathbb{E} f_0(Y)$ , so that  $\int_0^1 f_0^*(\sigma(u)) du = 0$ . This finally proves the second inequality.  $\square$

The Average Value-at-Risk is a special case of the infimum in (11). Indeed, it follows from the proof that the infimum is attained at a function of the form  $f_q(y) = q + \frac{1}{1-\alpha}(y - q)_+$  with conjugate

$$f_q^*(x) = \begin{cases} -q + qx & \text{if } 0 \leq x \leq \frac{1}{1-\alpha} \\ \infty & \text{else.} \end{cases}$$

It follows that  $\int_0^1 f_0^*(\sigma_\alpha(x)) dx = \int_0^\alpha f_0^*(0) dx + \int_\alpha^1 f_0^*\left(\frac{1}{1-\alpha}\right) dx = -\alpha q + \left(-q + \frac{q}{1-\alpha}\right)(1-\alpha) = 0$ , such that

$$\text{AV@R}_\alpha(Y) = \inf_{q \in \mathbb{R}} \mathbb{E} f_q(Y) = \inf_q q + \frac{1}{1-\alpha} \mathbb{E}(Y - q)_+. \tag{14}$$

Clearly, the infimum in (14) is in  $\mathbb{R}$ , a much smaller space than convex functions from  $\mathbb{R}$  to  $\mathbb{R}$ , as required in (11).

## 5 Implications for Stochastic Optimization

Portfolio optimization (i.e. asset allocation) is a typical problem in stochastic optimization. In this section we outline how the inf-representation (11) of spectral risk measures can be applied to efficiently evaluate stochastic optimization problems involving spectral risk measures.

### Problem Formulation

Stochastic optimization often considers the problem

$$\begin{aligned} & \text{minimize} && \mathbb{E} H(x, Y) \\ & \text{subject to} && x \in \mathbb{X}, \end{aligned} \tag{15}$$

where  $x$  can be chosen in a decision space  $\mathbb{X}$  and  $H$  is a loss function; for  $x$  given,  $\omega \mapsto H(x, Y(\omega))$  is a random variable for which the expectation in (15) can be maximized. (For the purpose of portfolio optimization, the function  $H$  is simply  $H = x^\top Y$ , where  $Y$  collect the returns of the individual stocks.)

If the function  $H$  should be evaluated subject to risk as well, then a frequently used substitute for (15) is

$$\begin{aligned} & \text{minimize} && \mathcal{R}(H(x, Y)) \\ & \text{subject to} && x \in \mathbb{X} \end{aligned}, \tag{16}$$

where  $\mathcal{R}$  is a convex risk measure with the usual dual representation

$$\mathcal{R}(Y) = \sup \{ \mathbb{E} Y Z : \mathcal{R}^*(Z) < \infty \}.$$

Combining these formulas leads to the combined problem

$$\begin{aligned} & \text{minimize} && \sup \mathbb{E} H(x, Y) \cdot Z \\ & \text{subject to} && x \in \mathbb{X} \\ & && \mathcal{R}^*(Z) < \infty. \end{aligned} \tag{17}$$

This problem formulation (17) – a mini-max problem – is not favorable for numerical optimizations, as for any function evaluation for a given  $x$  to compute the minimum a complete maximum has to be evaluated first. Provided that there are efficient mathematical estimates for the particular problem this may to be done with reduced precision, as otherwise the results for the optimal  $x \in \mathbb{X}$  is unpredictably wrong; if no such estimates are available, then the supremum in (17) has to be computed precisely – impossible for numerical computations.

For the Average Value-at-Risk,

$$\text{AV@R}_\alpha(Y) = \frac{1}{1-\alpha} \int_\alpha^1 F_Y^{-1}(p) dp = \sup \left\{ \mathbb{E} Y Z : 0 \leq Z \leq \frac{1}{1-\alpha} \right\},$$

none of these formulations are useful in (16) neither. However, the representation

$$\text{AV@R}_\alpha(Y) = \min_q q + \frac{1}{1-\alpha} \mathbb{E}(Y - q)_+$$

can be employing immediately, as the problem (16) rewrites

$$\begin{aligned} & \text{minimize} && q + \frac{1}{1-\alpha} \mathbb{E}(H(x, Y) - q)_+ \\ & \text{subject to} && x \in \mathbb{X} \\ & && q \in \mathbb{R}, \end{aligned} \tag{18}$$

which is just a minimization problem with the same objective and optimal value as (17), but much easier to handle from implementation and numerical point of view.

For a spectral risk measure problem (16) rewrites as

$$\begin{aligned} & \text{minimize} && \mathbb{E}f(H(x, Y)) + \int_0^1 f^*(\sigma(u)) du \\ & \text{subject to} && x \in \mathbb{X} \\ & && f \text{ (convex),} \end{aligned} \tag{19}$$

which is – in contrast to (16) – a straight forward optimization problem again, which allows an immediate implementation. The price, which is to pay in contrast to the simple Average Value-at-Risk, is that an entire function has to be looked up in (19), whereas just an additional number ( $q$ ) appears in (18).

## Implementation

For numerical implementations to find approximations of (19) it should be mentioned that  $\sigma \in L^\infty$  can be approximated by step functions  $\bar{\sigma}_n$  such that  $\sigma \leq \bar{\sigma}_n$ , where  $\bar{\sigma}_n = \sum_{i=1}^n \sigma_i \mathbb{1}_{[\alpha_{i-1}, \alpha_i)}$  ( $0 = \alpha_0 < \dots < \alpha_i < \alpha_{i+1} < \dots < \alpha_n = 1$ ,  $0 < \sigma_i < \sigma_{i+1}$ ). In this situation the range of the approximation  $\bar{\sigma}_n$  is the finite set  $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$  and

$$\mathcal{R}_\sigma(Y) = \int_0^1 F_Y^{-1}(\alpha) \sigma(\alpha) d\alpha \leq \int_0^1 F_Y^{-1}(\alpha) \bar{\sigma}_n(\alpha) d\alpha.$$

The corresponding measure on  $[0, 1]$  is  $\mu_{\sigma_n}(A) = \sigma_1 \delta_0 + \sum_{i=1}^n (1 - \alpha_i) (\sigma_{i+1} - \sigma_i) \delta_{\alpha_i} =: \sum_{i=0}^{n-1} \mu_i \delta_{\alpha_i}$ . From the proof of Theorem 16 it follows that one may choose the ansatz

$$f_n(y) := \mu_0 y + \sum_{i=1}^n \mu_i \left( q_i + \frac{1}{1 - \alpha_i} (y - q_i)_+ \right)$$

to solve the approximating problem

$$\begin{aligned} & \text{minimize} && \mathbb{E}f_n(H(x, Y)) \\ & \text{subject to} && x \in \mathbb{X} \\ & && q_1 \leq q_2 \leq \dots \leq q_n. \end{aligned}$$

The resulting value consequently is an upper bound of the problem (19).

## 6 Summary

This article outlines new descriptions of spectral risk measures. Spectral risk measures constitute an integral subclass of risk measures, as every risk measure can be written as a supremum of spectral risk measures.

The first representation derived is built as a supremum, based on conjugate duality. The other representation, which is the central result of this article, is described as an infimum. This description makes spectral risk measures eligible for successful future use in stochastic optimization.

## 7 Acknowledgment

We wish to thank the referees for their constructive criticism.

## Appendix

For reference and the sake of completeness we list the following elementary result for affine linear transformations of the convex conjugate function.

**Lemma 20.** *The convex conjugate of the function  $g(x) := \alpha + \beta x + \gamma \cdot f(\lambda x + c)$  for  $\gamma > 0$  and  $\lambda \neq 0$  is*

$$g^*(y) = -\alpha - c \frac{y - \beta}{\lambda} + \gamma \cdot f^*\left(\frac{y - \beta}{\lambda \gamma}\right).$$

*Proof.* Just observe that

$$\begin{aligned} g^*(y) &= \sup_x yx - g(x) \\ &= \sup_x yx - \alpha - \beta x - \gamma \cdot f(\lambda x + c) \\ &= \sup_x y \frac{x - c}{\lambda} - \alpha - \beta \frac{x - c}{\lambda} - \gamma \cdot f(x) \\ &= -\alpha - c \frac{y - \beta}{\lambda} + \sup_x x \frac{y - \beta}{\lambda} - \gamma \cdot f(x) \\ &= -\alpha - c \frac{y - \beta}{\lambda} + \gamma \cdot \sup_x x \frac{y - \beta}{\lambda \gamma} - f(x) \\ &= -\alpha - c \frac{y - \beta}{\lambda} + \gamma \cdot f^*\left(\frac{y - \beta}{\lambda \gamma}\right), \end{aligned} \tag{20}$$

where we have replaced  $x$  by  $\frac{x-c}{\lambda}$  in (20). □