

Asymptotics for penalized spline estimators in quantile regression

TAKUMA YOSHIDA¹

¹*Graduate School of Science and Engineering
Shimane University, Matsue 690-8504, Japan*

Abstract

Quantile regression predicts the τ -quantile of the conditional distribution of a response variable given the explanatory variable for $\tau \in (0, 1)$. The aim of this paper is to establish the asymptotic distribution of the quantile estimator obtained by penalized spline method. A simulation and an exploration of real data are performed to validate our results.

Keywords Asymptotic normality, B -spline, Penalized spline, Quantile regression.

1 Introduction

Regression analysis is one of the most important tools used to investigate the relationship between a response Y and a predictor X . Many major studies of regression have been concerned with the estimation of the conditional mean function of Y given a predictor $X = x$. On the other hand, the estimation of the conditional quantile function of Y given x has gained momentum in recent years. This analysis is called quantile regression. In quantile regression, the purpose is to estimate an unknown function $\eta_\tau(x)$ that satisfies

$$P(Y < \eta_\tau(x) | X = x) = \tau$$

for a given $\tau \in (0, 1)$. When $\tau = 0.5$, $\eta_\tau(x)$ is the conditional median of Y . One established advantage of quantile regression as compared to mean regression is that the estimators are more robust against outliers in the response measurements. Quantile regression models have been suggested by Koenker and Bassett (1978). Many authors have studied quantile regression based on the parametric method, its asymptotic theories, the computational aspects and other properties, and these developments have been summarized by Koenker (2005) and Hao and Naiman (2007). The nonparametric methods for quantile regression have also been studied extensively. Many authors have explored the topic in relation to kernel methods, including Fan et al. (1994), Yu and Jones (1998), Takeuchi et al. (2006), Kai et al. (2011). On the other hand, Hendricks and Koenker (1992) and Koenker et al. (1994) used the low-rank regression splines method and the smoothing splines method, respectively. Pratesi et al. (2009) and Reiss and Huang (2012) utilized the penalized spline smoothing method. This paper focuses on penalized splines. Compared with unpenalized splines and smoothing splines, an advantage

of the penalized spline methods is follows. Although the smoothing spline estimator gives the predictor with fitness and smoothness, the computational cost to construct the estimator is high. In unpenalized regression spline methods, on the other hand, it is known that the estimator tends to have a wiggle curve, but the computational cost is lower than that of smoothing spline methods. The penalized spline estimator, however, gives the curve with fitness and smoothness and its computational cost is lower than that of smoothing spline methods. Thus, penalized splines can be considered an efficient technique.

Previous results of asymptotic studies of nonparametric quantile regressions include the following. Fan et al. (1994) showed the asymptotic normality of the kernel estimator. Yu and Jones (1998) proposed a new kernel estimator and studied its asymptotic results. He and Shi (1994) showed the convergence rate of the unpenalized regression spline estimator. Portnoy (1997) discussed asymptotics for smoothing spline estimators. However, the asymptotics for the penalized spline estimator of quantile regression have not yet been studied.

In this paper, we show the asymptotic distribution of the penalized spline estimator for quantile regression with a low-rank B -spline model and the difference penalty. The penalized spline estimator of $\eta_\tau(x)$ for a given τ is defined as the minimizer of the convex loss function, which is the check function ρ_τ with an additional difference penalty. To establish the asymptotic distribution of the penalized spline estimator, we need to derive two biases (i) the model bias between the true function $\eta_\tau(x)$ and the B -spline model, and (ii) the bias arising from using the penalty term. By showing the asymptotic form of these two biases, the resulting asymptotic bias of the penalized spline estimator can be obtained. Finally, together with the asymptotic variance of the estimator, we show the asymptotic normality of the penalized spline quantile estimator.

This paper is organized as follows. In Section 2, we define the penalized spline quantile estimator for a given τ . In terms of our estimation method, we mainly focus on the penalized iteratively reweighted least squares method. Section 3 provides the asymptotic bias and variance as well as the asymptotic distribution of the penalized spline quantile estimator. Furthermore, the related properties are described. In Section 4, numerical studies are conducted. Related discussion and issues for future research are provided in Section 5. Finally, proofs for the theoretical results are all given in the Appendix.

2 Penalized spline estimator in quantile regression

For a given dataset $\{(y_i, x_i) : i = 1, \dots, n\}$, consider the conditional $100\tau\%$ quantile of response Y_i given $X_i = x_i$ as

$$P(Y_i < \eta_\tau(x_i) | X_i = x_i) = \tau,$$

where $\tau \in (0, 1)$ and $\eta_\tau(x_i)$ is an unknown true conditional quantile function of Y_i given $X_i = x_i$. It is easy to show that the true function satisfies

$$\eta_\tau(x) = \operatorname{argmin}_a E[\rho_\tau(Y - a(x)) | X = x].$$

Here, ρ_τ is the check function provided by Koenker and Bassett (1978), given as

$$\rho_\tau(u) = u(\tau - I(u < 0)),$$

where $I(u < b)$ is the indicator function of $(-\infty, b)$. We want to estimate $\eta_\tau(x)$ using penalized spline methods. To approximate $\eta_\tau(x)$, we consider the B -spline model

$$s_\tau(x) = \sum_{k=-p+1}^K B_k^{[p]}(x)b_k(\tau),$$

where $B_k^{[p]}(x)(k = -p+1, \dots, K)$ are the p th degree B -spline basis functions defined recursively as

$$\begin{aligned} B_k^{[0]}(x) &= \begin{cases} 1, & \kappa_{k-1} < x \leq \kappa_k, \\ 0, & \text{otherwise,} \end{cases} \\ B_k^{[p]}(x) &= \frac{x - \kappa_{k-1}}{\kappa_{k+p-1} - \kappa_{k-1}} B_k^{[p-1]}(x) + \frac{\kappa_{k+p} - x}{\kappa_{k+p} - \kappa_k} B_{k+1}^{[p-1]}(x), \end{aligned}$$

where $\kappa_k(k = -p+1, \dots, K+p)$ are knots and $b_k(\tau)(k = -p+1, \dots, K)$ are unknown parameters. We denote $B_k^{[p]}(x)$ as $B_k(x)$ unless the degrees of B -splines are specified. Details and many properties of the B -spline function are clarified by de Boor (2001). The estimator of $\mathbf{b}(\tau) = (b_{-p+1}(\tau) \cdots b_{K_n}(\tau))^T$ is defined as

$$\begin{aligned} \hat{\mathbf{b}}(\tau) &= (\hat{b}_{-p+1}(\tau) \cdots \hat{b}_{K_n}(\tau))^T \\ &= \underset{\mathbf{b}(\tau)}{\operatorname{argmin}} \left\{ \sum_{i=1}^n \rho_\tau(y_i - \mathbf{B}(x_i)^T \mathbf{b}(\tau)) + \frac{\lambda_\tau}{2} \mathbf{b}(\tau)^T D_m^T D_m \mathbf{b}(\tau) \right\}, \end{aligned} \quad (1)$$

where $\mathbf{B}(x_j) = (B_{-p+1}(x_j) \cdots B_{K_n}(x_j))^T$, $\lambda_\tau (> 0)$ is the smoothing parameter and $(K_n + p - m) \times (K_n + p)$ th matrix D_m is the m th difference matrix, which is defined as $D_m = (d_{ij}^{(m)})_{ij}$, where $d_{ij}^{(m)} = (-1)^{|i-j|} {}_m C_{|i-j|}$ for $i \leq j \leq m+1$, and 0 for otherwise. It is well known that the difference penalty in (1) is very useful in mean regression and can be regarded as the controller of the smoothness of $s_\tau(x)$ because we can interpret $\mathbf{b}(\tau)^T D_m^T D_m \mathbf{b}(\tau) \approx K_n^{2m-1} \int_0^1 \{s_\tau^{(m)}(x)\}^2 dx$ (see, Eilers and Marx (1996)). Although Reiss and Huang (2012) used the penalty $\int_0^1 \{s_\tau^{(m)}(x)\}^2 dx$, this penalty contains an integral and hence the computational difficulty for the resulting estimator grows. Therefore, this paper proposes using $\mathbf{b}(\tau)^T D_m^T D_m \mathbf{b}(\tau)$ as the penalty. In fact, $\hat{\mathbf{b}}(\tau)$ is obtained via linear-programming methods, such as simplex methods or interior points methods (see Koenker and Park (1996), Koenker (2005)). On the other hand, it is known that the iteratively reweighted least squares (IRLS) method is a useful in nonparametric quantile regression. The penalized spline estimator obtained via IRLS was also studied and detailed by Reiss and Huang (2012).

Since IRLS is important for obtaining the estimator, we now provide the complete algorithm. For a given λ_τ , the k -steps iterated estimator $\hat{\mathbf{b}}^{(k)}(\tau)$ is defined as follows:

$$\hat{\mathbf{b}}^{(k)}(\tau) = (Z^T W^{(k-1)} Z + \lambda_\tau D_m^T D_m)^{-1} Z^T W^{(k-1)} \mathbf{y},$$

where $\mathbf{y} = (y_1 \cdots y_n)^T$, $Z = (B_{-j+p}(x_i))_{ij}$, $W^{(k)} = \text{diag}[w_1^{(k)} \cdots w_n^{(k)}]$ and

$$w_i^{(k)} = \begin{cases} \frac{\tau - I(y_i - \mathbf{B}(x_i)^T \hat{\mathbf{b}}^{(k-1)}(\tau) < 0)}{2(y_i - \mathbf{B}(x_i)^T \hat{\mathbf{b}}^{(k-1)}(\tau))} & |y_i - \mathbf{B}(x_i)^T \hat{\mathbf{b}}^{(k-1)}(\tau)| > \alpha, \\ \frac{\tau(y_i - \mathbf{B}(x_i)^T \hat{\mathbf{b}}^{(k-1)}(\tau))}{\alpha} & 0 \leq y_i - \mathbf{B}(x_i)^T \hat{\mathbf{b}}^{(k-1)}(\tau) \leq \alpha, \\ \frac{(1 - \tau)(y_i - \mathbf{B}(x_i)^T \hat{\mathbf{b}}^{(k-1)}(\tau))}{\alpha} & -\alpha \leq y_i - \mathbf{B}(x_i)^T \hat{\mathbf{b}}^{(k-1)}(\tau) \leq 0, \end{cases}$$

for small $\alpha > 0$ and the initial $W^{(0)}$. As $k \rightarrow \infty$, it can be shown that $\lim_{k \rightarrow \infty} \hat{\mathbf{b}}^{(k)}(\tau)$ is approximately equivalent to the minimizer of (1). By using $\hat{\mathbf{b}}(\tau)$, the penalized spline estimator of $\eta_\tau(x)$ is defined as

$$\hat{\eta}_\tau(x) = \sum_{k=-p+1}^K B_k^{[p]}(x) \hat{b}_k(\tau) = B(x)^T \hat{\mathbf{b}}(\tau).$$

3 Asymptotic theory

In this section, we show the asymptotic property of $\hat{\eta}_\tau(x)$. Then, we assume that the number of knots K and smoothing parameter λ_τ are dependent on n , and we write K_n and $\lambda_{\tau,n}$, respectively. For simplicity, we write $\lambda_n = \lambda_{\tau,n}$. We give some assumptions regarding the asymptotics of the penalized spline quantile estimator.

Assumptions

1. The explanatory X is distributed as $Q(x)$ on $[0, 1]$.
2. The knots for the B -spline basis are equidistantly located as $\kappa_k = k/K_n (k = -p + 1, \dots, K_n + p)$ and the number of knots satisfies $K_n = o(n^{1/2})$.
3. There exists $\gamma \geq 0$ such that $E[|(\tau - I(Y < \eta_\tau(x)))|^{2+\gamma} | X = x] < \infty$.
4. The order of the difference matrix is $m \leq p + 1$.
5. The smoothing parameter λ_n is a positive sequence such that λ_n^{-1} is larger than the maximum eigenvalue of $G(\tau)^{-1/2} D_m^T D_m G(\tau)^{-1/2}$.

To describe the asymptotic form of $\hat{\eta}_\tau(x)$, we introduce the following symbols and notations. Define the $(K_n + p)$ th square matrix $G = (G_{ij})_{ij}$ by

$$G_{ij} = \int_0^1 B_{-p+i}(u) B_{-p+j}(u) dQ(u)$$

and the $(K_n + p)$ th square matrix $G(\tau)$ as having the (i, j) -component

$$G_{ij}(\tau) = \int_0^1 f(\eta_\tau(u) | u) B_{-p+i}(u) B_{-p+j}(u) dQ(u),$$

where $f(y|x)$ is the conditional density function of Y given $X = x$.

Let $\mathbf{b}^*(\tau)$ be a best L_∞ approximation to the true function $\eta_\tau(x)$, which satisfies

$$\sup_{x \in (0,1)} |\eta_\tau(x) + b_\tau^a(x) - \mathbf{B}(x)' \mathbf{b}^*(\tau)| = o(K_n^{-(p+1)}), \quad (2)$$

where

$$b_\tau^a(x) = -\frac{\eta_\tau^{(p+1)}(x)}{K_n^{p+1}(p+1)!} \sum_{k=1}^{K_n} I(\kappa_{k-1} \leq x < \kappa_k) \text{Br}_{p+1} \left(\frac{x - \kappa_{k-1}}{K_n^{-1}} \right),$$

$I(a < x < b)$ is the indicator function of an interval (a, b) and $\text{Br}_p(x)$ is the p th Bernoulli polynomial (see Zhou et al. (1998)). Next, we use $\eta_\tau^*(x) = \mathbf{B}(x)^T \mathbf{b}^*(\tau)$.

The penalized spline quantile estimator can be decomposed as

$$\hat{\eta}_\tau(x) - \eta_\tau(x) = \hat{\eta}_\tau(x) - \eta_\tau^*(x) + \eta_\tau^*(x) - \eta_\tau(x) = \hat{\eta}_\tau(x) - \eta_\tau^*(x) + b_\tau^a(x) + o(K_n^{-(p+1)}).$$

We investigate the asymptotic distribution of $\hat{\eta}_\tau(x) - \eta_\tau^*(x)$ in the following Proposition.

Proposition 1. *Let $\eta_\tau(\cdot) \in C^{p+1}$. Furthermore suppose $K_n = O(n^{1/(2p+3)})$ and $\lambda_n = O(n^\nu)$, $\nu \leq (p+m+1)/(2p+3)$. Then under the Assumptions, for $x \in (0, 1)$, as $n \rightarrow \infty$,*

$$\sqrt{\frac{n}{K_n}} \{\hat{\eta}_\tau(x) - \eta_\tau^*(x) - b_\tau^\lambda(x)\} \xrightarrow{D} N(0, \Phi_\tau(x)),$$

where

$$\begin{aligned} b_\tau^\lambda(x) &= -\frac{\lambda_n}{n} \mathbf{B}(x)^T (G(\tau) + (\lambda_n/n) D_m^T D_m)^{-1} D_m^T D_m \mathbf{b}^*(\tau) = O(n^{-(p+1)/(2p+3)}), \\ \Phi_\tau(x) &= \lim_{n \rightarrow \infty} \frac{\tau(1-\tau)}{K_n} \mathbf{B}(x)^T (G(\tau) + (\lambda_n/n) D_m^T D_m)^{-1} G(G(\tau) + (\lambda_n/n) D_m^T D_m)^{-1} \mathbf{B}(x). \end{aligned}$$

The following Theorem, which is the main result in this paper, can be obtained straightforwardly from Proposition 1.

Theorem 1. *Under the same assumptions as Proposition 1, for $x \in (0, 1)$, as $n \rightarrow \infty$,*

$$\sqrt{\frac{n}{K_n}} \{\hat{\eta}_\tau(x) - \eta_\tau(x) - b_\tau^a(x) - b_\tau^\lambda(x)\} \xrightarrow{D} N(0, \Phi_\tau(x)),$$

where $b_\tau^\lambda(x)$ and $\Phi_\tau(x)$ are those given in Proposition 1.

Remark 1 Under the same assumption as Theorem 1, the rate of convergence of the mean squared error (MSE) of $\hat{\eta}_\tau(x)$ becomes

$$E [\{\hat{\eta}_\tau(x) - \eta_\tau(x)\}^2] = O(n^{-(2p+2)/(2p+3)}).$$

This rate is the same as that of the penalized spline estimator in mean regression (see, Kauer- mann et al. (2009)).

Remark 2 For the unpenalized regression spline quantile estimator, its asymptotic normality is obtained through Theorem 1 with $\lambda_n = 0$.

Remark 3 When the true quantile function has a polynomial form $\eta_\tau(x) = a_0 + a_1x + \dots + a_qx^q$ ($q \leq p$), $\eta_\tau(x) = \eta_\tau^*(x)$ is satisfied since the q th polynomial model can be expressed as the linear combination of the p th B -spline bases $\{B_k^{[p]} : k = -p + 1, \dots, K_n\}$ (see de Boor (2001)). Therefore, in this case, the model bias becomes 0, indicating that the regression spline quantile estimator is unbiased. We can definitely show that $E[\psi_\tau(U_i)|\mathbf{X}_n] = 0$ in the proof of Lemma 2.

Remark 4 Let ε_i ($i = 1, \dots, n$) be independently and identically distributed as the density $f_\varepsilon(\varepsilon)$ and assume that X_i and ε_i are independent. Consider the data $\{(y_i, x_i) : i = 1, \dots, n\}$ with $Y_i = \eta(x_i) + \varepsilon_i$. Then the conditional 100 τ % quantile of Y_i given $X_i = x_i$ can be written as $\eta_\tau(x_i) = \eta(x_i) + F_\varepsilon^{-1}(\tau)$, where $F_\varepsilon^{-1}(\tau)$ is the 100 τ % quantile of ε_i . For any $\tau \in (0, 1)$, $\eta_\tau^{(p+1)}(x) = \eta^{(p+1)}(x)$, with which $b_\tau^a(x)$ is unchanged by τ . Next, we obtain $G(\tau) = f_\varepsilon(F_\varepsilon^{-1}(\tau))G$ since $f(\eta_\tau(x)|x)$ is equal to $f_\varepsilon(F_\varepsilon^{-1}(\tau))$. Furthermore, $\mathbf{b}^*(\tau)$ can be written as $\mathbf{b}^*(\tau) = \mathbf{b}^* + F_\varepsilon^{-1}(\tau)\mathbf{1}$, where \mathbf{b}^* is the best L_∞ approximation of $\eta(x)$ defined in the same manner as $\mathbf{b}^*(\tau)$ and $\mathbf{1}$ is a $(K_n + p)$ vector with all components equal to 1. Since all components of $D_m\mathbf{1}$ are vanishing, for $\tau \in (0, 1)$, we have

$$b_\tau^\lambda(x) = -\frac{\lambda_n}{nf_\varepsilon(F_\varepsilon^{-1}(\tau))}\mathbf{B}(x)^T \left(G + \frac{\lambda_n}{nf_\varepsilon(F_\varepsilon^{-1}(\tau))}D_m^TD_m \right)^{-1} D_m^TD_m\mathbf{b}^*.$$

The asymptotic variance of $\hat{\eta}_\tau(x)$ can be written as

$$\Phi_\tau(x) = \lim_{n \rightarrow \infty} \alpha_n(\tau)\mathbf{B}(x)^T \left(G + \frac{\lambda_n}{nf_\varepsilon(F_\varepsilon^{-1}(\tau))}D_m^TD_m \right)^{-1} G \left(G + \frac{\lambda_n}{nf_\varepsilon(F_\varepsilon^{-1}(\tau))}D_m^TD_m \right)^{-1} \mathbf{B}(x),$$

where

$$\alpha_n(\tau) = \frac{\tau(1-\tau)}{\{f_\varepsilon(F_\varepsilon^{-1}(\tau))\}^2 K_n}.$$

When the sample size is sufficiently large under the same assumptions as Theorem 1 and $m < p + 1$, the influences of τ on $b_\tau^\lambda(x)$ and $\Phi_\tau(x)$ appear only as $1/f_\varepsilon(F_\varepsilon^{-1}(\tau))$ and $\tau(1-\tau)/\{f_\varepsilon(F_\varepsilon^{-1}(\tau))\}^2$, respectively. In general, if the density of ε_i is symmetrical at $\varepsilon = 0$, the asymptotic bias and variance of $\hat{\eta}_\tau(x)$ are small at $\tau = 0.5$. Figure 1 shows $1/f_\varepsilon(F_\varepsilon^{-1}(\tau))$ and $\tau(1-\tau)/\{f_\varepsilon(F_\varepsilon^{-1}(\tau))\}^2$ with normal and Cauchy distributions.

We observe that $b_\tau^\lambda(x)$ and $\Phi_\tau(x)$ are smallest at $\tau = 0.5$. For $\Phi_\tau(x)$ near $\tau = 0$ or $\tau = 1$, the effect of τ becomes small.

Remark 5 Claeskens et al. (2009) studied the asymptotics of penalized spline estimators in mean regression, with the estimator $\hat{\eta}(x) = \mathbf{B}(x)^T \hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the minimizer of

$$(\mathbf{y} - Z\mathbf{b})'(\mathbf{y} - Z\mathbf{b}) + \mu_n \int_0^1 \{s^{(m)}(x)\}^2 dx. \quad (3)$$

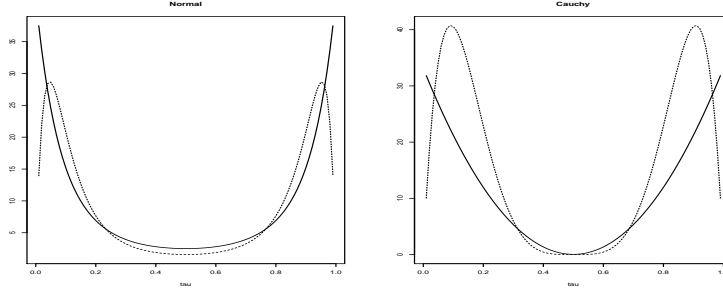


Figure 1: Plots for $1/f_\varepsilon(F_\varepsilon^{-1}(\tau))$ (solid) and $\tau(1-\tau)/\{f_\varepsilon(F_\varepsilon^{-1}(\tau))\}^2$ (dashed). The left panel shows the standard normal distribution and the right panel shows the Cauchy distribution with location 0 and scale 0.01.

Here, $s(x) = \mathbf{B}(x)^T \mathbf{b}$, $\mathbf{b} \in \mathbb{R}^{K_n+p}$ and μ_n is the smoothing parameter. They developed the asymptotics for $\hat{\eta}(x)$ under two scenarios: (a) $K_q = K_q(n, K_n, \mu_n)$, which as given in their paper is less than 1, or (b) $K_q \geq 1$. Assumption 5 of this paper is equal to the condition $K_q < 1$. Together with the approximation property that $\lambda_n \mathbf{b}(\tau)^T D_m^T D_m \mathbf{b}(\tau) \approx \mu_n K_n^{2m-1} \int_0^1 \{s_\tau^{(m)}(x)\}^2 dx$, the results of this paper can be regarded as the quantile regression version of Theorem 2 (a) of Claeskens et al. (2009).

Remark 6 To construct the penalized spline estimator of $\eta_\tau(x)$, we can also use the truncated spline $c_\tau(x) = \mathbf{C}(x)^T \boldsymbol{\theta}(\tau)$ as an approximation to $\eta_\tau(x)$, where $\mathbf{C}(x) = [1 \ x \ \cdots \ x^p \ (x - \kappa_1)_+^p \ \cdots \ (x - \kappa_{K_n-1})_+^p]$, $(x)_+ = \max\{x, 0\}$, and $\boldsymbol{\theta}(\tau) \in \mathbb{R}^{K_n+p}$ is an unknown parameter vector. Pratesi et al. (2009) obtained the estimator $\tilde{\eta}_\tau(x) = \mathbf{C}(x)^T \tilde{\boldsymbol{\theta}}(\tau)$, where $\tilde{\boldsymbol{\theta}}(\tau)$ is the minimizer of

$$\sum_{i=1}^n \rho_\tau(y_i - c_\tau(x_i)) + \mu_n \boldsymbol{\theta}(\tau)^T \Theta \boldsymbol{\theta}(\tau), \quad (4)$$

where μ_n is the smoothing parameter and $\Theta = \text{diag}[O_{p+1} \ I_{K_n-1}]$. By the equivalence property between the B -spline model and truncated model, there exists a $(K_n + p)$ th square and nonsingular matrix L such that $\mathbf{B}(x) = L\mathbf{C}(x)$. Therefore $c_\tau(x)$ can be written as

$$c_\tau(x) = \mathbf{C}(x)^T \boldsymbol{\theta} = \mathbf{B}(x)^T L^{-1} \boldsymbol{\theta}(\tau) = \mathbf{B}(x)^T \mathbf{b}(\tau),$$

where $\mathbf{b}(\tau) = L^{-1} \boldsymbol{\theta}(\tau)$. Furthermore, the penalty term in (4) satisfies from Claeskens et al. (2009)

$$\boldsymbol{\theta}(\tau)^T \Theta \boldsymbol{\theta}(\tau) = K_n^{2p} \mathbf{b}(\tau)^T D_{p+1}^T D_{p+1} \mathbf{b}(\tau)$$

The asymptotic distribution of $\tilde{\eta}_\tau(x) = \mathbf{C}(x)^T \tilde{\boldsymbol{\theta}}(\tau)$ can be obtained by showing that of $\mathbf{B}(x)^T \tilde{\mathbf{b}}$, where $\tilde{\mathbf{b}}$ is the minimizer of

$$\sum_{i=1}^n \rho_\tau(y_i - \mathbf{B}(x_i)^T \mathbf{b}(\tau)) + \mu_n K_n^{2p} \mathbf{b}(\tau)^T D_{p+1}^T D_{p+1} \mathbf{b}(\tau).$$

Then, the asymptotic distribution of $\tilde{\eta}_\tau(x)$ can be obtained using Theorem 1 under $m = p + 1$ and $\lambda_n = \mu_n K_n^{2p}$. Thus, we obtain the asymptotic distribution of the penalized truncated spline quantile estimator.

Remark 7 Under some weakly condition, the local p th polynomial quantile estimator $\tilde{\eta}_\tau(x)$ has an asymptotic order

$$E[\{\tilde{\eta}_\tau(x) - \eta_\tau(x)\}^2] = O(n^{-2(p+1)/(2p+3)})$$

(see Fan et al. (1994) and Ghouch and Genton (2009)) and, hence, it can be said that the rate of convergence of the p th B -spline quantile estimator and the local p th polynomial quantile estimator are the same. We note the bias of these estimators with $p = 1$. From Fan et al. (1994), the asymptotic bias of the local linear quantile estimator is

$$b_\tau^\ell(x) = -\frac{h_n^2 \eta_\tau^{(2)}(x)}{2} \int_{\mathbb{R}} z^2 K(z) dz,$$

where $K(z)$ is the second order kernel function and h_n is the bandwidth. If K_n^{-1} is equal to h_n , then the difference between $b_\tau^a(x)$ and $b_\tau^\ell(x)$ is only

$$\sum_{j=1}^{K_n} I(\kappa_{j-1} \leq x < \kappa_j) \text{Br}_2 \left(\frac{x - \kappa_{j-1}}{K_n^{-1}} \right) \quad \text{and} \quad \int_{\mathbb{R}} z^2 K(z) dz. \quad (5)$$

It is easy to show that $\text{Br}_2(x) = x^2 - x + 1/6 < 1/5$ for $x \in [0, 1]$, while we have $\int_{\mathbb{R}} z^2 K_G(z) dz = 1$ for the Gaussian kernel $K_G(z)$ and $\int_{\mathbb{R}} z^2 K_E(z) dz = 1/5$ for the Epanechnikov kernel $K_E(z)$. Therefore the bias of the regression spline estimator is smaller than that of the local linear estimator in this situation.

4 Numerical study

4.1 Simulation

In this section, we show numerical simulation to confirm the performance as well as the asymptotic normality of the penalized spline quantile estimator claimed in Theorem 1. The explanatory x_i is generated from a uniform distribution on the interval $[0, 1]$. The response Y_i is created by $Y_i = \eta(x_i) + \varepsilon_i$, where $\eta(x) = \sin(2\pi x)$. The errors ε_i 's are independently distributed via (i) a normal distribution with mean 0 and variance $(0.1)^2$, (ii) an exponential distribution with mean 2 and (iii) a Cauchy distribution with location 0 and scale 0.01. In this simulation, to obtain the penalized spline quantile estimator, we use $(p, m) = (3, 2)$ and (K_n, λ_n) is given via the generalized approximate cross-validation (GACV) discussed by Yuan (2006). For comparison, we construct the unpenalized regression spline quantile estimator with linear spline bases ($p = 1$) and the local linear quantile estimator. The penalized spline estimator, regression spline estimator, and local linear estimator are denoted as P-cubic, R-linear and L-linear, respectively. The number of knots of R-linear and the bandwidth of L-linear are given by GACV.

Table 1: Results of MISE for $n = 100$ and $n = 1000$. All entries for MISE are 10^3 times their actual values.

$n = 100$	Normal			Exponential			Cauchy		
τ	P-cubic	R-linear	L-linear	P-cubic	R-linear	L-linear	P-cubic	R-linear	L-linear
0.01	11.89	20.16	20.81	5.21	11.15	11.78	4704.18	6667.28	4122.68
0.1	3.78	4.55	5.03	6.26	9.85	12.50	206.43	340.47	289.05
0.25	3.23	3.27	3.99	10.72	16.68	13.42	18.46	42.15	86.46
0.5	2.87	3.34	3.60	20.26	31.66	27.92	18.88	35.22	43.33
$n = 1000$	Normal			Exponential			Cauchy		
τ	P-cubic	R-linear	L-linear	P-cubic	R-linear	L-linear	P-cubic	R-linear	L-linear
0.01	1.23	1.77	1.94	0.22	0.31	0.31	160.52	910.09	1178.74
0.1	0.46	1.45	0.67	1.08	1.30	1.08	19.53	24.99	53.93
0.25	0.44	1.84	0.52	2.16	2.77	1.91	2.08	7.08	2.17
0.5	0.12	0.34	0.27	3.66	5.60	3.19	0.20	4.81	1.38

Let

$$\text{MSE}_j = \frac{1}{R} \sum_{r=1}^R \{\hat{\eta}_{\tau,r}(z_j) - \eta_{\tau}(z_j)\}^2, \quad \text{MISE} = 100^{-1} \sum_{j=1}^{100} \text{MSE}_j,$$

where $z_j = j/J$, $J = 100$ and $\hat{\eta}_{\tau,r}(z_j)$ is the estimator for the r th repetition. For $\tau = 0.01, 0.1, 0.25$ and 0.5 , we calculate the mean integrated squared error (MISE). We then use sample sizes $n = 100$ and 1000 and the number of repetitions $R = 1000$.

Next, from P-cubic, we calculate

$$U_{\tau,r}(x) = \frac{\hat{\eta}_{\tau,r}(x) - \eta_{\tau}(x)}{\hat{\Phi}_{\tau,r}(x)}, \quad r = 1, \dots, R,$$

where

$$\hat{\Phi}_{\tau,r}(x) = \tau(1 - \tau) \mathbf{B}(x)^T (Z^T \hat{R}_r Z + \lambda_n D_m^T D_m)^{-1} Z^T Z (Z^T \hat{R}_r Z + \lambda_n D_m^T D_m)^{-1} \mathbf{B}(x),$$

$\hat{R}_r = \text{diag}[\hat{f}_r(\hat{\eta}_{\tau,r}(x_i)|x_i)]$ and $\hat{f}_r(y|x)$ is the conditional kernel density estimate given $X = x$. Then we construct the density estimate of $U_{\tau} \equiv \{U_{\tau,1}(x), \dots, U_{\tau,R}(x)\}$ at $x = 0.5$ and compare with the density of $N(0, 1)$. To obtain $\hat{f}_r(y|x)$ and U_{τ} , the normal kernel and the bandwidth discussed by Sheather and Jones (1991) are utilized.

Table 1 shows the MISE for $\tau = 0.01, 0.1, 0.25$ and 0.5 . For P-cubic with normal error, the performance of the quantile estimator is good even if $\tau = 0.01$. It is well known that the Cauchy distribution is a pathological distribution. However, the MISE of P-cubic with the Cauchy distribution is sufficiently small, indicating that the quantile estimator is robust. For the boundary τ , on the other hand, the MISE of the estimators is worse than that with interior τ . For the normal and Cauchy models, the median estimator has better behavior than those

with $\tau = 0.01, 0.1$ and 0.25 . On the other hand, for the exponential model, the median estimator has a larger MISE than $\hat{\eta}_\tau(x)$ with other values of τ . The reason for this is that the density $f(\varepsilon)$ of exponential distribution is monotonically decreasing and its peak is at $\varepsilon = 0$, which leads to many responses Y_i 's being dropped near $\eta(x_i) + F_\varepsilon^{-1}(\tau)$ with small τ . We note the performance of the penalized spline estimator for $\tau > 0.5$. When a normal or Cauchy error is used, it appears that the MISE of $\hat{\eta}_\tau(x)$ and that of $\hat{\eta}_{1-\tau}(x)$ become similar since $Y_i|x_i$ has a symmetrical density function at $\eta(x_i)$. For an exponential error, the closer τ is to 0, the smaller the MISE of $\hat{\eta}_\tau(x)$ will become. Overall, P-cubic has better behavior than R-linear and L-linear. However, for the exponential distribution and $n = 1000$, the MISE of L-linear is slightly smaller than that of P-cubic. Additionally, the performance of L-linear is slightly superior to that of R-linear. This indicates that the variance of L-linear is less than that of R-linear (see Remark 7).

In Figure 2, the density estimate of U_τ for $\tau = 0.1$ and 0.5 and the density of $N(0, 1)$ for each error are illustrated. In all errors, we can see that the density estimate of $U_{0.5}(x)$ becomes close to $N(0, 1)$ as n increases. For a normal distribution with $n = 1000$, the density estimate $U_{0.5}$ and $N(0, 1)$ are similar. In both errors, we see that the speed of convergence of $U_{0.5}$ is faster than that of $U_{0.1}$.

Remark 8 We have confirmed the behavior of the penalized splines with $p = 1$ (P-linear) and the regression splines with $p = 3$ (R-cubic) though this is not shown in this paper for reasons of space. The MISE of P-linear and R-cubic are similar to the P-cubic and R-linear, respectively. For spline smoothing, it is generally known that the pair of the ‘cubic’ spline and the second difference penalty are particularly useful in data analysis. Therefore we mainly focused on $(p, m) = (3, 2)$ in this simulation.

4.2 Application

In this section, we apply the penalized spline quantile estimator to real data. In all examples, we use $(p, m) = (3, 2)$ and (K_n, λ_n) is chosen via GACV.

Figure 3 showed the penalized spline quantile estimators ($\tau = 0.1, \dots, 0.9$) for bone mineral density (BMD) data. This data was presented by Hastie et al. (2009). Takeuchi et al. (2006) applied the kernel estimator to the same data. Compared with Figure 2 (b) of their paper, the penalized splines have a somewhat smooth curve.

Next, the confidence interval of $\eta_\tau(x)$ is illustrated. The $100\alpha\%$ confidence interval of $\eta_\tau(x)$ based on the asymptotic result of $\hat{\eta}_\tau(x)$ is obtained as

$$\left[\hat{\eta}_\tau(x) - \hat{b}_\tau^a(x) - \hat{b}_\tau^\lambda(x) - z_{1-\alpha/2} \sqrt{\hat{\Phi}_\tau(x)}, \hat{\eta}_\tau(x) - \hat{b}_\tau^a(x) - \hat{b}_\tau^\lambda(x) + z_{1-\alpha/2} \sqrt{\hat{\Phi}_\tau(x)} \right], \quad (6)$$

where $\hat{b}_\tau^a(x)$, $\hat{b}_\tau^\lambda(x)$ and $\hat{\Phi}_\tau(x)$ are the estimators of $b_\tau^a(x)$, $b_\tau^\lambda(x)$ and $\Phi_\tau(x)$, while $z_{1-\alpha/2}$ is a $(1 - \alpha/2)$ th normal percentile. As the estimator of $b_\tau^\lambda(x)$,

$$\hat{b}_\tau^\lambda(x) = -\lambda_n \mathbf{B}(x)^T (Z^T \hat{R} Z + \lambda_n D_m^T D_m)^{-1} D_m^T D_m \hat{\mathbf{b}}(\tau)$$

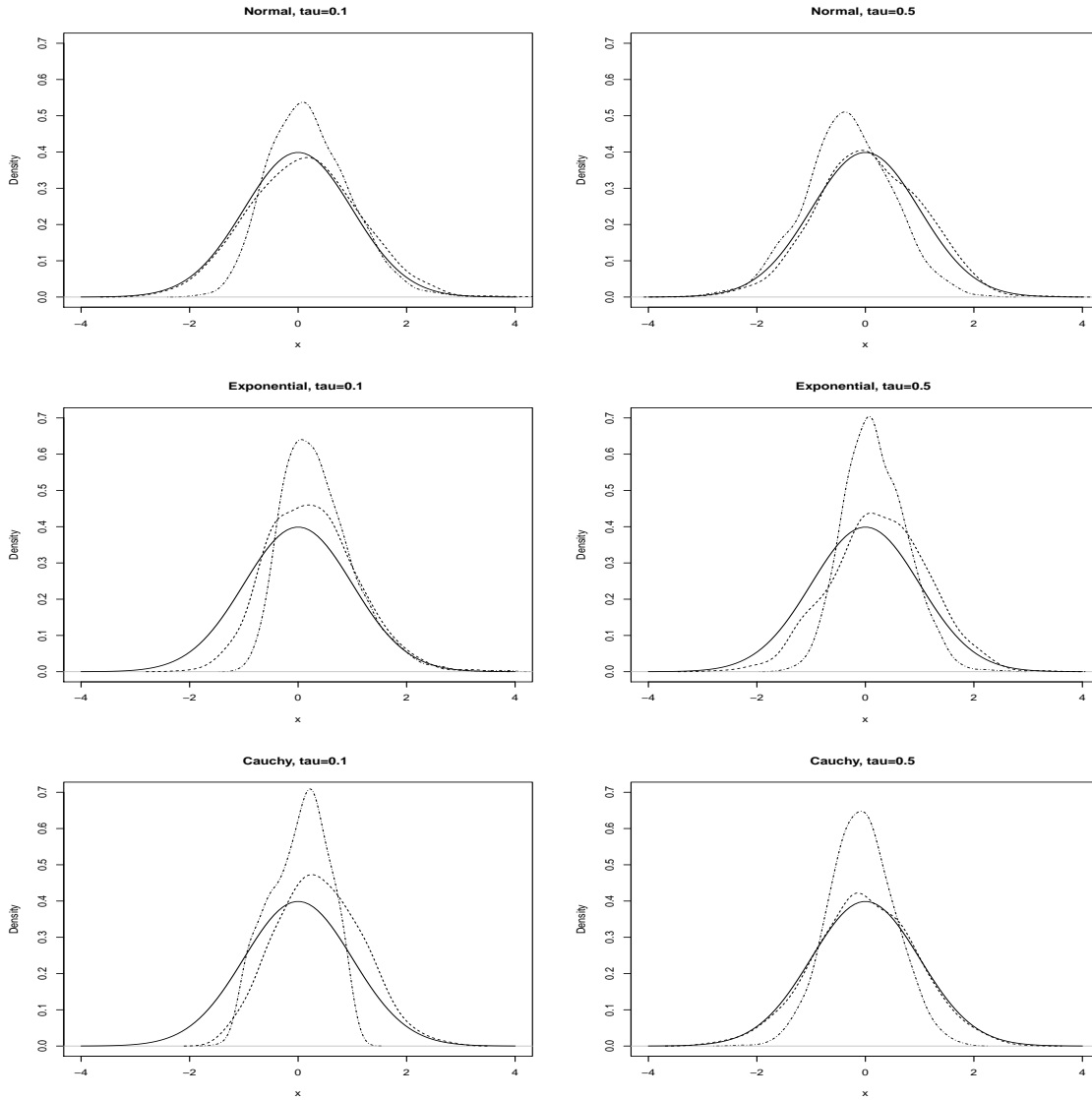


Figure 2: The density estimate of $U_\tau(x)$ for $n = 100$ (dot-dashed) and $n = 1000$ (dashed), and the density of $N(0, 1)$ (solid). The left panels are for $\tau = 0.1$ and the right panels are for $\tau = 0.5$. The upper, middle and bottom panels are for normal, exponential and Cauchy errors, respectively.

is used. We utilize $\hat{\Phi}_\tau(x)$ as given in the previous section. As the pilot estimator of $\eta_\tau^{(p+1)}(x)$ in $\hat{b}_\tau^a(x)$, we construct the $(p+1)$ th derivative of the penalized spline quantile estimator with the $(p+2)$ th B -spline model. Thus, we obtain (6).

In Figure 4, the 95% approximate confidence interval of $\eta_{0.5}(x)$ for motor cycle impact data is drawn. This dataset, with $\{(y_i, x_i) : i = 1, \dots, 132\}$ was given by Härdle (1990), where y_i is the acceleration (g) and x_i is the time (ms). For comparison, the 95% approximate confidence interval with uncorrected bias of $\eta_{0.5}(x)$ defined by

$$\left[\hat{\eta}_\tau(x) - 1.96\sqrt{\hat{\Psi}_\tau(x)}, \quad \hat{\eta}_\tau(x) + 1.96\sqrt{\hat{\Psi}_\tau(x)} \right]$$

is shown. The penalized spline estimator of the median has a curve with fitness and smoothness. In the area near $x = 20$, we see that there is a strong correction of the bias of $\hat{\eta}_{0.5}(x)$.

Finally, we compare the median estimator and the mean estimator for Boston housing data, with $\{(y_i, x_i) : i = 1, \dots, 506\}$, where y_i is the median value of owner-occupied homes in USD 1000s (given by MEDV) and x_i is the average number of rooms per dwelling (denoted RM). This dataset is available from Harrison and Rubinfeld (1979). Figure 5 shows the penalized spline quantile estimator of $\eta_{0.5}(x)$ (solid) and the penalized spline estimator

$$\hat{g}(x) = \mathbf{B}(x)^T (Z^T Z + \mu_n D_m^T D_m)^{-1} Z^T \mathbf{y}$$

of the conditional mean of Y : $g(x) = E[Y|X = x]$ (dashed), where μ_n is the smoothing parameter chosen by generalized cross-validation. At around $x = 5$ and the right-hand side of $x = 8$, the behavior of the median estimator and the mean estimator are different. We see that $\hat{g}(x)$ is affected by extreme points, such as $(x, y) = (4.97, 50)$ and $(x, y) = (8.78, 21.9)$. On the other hand, it appears that the influence of extreme values is limited for the median estimator.

5 Discussion

This paper have discussed the asymptotic theory of the penalized spline quantile estimator. We showed the asymptotic bias and variance as well as the asymptotic normality of the penalized spline quantile estimator. The results can be regarded as the quantile regression version of the Theorem 2 (a) of Claeskens et al. (2009).

As the further study, we may consider the asymptotic property of the penalized splines with multivariate covariate (x_1, \dots, x_d) . Doskum and Koo (2000) have studied the unpenalized spline quantile estimator in additive models, but the asymptotic results were not discussed. The additive model has the true quantile function as $\eta_\tau(x_1, \dots, x_d) = \sum_{i=1}^d \eta_{i\tau}(x_i)$. The aim is then to estimate $\eta_{i\tau}(x_i)$ for each i . Similar to the work of Doskum and Koo, we can construct the penalized spline estimator in additive models. In this field, the asymptotic results should be determined.

In relation to the serious problem of the nonparametric quantile regression, a phenomenon called the “quantile crossing” occurs (see Koenker (2005)). He (1997) and Takeuchi et al.

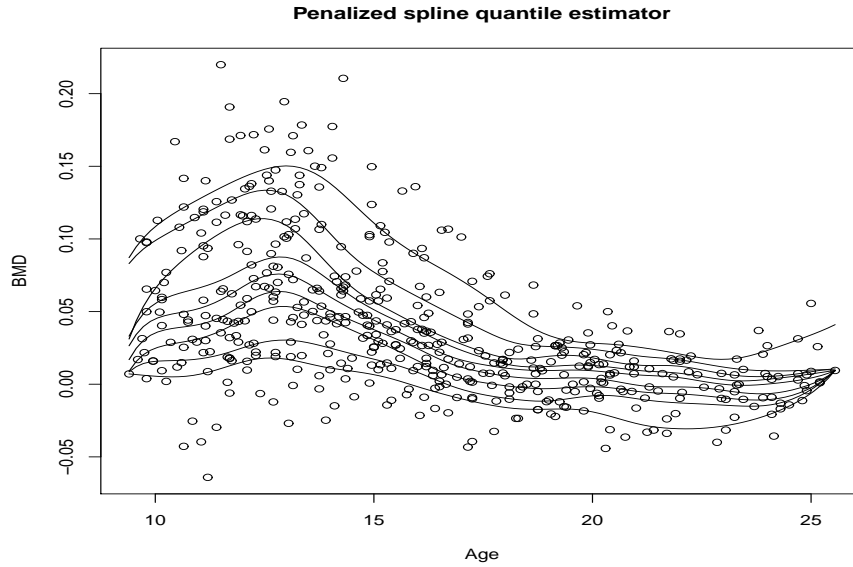


Figure 3: BMD data ($n = 485$) with $\hat{\eta}_\tau(x)$. The solid lines are for $\tau = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$ and 0.9 from the bottom to top.

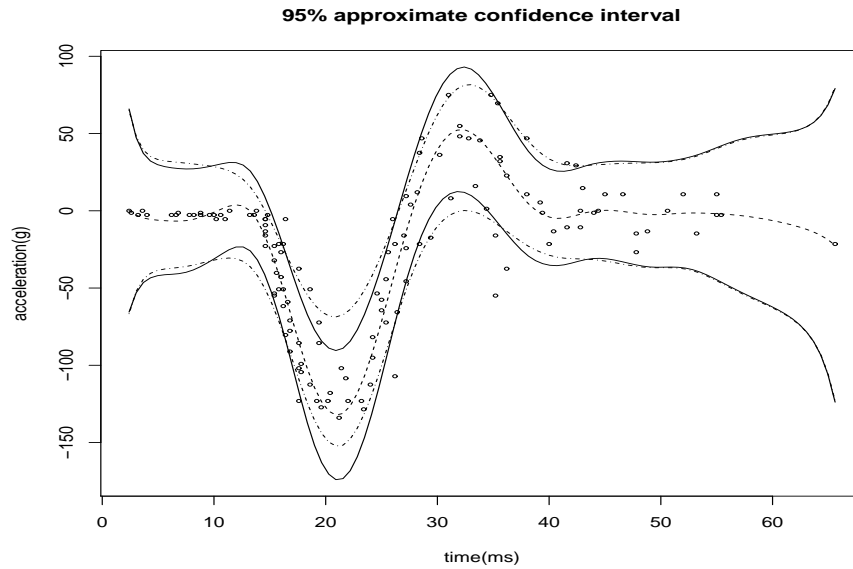


Figure 4: Motor cycle impact data ($n = 132$) with $\hat{\eta}_{0.5}(x)$ (dashed), the 95% approximate confidence intervals (solid) and the 95% approximate confidence intervals with uncorrected bias (dot-dashed).

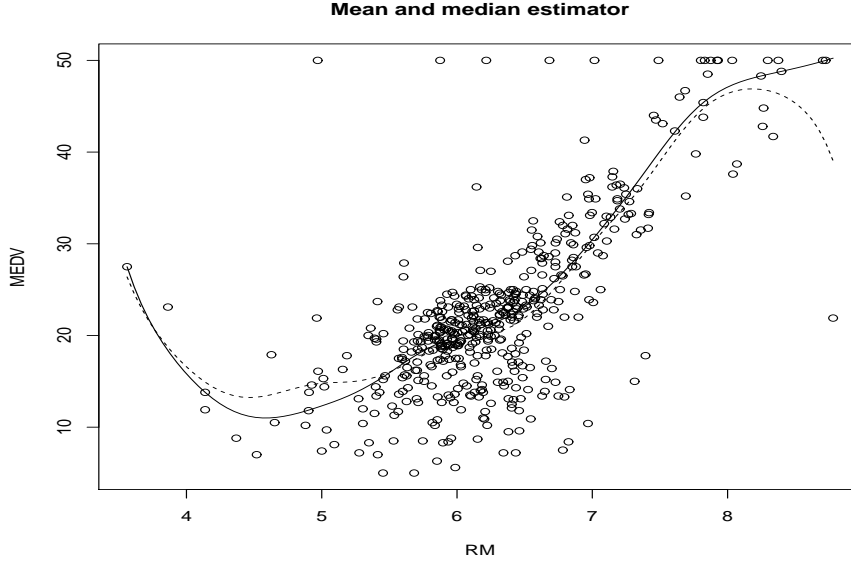


Figure 5: Boston housing data ($n = 506$) with the mean(dashed) and median(solid) estimators.

(2006) studied the original estimation methods of $\eta_\tau(x)$ without quantile crossing. However, the asymptotics for their estimators have not yet been developed. The asymptotic study of the penalized splines without quantile crossing would be an interesting topic for further study. In addition, by using the asymptotic results of the penalized spline estimator $\hat{\eta}_\tau(x)$, it may be possible to construct the penalized spline quantile estimator without quantile crossing although this is beyond the scope of this paper.

Appendix

For a random variable U_n , $E[U_n|\mathbf{X}_n]$ and $V[U_n|\mathbf{X}_n]$ denote the conditional expectation and variance of U_n given $(\mathbf{X}_1, \dots, \mathbf{X}_n) = (\mathbf{x}_1, \dots, \mathbf{x}_n)$, respectively. For the matrix $A = (a_{ij})_{ij}$, $\|A\|_\infty = \max_{ij} \{|a_{ij}|\}$. For random sequence $\{a_n\}$ and $\{b_n\}$, if $a_n/b_n = O_P(1)$, then it is written as $a_n \stackrel{as}{\sim} b_n$.

Lemma 1. Let $A = (a_{ij})_{ij}$ be $(K_n + p)$ matrix and let $H(\tau) = G(\tau) + (\lambda_n/n)D_m^T D_m$. Assume that $K_n \rightarrow \infty$ as $n \rightarrow \infty$, $\|A\|_\infty = O_P(K_n^\alpha)$. Then, under the Assumption, $\|AG\|_\infty = O(K_n^{\alpha-1})$ and $\|AH(\tau)^{-1}\|_\infty = O(K_n^{1+\alpha})$.

Lemma 1 can be proven similar to Lemma 1 of Claeskens et al. (2009). Then, Assumption 5 which guarantees $K_q < 1$ that given in their paper.

Lemma 2. Let $\psi_\tau(u) = \tau - I(u < 0)$, $u_i = y_i - \mathbf{B}(x_i)^T \mathbf{b}^*(\tau)$. Under the same assumption as

Theorem 1,

$$-\sqrt{\frac{K_n}{n}} \sum_{i=1}^n \mathbf{B}(x_i)^T \boldsymbol{\delta} \psi_\tau(u_i) \stackrel{as}{\approx} \sqrt{K_n} W^T \boldsymbol{\delta},$$

where $W \sim N(\mathbf{0}, \tau(1-\tau)G)$.

Proof of Lemma 2. Let

$$Z_n = -\sqrt{\frac{K_n}{n}} \sum_{i=1}^n \mathbf{B}(x_i)^T \boldsymbol{\delta} \psi_\tau(u_i).$$

We show the asymptotic distribution of Z_n by Lyapunov's theorem. First from the fact that $P(Y < \eta_\tau(x_i) | X_i = x_i) = \tau$, we have

$$\begin{aligned} E[\psi_\tau(U_i) | \mathbf{X}_n] &= \tau - E[I(Y_i < \mathbf{B}(x_i)^T \mathbf{b}^*(\tau)) | \mathbf{X}_n] \\ &= \tau - P(Y < \mathbf{B}(x_i)^T \mathbf{b}^*(\tau) | X_i = x_i) \\ &= \tau - P(Y < \eta_\tau(x_i) + b_a(x_i, \tau)(1 + o(1)) | X_i = x_i) \\ &= -b_a(x_i, \tau) f(\eta_\tau(x_i) | x_i) (1 + o(1)) \\ &= o(1). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &E \left[\left| \sqrt{\frac{K_n}{n}} \mathbf{B}(x_i)^T \boldsymbol{\delta} \{ \psi_\tau(U_i) - E[\psi_\tau(U_i) | \mathbf{X}_n] \} \right|^{2+\gamma} \middle| \mathbf{X}_n \right] \\ &= \left(\frac{K_n}{n} \right)^{(2+\gamma)/2} |\mathbf{B}(x_i)^T \boldsymbol{\delta}|^{2+\gamma} E[|\psi_\tau(U_i)|^{2+\gamma} + o(1) | \mathbf{X}_n] \\ &\leq O \left(\left(\frac{K_n}{n} \right)^{(2+\gamma)/2} \right). \end{aligned}$$

The straightforward calculation yields

$$\begin{aligned} V[Z_n | \mathbf{X}_n] &= \frac{K_n}{n} \sum_{i=1}^n \{ \mathbf{B}(x_i)^T \boldsymbol{\delta} \}^2 V[\psi_\tau(Y_i - \mathbf{B}(x_i)^T \mathbf{b}^*(\tau)) | \mathbf{X}_n] \\ &= \tau(1-\tau) \boldsymbol{\delta}^T \left(\frac{K_n}{n} \sum_{i=1}^n \mathbf{B}(x_i) \mathbf{B}(x_i)^T \right) \boldsymbol{\delta} \\ &= K_n \tau(1-\tau) \boldsymbol{\delta}^T G \boldsymbol{\delta} (1 + o_P(1)) \\ &= O(K_n) \end{aligned}$$

So it follows that

$$\begin{aligned} &\frac{1}{V[Z_n | \mathbf{X}_n]^{(2+\gamma)/2}} \sum_{i=1}^n E \left[\left| \sqrt{\frac{K_n}{n}} \mathbf{B}(x_i)^T \boldsymbol{\delta} \{ \psi_\tau(U_i) - E[\psi_\tau(U_i) | \mathbf{X}_n] \} \right|^{2+\gamma} \middle| \mathbf{X}_n \right] \\ &\leq O(K_n^{-(2+\gamma)/2}) O \left(n \left(\frac{K_n}{n} \right)^{(2+\gamma)/2} \right) \\ &= o(1) \end{aligned}$$

since $\gamma \geq 0$. This leads to

$$\frac{Z_n - E[Z_n|\mathbf{X}_n]}{V[Z_n|\mathbf{X}_n]} \xrightarrow{D} N(0, 1)$$

from Lyapunov's theorem. The expectation of Z_n can be calculated as

$$\begin{aligned} E[Z_n|\mathbf{X}_n] &= -\sqrt{\frac{K_n}{n}} \sum_{i=1}^n \mathbf{B}(x_i)^T \boldsymbol{\delta} E[\psi_\tau(U_i)|\mathbf{X}_n] \\ &= \sqrt{\frac{K_n}{n}} \sum_{i=1}^n \mathbf{B}(x_i)^T \boldsymbol{\delta} b_a(x_i, \tau) f(\eta_\tau(x_i)|x_i) (1 + o_P(1)) \\ &= \sqrt{nK_n} \int_0^1 \mathbf{B}(u)^T \boldsymbol{\delta} b_a(u, \tau) f(\eta_\tau(u)|u) du (1 + o_P(1)). \end{aligned}$$

From the proof of Lemma 6.10 of Argwall and Studen (1989), for $j = -p + 1, \dots, K_n$, we have

$$\int_0^1 B_j(u) b_a(u, \tau) f(\eta_\tau(u)|u) du (1 + o(1)) = o(K_n^{-(p+2)}),$$

by which $\sqrt{nK_n} o(K_n^{-(p+2)}) = o(1)$. Consequently, we have $E[Z_n|\mathbf{X}_n]/V[Z_n|\mathbf{X}_n] = o_P(1)$ and Lemma 2 holds. \square

Lemma 3. Let $w_{in} = \sqrt{K_n/n} \mathbf{B}(x_i)^T \boldsymbol{\delta}$ ($i = 1, \dots, n$) for $\boldsymbol{\delta} \in \mathbb{R}^{K_n+p}$. Then, under the assumptions,

$$\sum_{i=1}^n \int_0^{w_{in}} \{I(u_i \leq s) - I(u_i \leq 0)\} ds \approx \frac{K_n}{2} \boldsymbol{\delta}^T G(\tau) \boldsymbol{\delta}.$$

Proof of Lemma 3. Let

$$R_n = \sum_{i=1}^n \int_0^{w_{in}} \{I(u_i \leq s) - I(u_i \leq 0)\} ds.$$

Since

$$\begin{aligned} &E \left[\int_0^{w_{in}} \{I(u_i \leq s) - I(u_i \leq 0)\} ds \middle| \mathbf{X}_n \right] \\ &= \int_0^{w_{in}} E[\{I(U_i \leq s) - I(U_i \leq 0)\} | \mathbf{X}_n] ds \\ &= \int_0^{w_{in}} \{P(Y_i < \mathbf{B}(x_i)^T \mathbf{b}^*(\tau) + s | X_i = x_i) - P(Y_i < \mathbf{B}(x_i)^T \mathbf{b}^*(\tau) | X_i = x_i)\} ds \\ &= \sqrt{\frac{K_n}{n}} \int_0^{\mathbf{B}(x_i)^T \boldsymbol{\delta}} \left\{ P \left(Y_i < \mathbf{B}(x_i)^T \mathbf{b}^*(\tau) + t \frac{K_n}{n} \middle| X_i = x_i \right) - P(Y_i < \mathbf{B}(x_i)^T \mathbf{b}^*(\tau) | X_i = x_i) \right\} dt \\ &= \frac{K_n}{n} \int_0^{\mathbf{B}(x_i)^T \boldsymbol{\delta}} f(\mathbf{B}(x_i)^T \mathbf{b}^*(\tau) | x_i) t dt \\ &= \frac{K_n}{2n} f(\mathbf{B}(x_i)^T \mathbf{b}^*(\tau) | x_i) \{\mathbf{B}(x_i)^T \boldsymbol{\delta}\}^2. \end{aligned}$$

Therefore we obtain

$$\begin{aligned}
E[R_n|\mathbf{X}_n] &= \frac{K_n}{2n} \sum_{i=1}^n f(\mathbf{B}(x_i)^T \mathbf{b}^*(\tau)|x_i) \boldsymbol{\delta}^T \mathbf{B}(x_i) \mathbf{B}(x_i)^T \boldsymbol{\delta} \\
&= \frac{K_n}{2} \boldsymbol{\delta}^T \left(\frac{1}{n} \sum_{i=1}^n f(\eta_\tau(x_i) + o(1)|x_i) \mathbf{B}(x_i) \mathbf{B}(x_i)^T \right) \boldsymbol{\delta} \\
&= \frac{K_n}{2} \boldsymbol{\delta}^T G(\tau) \boldsymbol{\delta} (1 + o_P(1)).
\end{aligned}$$

Finally, we show $V[R_n|\mathbf{X}_n] = o_P(1)$. For $i = 1, \dots, n$, we have

$$\int_0^{w_{in}} \{I(u_i \leq s) - I(u_i \leq 0)\} ds \leq \sqrt{\frac{K_n}{n}} \mathbf{B}(x_i)^T \boldsymbol{\delta}.$$

Therefore the variance of R_n can be evaluated as

$$\begin{aligned}
V[R_n|\mathbf{X}_n] &\leq \sum_{i=1}^n E \left[\left(\int_0^{w_{in}} \{I(u_i \leq s) - I(u_i \leq 0)\} ds \right)^2 \middle| \mathbf{X}_n \right] \\
&\leq \sqrt{\frac{K_n}{n}} \max_{i=1, \dots, n} \{ \mathbf{B}(x_i)^T \boldsymbol{\delta} \} E[R_n|\mathbf{X}_n].
\end{aligned}$$

Since $E[R_n|\mathbf{X}_n] = O(K_n)$, we obtain $\sqrt{V[R_n|\mathbf{X}_n]}/E[R_n|\mathbf{X}_n] = o_P(1)$ and, hence, Lemma 3 holds. \square

Proof of Proposition 1. Let

$$\begin{aligned}
U_n(\boldsymbol{\delta}) &= \sum_{i=1}^n \left[\rho_\tau \left(u_i - \sqrt{\frac{K_n}{n}} \mathbf{B}(x_i)^T \boldsymbol{\delta} \right) - \rho_\tau(u_i) \right] \\
&\quad + \frac{\lambda_n}{2} \left(\mathbf{b}^*(\tau) + \sqrt{\frac{K_n}{n}} \boldsymbol{\delta} \right)^T D_m^T D_m \left(\mathbf{b}^*(\tau) + \sqrt{\frac{K_n}{n}} \boldsymbol{\delta} \right) - \frac{\lambda_n}{2} \mathbf{b}^*(\tau)^T D_m^T D_m \mathbf{b}^*(\tau),
\end{aligned}$$

where $u_i = y_i - \mathbf{B}(x_i)^T \mathbf{b}^*(\tau)$. Then the minimizer $\hat{\boldsymbol{\delta}}_n(\tau)$ of $U_n(\boldsymbol{\delta})$ can be obtained as

$$\hat{\boldsymbol{\delta}}_n(\tau) = \sqrt{\frac{n}{K_n}} (\hat{\mathbf{b}}(\tau) - \mathbf{b}^*(\tau)).$$

First we show the convergence point $U_0(\boldsymbol{\delta})$ of $U_n(\boldsymbol{\delta})$ for any $\boldsymbol{\delta} \in \mathbb{R}^{K_n+p}$. For the following discussion, we introduce the Knight's identity (see, Knight (1998)):

$$\rho_\tau(u - v) - \rho_\tau(u) = -v \psi_\tau(u) + \int_0^v \{I(u \leq s) - I(u \leq 0)\} ds, \quad (7)$$

where $\psi_\tau(u) = \tau - I(u < 0)$. By using (7), we can write $U_n(\boldsymbol{\delta})$ as

$$U_n(\boldsymbol{\delta}) = U_{1n}(\boldsymbol{\delta}) + U_{2n}(\boldsymbol{\delta}) + U_{3n}(\boldsymbol{\delta}) + U_{4n}(\boldsymbol{\delta}),$$

where

$$\begin{aligned}
U_{1n}(\boldsymbol{\delta}) &= -\sqrt{\frac{K_n}{n}} \sum_{i=1}^n \mathbf{B}(x_i)^T \boldsymbol{\delta} \psi_\tau(u_i), \\
U_{2n}(\boldsymbol{\delta}) &= \sum_{i=1}^n \int_0^{w_{in}} \{I(u_i \leq s) - I(u_i \leq 0)\} ds, \\
U_{3n}(\boldsymbol{\delta}) &= \frac{\lambda_n K_n}{2n} \boldsymbol{\delta}^T D_m^T D_m \boldsymbol{\delta}, \\
U_{4n}(\boldsymbol{\delta}) &= \lambda_n \sqrt{\frac{K_n}{n}} \mathbf{b}^*(\tau)^T D_m^T D_m \boldsymbol{\delta},
\end{aligned}$$

where $w_{in} = \sqrt{K_n/n} \mathbf{B}(x_i)^T \boldsymbol{\delta}$. From Lemma 1, $U_{1n}(\boldsymbol{\delta})$ satisfies

$$U_{1n}(\boldsymbol{\delta}) \stackrel{as}{\approx} -\sqrt{K_n} W^T \boldsymbol{\delta},$$

where $W \sim N(\mathbf{0}, \tau(1-\tau)G)$. Furthermore Lemma 2 and $U_{3n}(\boldsymbol{\delta})$ yield

$$U_{2n}(\boldsymbol{\delta}) + U_{3n}(\boldsymbol{\delta}) \stackrel{as}{\approx} \frac{K_n}{2} \boldsymbol{\delta}^T \left(G(\tau) + \frac{\lambda_n}{n} D_m^T D_m \right) \boldsymbol{\delta}.$$

Therefore, we obtain

$$U_n(\boldsymbol{\delta}) \stackrel{as}{\approx} U_0(\boldsymbol{\delta}) = -\sqrt{K_n} W^T \boldsymbol{\delta} + \lambda_n \sqrt{\frac{K_n}{n}} \mathbf{b}^*(\tau)^T D_m^T D_m \boldsymbol{\delta} + \frac{K_n}{2} \boldsymbol{\delta}^T \left(G(\tau) + \frac{\lambda_n}{n} D_m^T D_m \right) \boldsymbol{\delta}.$$

Because $U_0(\boldsymbol{\delta})$ is convex with respect to $\boldsymbol{\delta}$ and has unique minimizer, the minimizer $\hat{\boldsymbol{\delta}}_n(\tau)$ of $U_n(\boldsymbol{\delta})$ converge to $\boldsymbol{\delta}_0(\tau) = \operatorname{argmin}_{\boldsymbol{\delta}} \{U_0(\boldsymbol{\delta})\}$. This fact is detailed in Pollard(1991), Knight (1998) and Kato (2009). Hence we have

$$\sqrt{\frac{n}{K_n}} \{\hat{\boldsymbol{b}}(\tau) - \mathbf{b}^*(\tau)\} \stackrel{as}{\approx} \boldsymbol{\delta}_0(\tau) = \left(G(\tau) + \frac{\lambda_n}{n} D_m^T D_m \right)^{-1} \left(\frac{1}{\sqrt{K_n}} W - \frac{\lambda_n}{\sqrt{n} K_n} D_m^T D_m \mathbf{b}^*(\tau) \right).$$

Since $\hat{\eta}_\tau(x) - \eta_\tau^*(x) = \mathbf{B}(x)^T (\hat{\boldsymbol{b}}(\tau) - \mathbf{b}^*(\tau))$, we obtain for $x \in (0, 1)$, as $n \rightarrow \infty$,

$$\sqrt{\frac{n}{K_n}} \{\hat{\eta}_\tau(x) - \eta_\tau^*(x) - b_\tau^\lambda(x)\} \xrightarrow{D} N(0, \Phi_\tau(x))$$

by the definition of W . We can confirm with Lemma 1 that $\Phi_\tau(x) = O(1)$. Finally we show the asymptotic order of $b_\tau^\lambda(x)$. Let $\mathbf{B}^{[p]}(x) = (B_{-p+1}^{[p]}(x) \cdots B_{K_n}^{[p]}(x))^T$. By the properties of the derivative of the B -spline model, we have $s_\tau^{(m)}(x) = \partial^m s_\tau(x) / \partial x^m = K_n^m \mathbf{B}^{[p-m]}(x)^T D_m \mathbf{b}(\tau)$. Therefore we obtain $\mathbf{B}^{[p-m]}(x)^T \{K_n^m D_m \mathbf{b}^*(\tau)\} = \eta_\tau^{(m)}(x)(1 + o(1))$ for $m \leq p$. Since the asymptotic order of $\mathbf{B}^{[p-m]}(x)^T \{K_n^m D_m \mathbf{b}^*(\tau)\}$ and that of $\|K_n^m D_m \mathbf{b}^*(\tau)\|_\infty$ are the same as $O(1)$, $\|D_m \mathbf{b}^*(\tau)\|_\infty = O(K_n^{-m})$ is satisfied for $m \leq p$. In addition, similar to the proof of Theorem 1 of Kauermann et al. (2009), $\|D_{p+1} \mathbf{b}^*(\tau)\|_\infty = O(K_n^{-(p+1)})$ is fulfilled. Together with Lemma 1, we obtain

$$b_\tau^\lambda(x) = -\frac{\lambda_n}{n} \mathbf{B}(x)^T \left(G(\tau) + \frac{\lambda_n}{n} D_m^T D_m \right)^{-1} D_m^T D_m \mathbf{b}^*(\tau) = O(\lambda_n n^{-1} K_n^{1-m}) = O(n^{-(p+1)/(2p+3)}).$$

Thus Proposition 2 has been proven. \square

Proof of Theorem 1. Theorem 1 can be proven directly from Propositions 1. Under the condition $K_n = O(n^{1/(2p+3)})$, we have

$$\sqrt{\frac{n}{K_n}}\{\hat{\eta}_\tau(x) - \eta_\tau^*(x) - b_\tau^\lambda(x)\} = \sqrt{\frac{n}{K_n}}\{\hat{\eta}_\tau(x) - \eta_\tau^*(x) - b_\tau^a(x) + o(K_n^{-(p+1)}) - b_\tau^\lambda(x)\}$$

and $\sqrt{n/K_n}b_\tau^a(x) = O(\sqrt{n/K_n}K_n^{-(p+1)}) = O(1)$. This completes the proof. \square

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