An Analysis of Bodies Having Minimum Pressure Drag in Supersonic Flow: Exploring the Nonlinear Domain

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One of the most interesting problems in supersonic aerodynamics has been to find profile shapes that have minimum pressure drag, subject to a set of geometric constraints. Considerable work was done in this area in the fifties, all of which relied on a linearized flow model. We revisit this problem using a more sophisticated flow model. It is logical to expect the optimum profile shape to look different. We confirm this, but also note that the differences are very small. We then examine the equations of fluid flow and try to see why the linearized flow model works well for this problem and where the differences come from.

1 Problem Definition

The problem that we study in this paper is that of finding the two-dimensional (airfoils) and axisymmetric (bodies of revolution) profile sections that have minimum pressure drag in supersonic flow. We assume that the flow is inviscid, modelled by the nonlinear Euler equations. We also enforce the constraints that the ends are pointed and the enclosed area/volume is fixed.

2 Results from Classical Theory

Analytical solutions for the problem being studied have been obtained, assuming a linearized flow model. For the $2-d$ case the optimum profile is parabolic.

$$y(x) = 3Ax(1-x) \; , \quad \tau = \frac{3A}{2} \; , \tag{1}$$

where A is the area enclosed and τ is the thickness-chord ratio. The drag coefficient is given by

$$C_d = \frac{12A^2}{\sqrt{M^2-1}} \; . \tag{2}$$

For the axisymmetric case, the profile shapes that solve this problem are the well known Sears-Haack profiles, discovered independently by Sears(1947) and Haack(1947). The derivation of the Sears-Haack profiles is outlined in the book by Ashley and Landahl [1] and also in an article by Carlo Ferrari [2]. The Sears-Haack profile is given by

$$y(x) = \sqrt{\frac{16V}{3\pi^2}}\,[4x(1-x)]^{\frac{3}{4}} \;\;, \tau = \sqrt{\frac{64V}{3\pi^2}} \;, \tag{3}$$

where V is the enclosed volume and τ is the fineness ratio. The drag coefficient is given by

$$C_D = 24V \; . \tag{4}$$

As can be observed, these profile shapes have some interesting properties. Firstly, they are unique solutions to the optimization problem. Moreover, they are just a function of the enclosed area/volume and not the Mach Number.

3 Nonlinear Optimization via Control Theory

In this work we apply the adjoint method developed by Jameson and his associates during the last 15 years. ([3, 4, 5, 6]) The aerodynamic shape optimization problem involves minimizing(or maximizing) a given cost function, with parameters that define the shape of the body as the design variables, usually of the form

$$I = \int_{\mathcal{B}_\xi} \mathcal{M}(w,S)\, d\mathcal{B}_\xi \;, \tag{5}$$

where w is the vector of flow state variables and S_{ij} are the coefficients of the Jacobian matrix of the transformation from physical space to computational space. $\mathcal{M}(w,S)$ in our case is just C_p, the pressure coefficient. We also have the constraint that the state variables at the computational points have to satisfy the flow equations, irrespective of the shape of the boundary.

$$\int_{\mathcal{B}} n_i \phi^T f_i(w) d\mathcal{B} = \int_{\mathcal{D}} \frac{\partial \phi^T}{\partial x_i} f_i(w) d\mathcal{D} \;, \tag{6}$$

or, when transformed to computational space

$$\int_{\mathcal{B}_\xi} n_i \phi^T S_{ij} f_j(w) d\mathcal{B}_\xi = \int_{\mathcal{D}_\xi} \frac{\partial \phi^T}{\partial \xi_i} S_{ij} f_j(w) d\mathcal{D}_\xi \;, \tag{7}$$

where ϕ is any arbitrary test function.

Since equation (7) is true for any test function ϕ, we can choose ϕ to be the adjoint variable ψ. We can then add equation (7) to the cost function defined in (5) to form the following augmented cost function.

$$I = \int_{\mathcal{B}_\xi} \mathcal{M}(w,S)\, d\mathcal{B}_\xi + \int_{\mathcal{B}_\xi} n_i \psi^T S_{ij} f_j(w) d\mathcal{B}_\xi - \int_{\mathcal{D}_\xi} \frac{\partial \psi^T}{\partial \xi_i} S_{ij} f_j(w) d\mathcal{D}_\xi \;. \tag{8}$$

We then take a variation of the cost function described in (8)

$$\delta I = \int_{\mathcal{B}_\xi} \left(\frac{\partial \mathcal{M}}{\partial w} \delta w + \delta \mathcal{M}_{II} \right) d\mathcal{B}_\xi \tag{9}$$
$$+ \int_{\mathcal{B}_\xi} n_i \psi^T \left(S_{ij} \frac{\partial f_j(w)}{\partial w} \delta w + \delta S_{ij} f_j(w) \right) d\mathcal{B}_\xi$$
$$- \int_{\mathcal{D}_\xi} \frac{\partial \psi^T}{\partial \xi_i} \left(S_{ij} \frac{\partial f_j(w)}{\partial w} \delta w + \delta S_{ij} f_j(w) \right) d\mathcal{D}_\xi \ .$$

We choose ψ such that the variation in the cost function δI does not depend on the variation of the solution δw. ψ is then a solution of the adjoint equations

$$\frac{\partial \mathcal{M}}{\partial w} = -n_i \psi^T S_{ij} \frac{\partial f_j(w)}{\partial w} \text{ , on } \mathcal{B}_\xi \ , \tag{10}$$
$$\left(S_{ij} \frac{\partial f_j(w)}{\partial w} \right)^T \frac{\partial \psi}{\partial \xi_i} = 0, \quad \text{on } \mathcal{D}_\xi \ .$$

One thus obtains an expression for the change in the cost function of the form

$$\delta I = \int_{\mathcal{B}_\xi} \mathcal{G} \delta \mathcal{F} d\mathcal{B}_\xi \ , \tag{11}$$

where $\mathcal{F}(\xi)$ is a function defining the shape and $\mathcal{G}$ is the required gradient.

The gradient with respect to the design variables is obtained from the solutions to the adjoint equations by a reduced gradient formulation ([5]). This is modified to account for the area/volume constraints. In order to preserve the smoothness of the profile the gradient is smoothed by an implicit smoothing formula. This corresponds to redefining the gradient with respect to a weighted Sobolev inner product ([4]). The optimum is then found by a sequential procedure in which the shape is modified in a descent direction defined by the smoothed gradient at each step, and the flow solution and the gradient are recalculated after each shape change.

4 Results and Discussions

Optimum Profile Shapes

The results of the 2D optimization can be seen in Fig. 1 and the results of the axisymmetric optimization can be seen in Fig. 2. As can be observed, the nonlinear optimum profiles are slightly different from the classical optimum profiles. They have a more rearward point of maximum thickness. The primary difference between a linearized flow model and a nonlinear model is the appearance of shocks at the leading edge in the case of the nonlinear flow model. Reducing the included angle at the leading edge and moving the point of maximum thickness backward is consistent with reducing the magnitude of the leading edge shock. This results in a lower drag and at the same time brings the flow closer to the linear regime.

The difference is hardly noticable for small thickness-chord/fineness ratios. This is just an indicator to the fact that linear theory is a very good approximation for small fineness ratios. Moreover, the nonlinear optimum profiles for axisymmetric flow are a lot closer to their corresponding classical profiles than for 2D flow. This is because of the three-dimensional relieving effect experienced in axisymmetric flow.

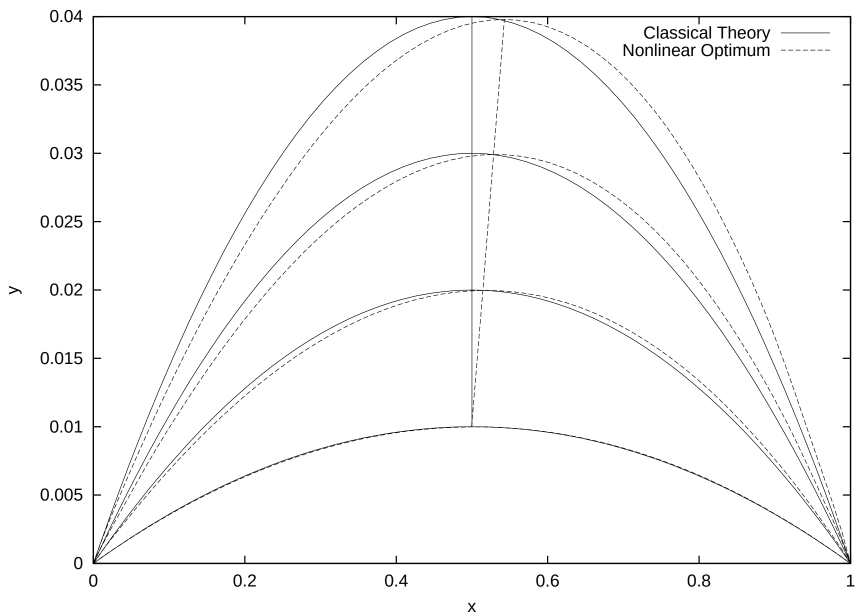

Fig. 1. Classical and Nonlinear Optimum Profiles for 2-D flow

Variation with Mach Number

The optimum profile for 2D flow changes with Mach number. The optimum shape for two Mach numbers is shown in Fig. 3. It is seen that the point of mazimum thickness is more backward for the higher Mach number. This again is consistent with our earlier argument that the main goal of the nonlinear optimization is to reduce the magnitude of the leading edge shock. Such a variation is not observed for axisymmetric optimum profiles. This is due to the fact that the drag coefficient is not sensitive to changes in Mach number in this case. This can also be seen from Eqn. 4.

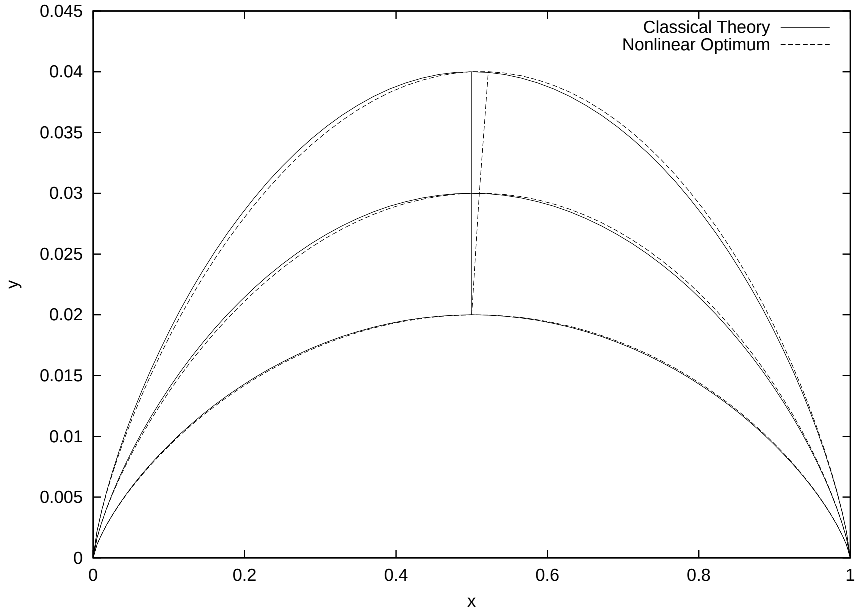

Fig. 2. Classical and Nonlinear Optimum Profiles for Axisymmetric flow

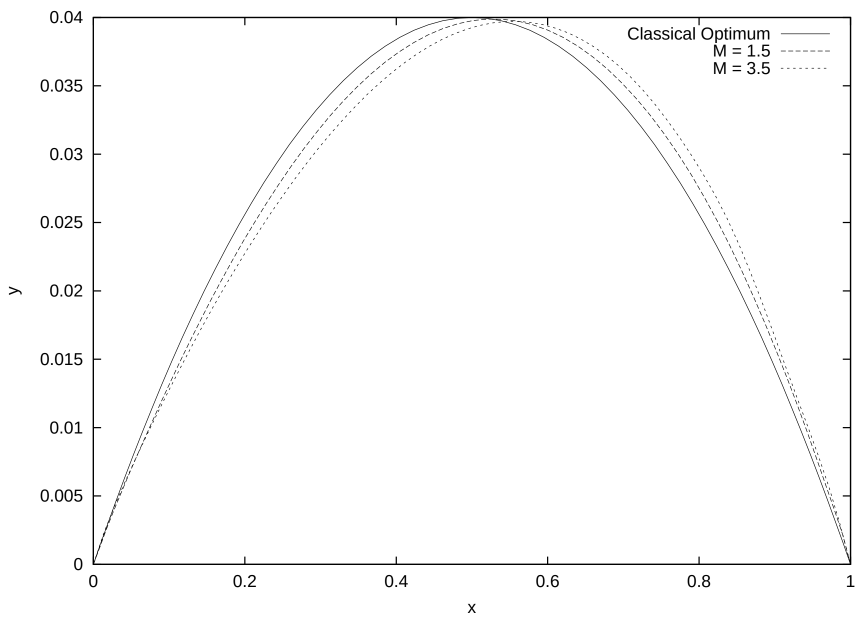

Fig. 3. Variation of 2D Optimum Profiles with Mach Number

Discussion of Results

The main assumptions of linear theory are that the flow is isentropic, irrotational, and the perturbation velocities in the axial and normal directions are very small compared to the free-stream velocity. We observe that these are valid assumptions except in the vicinity of the leading and the trailing edges, where we have stagnation points. Here we have shocks that cause entropy jumps. Moreover, the perturbation velocities are no longer small enough. Thus we expect to see the biggest change at these points. This is found to be true.

5 Conclusions

The minimum pressure drag problem was solved using a nonlinear flow model. It is observed that the optimum shapes thus obtained are slightly different from the classical results. The difference is even smaller for the case of axisymmetric flow. It can be concluded that linearized theory provides a very good approximation of the flow field for the regimes considered and thus the optimum shapes obtained using a nonlinear flow model will be very close to the classical results based on linear theory.

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