Chapter 9 Graphs

9.4 Connectivity

1. Introduction

- Introduction
  - read page 621
- Paths
  - Definition 1 (page 622)
    - Let n be a nonnegative integer and G an undirected graph. A path of length n from u to v in G is a sequence of n edges $e_1,\ldots,e_n$ of G such that $f(e_1)=\{x_0, x_1\}$, $f(e_2)=\{x_1, x_2\}$, ..., $f(e_n)=\{x_{n-1}, x_n\}$, where $x_0=u$ and $x_n=v$. When the graph is simple, we denote this path by its vertex sequence (since listing these vertices uniquely determines the path).

2. Paths

- Definition 1 (page 622)
  - The path is a circuit if it begins and ends at the same vertex, that is, if $u=v$, and has length greater than zero. The path or circuit is said to pass through the vertices $x_1, x_2, \ldots, x_{n-1}$ or traverse the edges $e_1, e_2, \ldots, e_n$.
  - A path or circuit is simple if it does not contain the same edge more than once

2. Paths

- Example 1 (page 622)
  - In the simple graph shown in Figure 1, a, d, c, f, e is a simple path of length 4, since {a,d},{d,c}, {c,f} and {f,e} are all edges.
  - However, d,e,c,a is not a path, since {e,c} is not an edge.
  - Note that b,c,f,e,b is a circuit of length 4 since {b,c},{c,f},{f,e}, and {e,b} are edges, and this path begins and ends at b. The path a,b,e,d,a,b, which is of length 5, is not simple since it contains the edge {a,b} twice.

2. Paths

- Definition 2 (page 623)
  - Let n be a nonnegative integer and G a directed multigraph.
  - A path of length n from u to v in G is a sequence of edges $e_1, e_2, \ldots, e_n$ of G such that $f(e_1) = (x_0,x_1)$, $f(e_2) = (x_1,x_2),\ldots, f(e_n) = (x_{n-1},x_n)$, where $x_0 = u$ and $x_n = v$.
  - When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_0, x_1, x_2, \ldots, x_n$.

2. Paths

- Definition 2 (page 623)
  - A path of length greater than zero that begins and ends at the same vertex is called a circuit or cycle.
  - A path or circuit is called simple if it does not contain the same edge more than once.

3. Connectedness in Undirected Graphs

- Definition 3 (page 624)
  - An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph.
- Example 5 (page 624)
  - The graph G_1 in Figure 2 is connected, since for every pair of distinct vertices there is a path between them (the reader should verify this).
  - However, the graph G_2 in Figure 2 is not connected. For instance, there is no path in G2 between vertices a and d.

3. Connectedness in Undirected Graphs

- Theorem 1 (page 625)
  - There is a simple path between every pair of distinct vertices of a connected undirected graph.
  - Proof:
    - Please read book (page 625).
  - Remark
    - A graph that is not connected is the union of two or more connected subgraphs, each pair of which has no vertex in common.
    - These disjoint connected subgraphs are called the connected components (连通分枝) of the graph

3. Connectedness in Undirected Graphs

- Example 6 (page 625)
  - What are the connected components of the graph H shown in Figure 3?
  - Solution
    - The graph H is the union of three disjoint connected subgraphs H_1, H_2, and H_3 shown in Figure 3. These three subgraphs are the connected components of H.

3. Connectedness in Undirected Graphs

- Some Concepts (page 625)
  - Sometimes the removal of a vertex and all edges incident with it produces a subgraph with more connected components than in the original graph. Such vertices are called cut vertices (割点).
  - An edge whose removal produces a graph with more connected components than in the original graph is called a cut edge or bridge (割边或桥).

4. Connectedness in Directed Graphs

- Definition 4 (page 626)
  - A directed graph is strongly connected (强连通) if there is a path from a to b and from b to a whenever a and b are vertices in the graph.
- Definition 5 (page 626)
  - A directed graph is weakly connected (弱连通) if there is a path between every two vertices in the underlying undirected graph.

4. Connectedness in Directed Graphs

- Example 9 (page 626)
  - Are the directed graphs G and H shown in Figure 5 strongly connected? Are they weakly connected?
  - Solution
    - G is strongly connected because there is a path between any two vertices in this directed graph (the reader should verify this). Hence, G is also weakly connected.

4. Connectedness in Directed Graphs

- Example 9 (page 626)
  - Solution(cont.)
    - The Graph H is not strongly connected. There is no directed path from a to b in this graph. However, H is weakly connected, since there is a path between any two vertices in the underlying undirected graph of H (the reader should verify this).

4. Connectedness in Directed Graphs

- Example 10 (page 627)
  - The graph H in Figure 5 has three strongly connected components, consisting of the vertex a; the vertex e; and the graph consisting of the vertices b,c, and d and edges (b,c), (c,d),and (d,b).

5. Paths and Isomorphism

- Introduction
  - the existence of a simple circuit of a particular length

    ------------- useful invariant

- Example 12 (page 627)
  - Determine whether the graphs G and H shown in Figure 6 are isomorphic.
  - Solution
    - Both G and H have six vertices and eight edges. Each has four vertices of degree three, and two vertices of degree two. So, the three invariants—number of vertices, number of edges, and degrees of vertices—all agree for the two graphs

5. Paths and Isomorphism

- Example 12 (page 627)
  - Solution (cont.)
    - However, H has a simple circuit of length three, namely, v1,v2,v6,v1 whereas G has no simple circuit of length three, as can be determined by inspection (all simple circuits in G have length at least four). Since the existence of a simple circuit of length three is an isomorphic invariant, G and H are not isomorphic.

5. Paths and Isomorphism

- Example 13 (page 628)
  - Determine whether the graphs G and H shown in Figure 7 are isomorphic.
  - Solution:
    - Both G and H have five vertices and six edges, both have two vertices of degree three and three vertices of degree two, and both have a simple circuit of length three, a simple circuit of length four, and a simple circuit of length five. Since all these isomorphic invariants agree, G and H may be isomorphic.

5. Paths and Isomorphism

- Example 13 (page 628)
  - Solution:
    - To find a possible isomorphism, we can follow paths that go through all vertices so that the corresponding vertices in the two graphs have the same degree.
    - For example, the paths u_1, u_4, u_3, u_2, u_5 in G and v_3, v_2, v_1, v_5, v_4 in H both go through every vertex in the graph, start at a vertex of degree three, go through vertices of degrees two, three, and two, respectively, and end at a vertex of degree two.

5. Paths and Isomorphism

- Example 13 (page 628)
  - Solution:
    - By following these paths through the graphs, we define the mapping f with $f(u_1) = v_3$, $f(u_4) = v_2$, $f(u_3) = v_1$, $f(u_2) = v_5$, and $f(u_5) = v_4$.
    - The reader can show that f is an isomorphism, so that G and H are isomorphic, either by showing that f preserves edges, or by showing that with the appropriate orderings of vertices the adjacency matrices of G and H are the same.

6. Counting Paths between Vertices

- Theorem 2 (page 628)
  - Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$ (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i,j)th entry of A^r.

6. Counting Paths between Vertices

- Example 14 (page 629)
  - How many paths of length 4 are there from a to d in the simple graoh G in Figure 8?
  - Solution
    - The adjacency matrix of G (ordering the vertices as a, b, c, d) is

$$A = \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{matrix}$$

6. Counting Paths between Vertices

- Example 14 (page 629)
  - Solution
    - Hence, the number of paths of length 4 from a to d is the (1,4)th entry of A^4. Since

$$A^4 = \begin{matrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{matrix}$$

    - There are exactly eight paths of length 4 from a to d. By inspection of the graph, we see that

      a,b,a,b,d; a,b,a,c,d; a,b,d,b,d; a,b,d,c,d; a,c,a,b,d; a,c,a,c,d; a,c,d,b,d; a,c,d,c,d

      are the eight paths from a to d.

Homework

- Page 629~633
  - 4, 14(a), 15(read), 18, 20, 22,24(c), 25(c)(read), 32