Modern Control Theory

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Chapter 3:

Motion Analysis and Discretization

3.1 The concept of matrix exponent

3.2 Calculation of matrix exponent

3.3 Controlled motion of linear time invariant system

3.4 The state space description of discrete system

3.5 The controlled motion of linear time invariant discrete system

3.6 Discretization of linear continuous system

3.1 The concept of matrix exponent

3.1.1 The free motion of linear time invariant system

Free motion: The motion caused only by initial condition, with input u=0.

Equation of free motion:

$$\dot{X} = AX, \quad X(t_0) = X_0$$

homogeneous state equation

At $X(t_0) = X_0, \ [t, \ \infty)$ **, the solution of homogeneous state equation can be represented as**

$$X(t) = \Phi(t - t_0)X_0$$

where $\Phi(t - t_0)$ **is state transition matrix , satisfy**

$$\dot{\Phi}(t - t_0) = A\Phi(t - t_0)$$

$$\Phi(0) = I$$

Proving of free motion solution $X(t)=\Phi(t-t_0)X_0$

$$\dot{X}(t)=\dot{\Phi}(t-t_0)X_0=A\Phi(t-t_0)X_0=AX$$

$$X(t_0)=\Phi(t-t_0)X_0=\Phi(0)X_0=X_0$$

Notice:

(1) Solution of free motion can be represented uniform form with state transition matrix .X(t) is state transition of X(0).

(2) If state transition matrix is determined ,the Solution of free motion can be known. So the solution is determined uniquely by state transition matrix.

(3) For linear time invariant system ,

$$\Phi(t-t_0)=e^{A(t-t_0)}$$

3.1.2 Property of state transition matrix (LTI)

（1）Invertibility $\Phi^{-1}(t-t_0)=\Phi(t_0-t)$

Prove: $\Phi(t-t_0)\Phi(t_0-t)=e^{A(t-t_0)}e^{A(t_0-t)}=e^{A\cdot 0}=I$

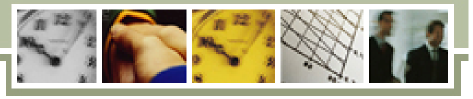
(2) Decomposability $\Phi(t_1+t_2)=\Phi(t_1)\Phi(t_2)$

Prove: $\Phi(t_1+t_2)=e^{A(t_1+t_2)}=e^{At_1}\cdot e^{At_2}=\Phi(t_1)\Phi(t_2)$

(3) Transitivity $\Phi(t_2-t_1)\Phi(t_1-t_0)=\Phi(t_2-t_0)$

Prove:
$$\Phi(t_2-t_1)\Phi(t_1-t_0)=\Phi(t_2)\Phi(-t_1)\Phi(t_1)\Phi(-t_0)$$
$$=\Phi(t_2)\Phi(0)\Phi(-t_0)=\Phi(t_2)\Phi(-t_0)=\Phi(t_2-t_0)$$

3.2 Calculation of matrix exponent



3.2.1 Base on the concept of matrix exponent

$$e^{At} = I + At + \frac{A^2}{2!}t^2 + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}t^k$$

3.2.2 Use Laplace inverse transform

$$e^{At} = L^{-1}\left[(sI - A)^{-1}\right]$$

Prove:

$$e^{At} = I + At + \frac{A^2}{2!}t^2 + \cdots$$

$$L\left[e^{At}\right] = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots = (sI - A)^{-1}$$

$$e^{At} = L^{-1}\left[(sI - A)^{-1}\right]$$

3.2.3 Transform to finite polynomial

—Cayley-Hamilton theorem

$$e^{At} = a_0(t)I + a_1(t)A + \cdots + a_{n-1}(t)A^{n-1}$$

$a_0(t), a_1(t), \cdots, a_{n-1}(t)$ are functions of t.

(1) If the eigenvalues of matrix A are distinct

$$\begin{bmatrix} a_0(t) \\ a_1(t) \\ \vdots \\ a_{n-1}(t) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix}$$

(2) If the eigenvalue (λ_1)of matrix A is n-repeated root

$$\begin{bmatrix} a_{n-1}(t) \\ a_{n-2}(t) \\ \vdots \\ a_1(t) \\ a_0(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & (n-1)\lambda_1 \\ \vdots & \vdots & \iddots & & \vdots \\ 0 & 0 & & & \frac{(n-1)(n-2)}{2!}\lambda_1^{n-3} \\ 0 & 1 & 2\lambda_1 & \cdots & \frac{n-1}{1!}\lambda_1^{n-2} \\ 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{(n-1)!}t^{n-1}e^{\lambda_1 t} \\ \frac{1}{(n-2)!}t^{n-2}e^{\lambda_1 t} \\ \vdots \\ \\ \frac{1}{1!}te^{\lambda_1 t} \\ e^{\lambda_1 t} \end{bmatrix}$$

3.2.4 Through nonsingular transformation

(1) If the eigenvalues of matrix A are distinct

$$e^{At} = P\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}P^{-1}$$

P is a matrix which make A diagonal canonical form

(2) If the eigenvalue (λ_1)of matrix A is n-repeated root

$$e^{At} = Q\begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \cdots & \frac{\mathbf{1}}{(n-\mathbf{1})!}t^{n-\mathbf{1}}e^{\lambda_1 t} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & te^{\lambda_1 t} \\ \mathbf{0} & \cdots & \mathbf{0} & e^{\lambda_1 t} \end{bmatrix}Q^{-1}$$

Q is a matrix which make A Jordan canonical form

Example: Find matrix exponent e^{At} of $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Sol:

(1) Use Laplace inverse transform $e^{At} = L^{-1}\left[(sI - A)^{-1}\right]$

$$[sI - A]^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{(s^2+3s+2)}$$

$$= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$e^{At} = L^{-1}\begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} = L^{-1}\begin{bmatrix} \frac{2}{s+1} + \frac{-1}{s+2} & \frac{1}{s+1} + \frac{-1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

(2) Use Cayley-Hamilton theorem

$$e^{At} = a_0(t)I + a_1(t)A$$

$$|\lambda I - A| = \det\begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix} = (\lambda+1)(\lambda+2)$$

$$\lambda_1 = -1, \qquad \lambda_2 = -2.$$

$$\begin{bmatrix} a_0(t) \\ a_1(t) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix}$$

$$\begin{bmatrix} a_0(t) \\ a_1(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$e^{At} = a_0(t)I + a_1(t)A$$

$$= (2e^{-t} - e^{-2t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t})\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

(3) Through nonsingular transformation

$$e^{At} = P\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}P^{-1}$$

$$P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$P^{-1} = -\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

3.3 Controlled motion of linear time invariant system

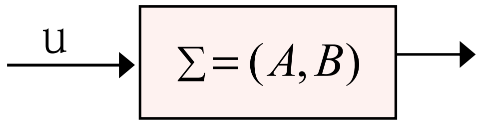
Concept: motion of linear time invariant system under control action

Equation:

$$\dot{X} = AX + Bu, \quad X(t_0) = X_0$$

Nonhomogeneous state equation

Block diagram:



$$\dot{X} = AX + Bu$$
$$X(t_0) = X_0$$

The solution of nonhomogeneous state equation (theorem) :

If the solution of The solution of nonhomogeneous state equation exists, the solution must have the follow form.

$$t_0 = 0,\ \ X(t) = \Phi(t)X_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau \qquad t \in [0, \infty)$$

$$t_0 \neq 0,\ \ X(t) = \Phi(t-t_0)X_0 + \int_{t_0}^t \Phi(t-\tau)Bu(\tau)d\tau \qquad t \in [t_0, \infty)$$

Prove: $\dot{X} = AX + Bu$

$$\dot{X} - AX = Bu$$

Left –multiplying the two sides of equation above with e^{-At}

$$e^{-At}\left[\dot{X} - AX\right] = e^{-At}Bu$$

$$\tfrac{d}{dt}\left[e^{-At}X\right] = e^{-At}Bu$$

Doing integral

$$e^{-A\tau}X(\tau)\Big|_0^t = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$e^{-At}X(t) - X(0) = \int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$X(t) = e^{At}X(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$= \Phi(t)X(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau \qquad t \in [0,\infty)$$

Example: The state equation is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u \qquad (t\geq 0)$$

where $u(t)=1(t)$, find the solution of equation

Sol: From the example above

$$\Phi(t)=e^{At}=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

$$X(t)=\Phi(t)X_0+\int_0^t\Phi(t-\tau)Bu(\tau)d\tau$$

$$=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$$

$$+\int_0^t\begin{bmatrix}2e^{-(t-\tau)}-e^{-2(t-\tau)} & e^{-(t-\tau)}-e^{-2(t-\tau)}\\ -2e^{-(t-\tau)}+2e^{-2(t-\tau)} & -e^{-(t-\tau)}+2e^{-2(t-\tau)}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}d\tau$$

$$
=\begin{bmatrix} (2e^{-t}-e^{-2t})x_1(0)+(e^{-t}-e^{-2t})x_2(0) \\ (-2e^{-t}+2e^{-2t})x_1(0)+(-e^{-t}+2e^{-2t})x_2(0) \end{bmatrix}+\int_0^t \begin{bmatrix} e^{-(t-\tau)}-e^{-2(t-\tau)} \\ -e^{-(t-\tau)}+2e^{-2(t-\tau)} \end{bmatrix} d\tau
$$

$$
=\begin{bmatrix} (2e^{-t}-e^{-2t})x_1(0)+(e^{-t}-e^{-2t})x_2(0) \\ (-2e^{-t}+2e^{-2t})x_1(0)+(-e^{-t}+2e^{-2t})x_2(0) \end{bmatrix}+\begin{bmatrix} \frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t} \\ e^{-t}-e^{-2t} \end{bmatrix}
$$

$$
=\begin{bmatrix} (2e^{-t}-e^{-2t})x_1(0)+(e^{-t}-e^{-2t})x_2(0)+\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t} \\ (-2e^{-t}+2e^{-2t})x_1(0)+(-e^{-t}+2e^{-2t})x_2(0)+e^{-t}-e^{-2t} \end{bmatrix}
$$

The state transition matrix are known , find system matrix

Method： (1) $A = \dot{\Phi}(t)\Phi(-t)$

Prove：

$$\dot{\Phi}(t) = A(t)\Phi(t)$$

$$A(t) = \dot{\Phi}(t)\Phi^{-1}(t) = \dot{\Phi}(t)\Phi(-t)$$

(2) $A = \dot{\Phi}(0)$

Prove：

$$\Phi(t) = e^{At}$$

$$\dot{\Phi}(t) = Ae^{At}$$

$$t = 0,\ \dot{\Phi}(0) = Ae^{A0} = A$$

(3) $A = sI - \left[L\left[\Phi(t) \right] \right]^{-1}$

Prove：

$$\Phi(t) = L^{-1}\left[(sI - A)^{-1} \right]$$

$$(sI - A)^{-1} = L\left[\Phi(t) \right]$$

$$sI - A = \left[L\left[\Phi(t) \right] \right]^{-1}$$

$$A = sI - \left[L\left[\Phi(t) \right] \right]^{-1}$$

Example：

Known $\Phi(t) = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & (1-2t)e^{-2t} & 4te^{-2t} \\ 0 & -te^{-2t} & (1+2t)e^{-2t} \end{bmatrix}$. Find A.

Sol: (1) $A=\dot{\Phi}(t)\Phi(-t)$

$$A=\begin{bmatrix}-e^{-t} & 0 & 0\\ 0 & (-4+4t)e^{-2t} & (4-8t)e^{-2t}\\ 0 & (-1+2t)e^{-2t} & -4te^{-2t}\end{bmatrix}\cdot\begin{bmatrix}e^{t} & 0 & 0\\ 0 & (1+2t)e^{2t} & -4te^{2t}\\ 0 & te^{2t} & (1-2t)e^{2t}\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & -4 & 4\\ 0 & -1 & 0\end{bmatrix}$$

(2) $A=\dot{\Phi}(0)$

$$A=\dot{\Phi}(0)=\begin{bmatrix}-e^{-t} & 0 & 0\\ 0 & (-4+4t)e^{-2t} & (4-8t)e^{-2t}\\ 0 & (-1+2t)e^{-2t} & -4te^{-2t}\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & -4 & 4\\ 0 & -1 & 0\end{bmatrix}$$

(3) $A = sI - \left[L\left[\Phi(t) \right] \right]^{-1}$

$$(SI - A)^{-1} = L\left[\Phi(t)\right] = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{1}{s+1} - \frac{2}{(s+2)^2} & \frac{4}{(s+2)^2} \\ 0 & \frac{-1}{(s+2)^2} & \frac{1}{s+2} + \frac{2}{(s+2)^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{s}{(s+2)^2} & \frac{4}{(s+2)^2} \\ 0 & \frac{-1}{(s+2)^2} & \frac{s+4}{(s+2)^2} \end{bmatrix}$$

$$(SI - A) = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ 0 & \frac{s}{(s+2)^2} & \frac{4}{(s+2)^2} \\ 0 & \frac{-1}{(s+2)^2} & \frac{s+4}{(s+2)^2} \end{bmatrix}^{-1} = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+4 & -4 \\ 0 & 1 & s \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

3.4 The state space description of discrete system

	Continuous system	Discrete system
Time-domain	differential equation	difference equation
Frequency-domain	s-domain transfer function	z-domain pulse transfer function
Transform	Laplace transform	Z transform

Differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \cdots + b_{n-1}\dot{u} + b_n u$$

Difference equation

$$\begin{aligned} & y(k+n) + a_1 y(k+n-1) + \cdots + a_{n-1} y(k+1) + a_n y(k) \\ & = b_0 u(k+n) + b_1 u(k+n-1) + \cdots + b_n u(k) \qquad (k = 0,1,2\cdots) \end{aligned}$$

k means *k*th sample moment

s-domain transfer function

$$W(s)=\frac{Y(s)}{U(s)}=\frac{b_o s^n+b_1 s^{n-1}+\cdots+b_{n-1}s+b_n}{s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n}$$

z-domain pulse transfer function

$$W(Z)=\frac{Y(Z)}{U(Z)}=\frac{b_0 z^n+b_1 z^{n-1}+\cdots+b_n}{z^n+a_1 z^{n-1}+\cdots+a_n}$$

3.4.1. Change scalar difference equation to state space description

3.4.1.1 Case 1-- difference equation do not contain difference of input variable

$$y(k+n)+a_1 y(k+n-1)+\cdots+a_{n-1}y(k+1)+a_n y(k)=b_n u(k)$$

(1)Select state variables

$$\begin{cases} x_1(k)=y(k) \\ x_2(k)=y(k+1) \\ x_3(k)=y(k+2) \\ \vdots \\ x_n(k)=y(k+n-1) \end{cases}$$

(2) Change state variable equation to 1th difference equations set

$$
\begin{cases}
x_1(k+1) = y(k+1) = x_2(k) \\
x_2(k+1) = y(k+2) = x_3(k) \\
\vdots \\
x_{n-1}(k+1) = y(k+n-1) = x_n(k) \\
x_n(k+1) = y(k+n) \\
\qquad = -a_n y(k) - a_{n-1} y(k+1) - \cdots - a_1 y(k+n-1) + b_n u(k) \\
\qquad = -a_n x_1(k) - a_{n-1} x_2(k) - \cdots - a_1 x_n(k) + b_n u(k)
\end{cases}
$$

$$y(k) = x_1(k)$$

(3)Write out state space description

$$
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_n \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}
$$

The general form of Discrete system state space description

$$
\begin{cases} X(k+1) = G(k)X(k) + H(k)u(k) \\ y(k) = C(k)X(k) + D(k)u(k) \end{cases}
$$

or

$$
\begin{cases} X(k+1) = G_k X_k + H_k u_k \\ y(k) = C_k X_k + D_k u_k \end{cases}
$$

3.4.1.2 Case 2-- difference equation contain difference of input variable

$$y(k+n)+a_1 y(k+n-1)+\cdots+a_{n-1} y(k+1)+a_n y(k)$$
$$=b_0 u(k+n)+b_1 u(k+n-1)+\cdots+b_n u(k) \qquad (k=0,1,2\cdots)$$

(1)Select state variables

$$\begin{cases} x_1(k)=y(k)-h_0 u(k) \\ x_2(k)=x_1(k+1)-h_1 u_1(k) \\ x_3(k)=x_2(k+1)-h_2 u(k) \\ \vdots \\ x_n(k)=x_{n-1}(k+1)-h_{n-1} u(k) \\ x_{n+1}(k)=x_n(k+1)-hu(k) \end{cases}$$

$$\begin{cases} h_0=b_0 \\ h_1=b_1-a_1 h_0 \\ h_2=b_2-a_1 h_1-a_2 h_0 \\ \vdots \\ h_n=b_n-a_1 h_{n-1}-\cdots-a_{n-1} h_1-a_n h_0 \end{cases}$$

(2) Change state variable equation to1th difference equations set

$$
\begin{cases}
x_1(k+1) = x_2(k) + h_1 u(k) \\
x_2(k+1) = x_3(k) + h_2 u(k) \\
\vdots \\
x_{n-1}(k+1) = x_n(k) + h_{n-1} u(k) \\
x_n(k+1) = -a_n x_1(k) - a_{n-1} x_2(k) - \cdots - a_1 x_n(k) + h_n u(k)
\end{cases}
$$

$$
y(k) = x(k) + h_0 u(k)
$$

(3) Write out state space description

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-1} \\ h_n \end{bmatrix} u(k)$$

$$y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + h_0 u(k)$$

3.4.2. Change pulse transfer function to state space description

$$W(z)=\frac{Y(z)}{U(z)}=\frac{b_1 z^{n-1}+\cdots+b_{n-1}z+b_n}{z^n+a_1 z^{n-1}+\cdots+a_{n-1}z+a_n}$$

3.4.2.1 Case 1-- pulse transfer function have distinct poles

$$W(z)=\frac{Y(z)}{U(z)}=\frac{k_1}{z-z_1}+\frac{k_2}{z-z_2}+\cdots+\frac{k_n}{z-z_n} \qquad k_i=\lim_{z\to z_i} W(z)(z-z_i), \quad (i=1,2,\cdots n)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix}=\begin{bmatrix} z_1 & & & 0 \\ & z_2 & & \\ & & \ddots & \\ 0 & & & z_n \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}+\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}$$

3.4.2.2 Case 2-- pulse transfer function has n-repeated pole

$$W(z)=\frac{Y(z)}{U(z)}=\frac{k_{11}}{(z-z_1)^n}+\frac{k_{12}}{(z-z_1)^{n-1}}+\cdots+\frac{k_{1n}}{z-z_1}$$

$$k_{1i}=\lim_{z\to z_1}\frac{1}{(i-1)!}\frac{d^{i-1}}{dz^{i-1}}\left[W(z)(z-z_1)^n\right]\qquad (i=1,2,\cdots n)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix}=\begin{bmatrix} z_1 & 1 & & 0 \\ & z_1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & z_1 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}$$

3.5 The controlled motion of linear time invariant discrete system

Methods for Solving difference equation include iteration method and Z-transform method

3.5.1 Iteration method

3.5.1.1 Liner time variant discrete system

X(0) and u(k) (k=0,1,…,n) are known, the solution can be obtained:

$$
\begin{aligned}
x(1) &= G(0)X(0) + H(0)u(0) \\
x(2) &= G(1)X(1) + H(1)u(1) \\
x(3) &= G(2)X(2) + H(2)u(2) \\
&\vdots
\end{aligned}
$$

3.5.1.2 Liner time invariant discrete system

G, H are invariant, X(0) and u(k) (k=0,1,…,n) are known, then

$$x(1) = GX(0) + Hu(0)$$

$$x(2) = GX(1) + Hu(1) = G^2X(0) + GHu(0) + Hu(1)$$

$$x(3) = GX(2) + Hu(2) = G^3X(0) + G^2Hu(0) + GHu(1) + Hu(2)$$

$$X(k) = G^kX(0) + \sum_{i=0}^{k-1} G^{k-i-1}Hu(i)$$

Discussion of solution

(1) First part associate with initial valve X(0).

Second part associate with control action u(k).

(2) Kth state x(k) associate only with sampled value before k mo ment, not with kth sampled value .

(3) State transition matrix $\Phi(k) = G^k$ satisfy $\Phi(k+1) = G\Phi(k)$, $\Phi(0) = I$

Example: Known linear time invariant discrete system equation is

$$X(k+1)=GX(k)+Hu(k), \quad G=\begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \qquad H=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

t=0, $\qquad X(0)=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$k=0,1,2,\cdots \qquad u(k)=1$$

Find the solution of sate equation.

Sol: Using iteration method

$$X(1) = GX(0) + Hu(0) = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.84 \end{bmatrix}$$

$$X(2) = GX(1) + Hu(1) = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1.84 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.84 \\ -0.84 \end{bmatrix}$$

$$X(3) = GX(2) + Hu(2) = \begin{bmatrix} 0 & 1 \\ -0.16 & -1 \end{bmatrix} \begin{bmatrix} 2.84 \\ -0.84 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.16 \\ 1.386 \end{bmatrix}$$

$$X(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2.84 & 0.16 & \cdots \\ -1 & 1.84 & -0.84 & 1.386 & \cdots \end{bmatrix}$$

3.5.2 Z-transform method

$$X(k+1) = GX(k) + Hu(k)$$

Doing Z-transform

$$zX(z) - zX(\mathbf{0}) = GX(z) + HU(z)$$

$$(zI - G)X(z) = zX(\mathbf{0}) + HU(z)$$

$$X(z) = (zI - G)^{-1} zX(\mathbf{0}) + (zI - G)^{-1} HU(z)$$

Doing Z-inverse transform

$$X(k) = Z^{-1}\left[(zI - G)^{-1} z\right] X(0) + Z^{-1}\left[(zI - G)^{-1} HU(z)\right]$$

$$\therefore \qquad G^k = Z^{-1}\left[(zI-G)^{-1}z\right]$$

$$\sum_{i=0}^{k-1} G^{k-i-1}HU(i) = Z^{-1}\left[(zI-G)^{-1}HU(z)\right]$$

Example above: using Z-transform method

$$\phi(k) = G^k = Z^{-1}\left[(zI-G)^{-1}z\right]$$

$$(zI-G)^{-1} = \begin{bmatrix} z & -1 \\ 0.16 & z+1 \end{bmatrix}^{-1} = \frac{1}{z^2+z+0.16}\begin{bmatrix} z+1 & 1 \\ -0.16 & z \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{z+1}{(z+0.2)(z+0.8)} & \dfrac{1}{(z+0.2)(z+0.8)} \\ \dfrac{-0.16}{(z+0.2)(z+0.8)} & \dfrac{z}{(z+0.2)(z+0.8)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{4/3}{z+0.2} - \dfrac{1/3}{z+0.8} & \dfrac{5/3}{z+0.2} - \dfrac{5/3}{z+0.8} \\ \dfrac{0.8/3}{z+0.2} + \dfrac{0.8/3}{z+0.8} & \dfrac{1/3}{z+0.2} + \dfrac{4/3}{z+0.8} \end{bmatrix}$$

$$Z^{-1}\left(\frac{z}{z+a}\right)=(-a)^{k}$$

$$\phi(k)=G^{k}=Z^{-1}\left[(zI-G)^{-1}z\right]$$

$$=Z^{-1}\begin{bmatrix} \frac{4}{3}\left(\frac{z}{z+0.2}\right)-\frac{1}{3}\left(\frac{z}{z+0.8}\right) & \frac{5}{3}\left(\frac{z}{z+0.2}\right)-\frac{5}{3}\left(\frac{z}{z+0.8}\right) \\ \frac{0.8}{3}\left(\frac{z}{z+0.2}\right)+\frac{0.8}{3}\left(\frac{z}{z+0.8}\right) & \frac{1}{3}\left(\frac{z}{z+0.2}\right)+\frac{4}{3}\left(\frac{z}{z+0.8}\right) \end{bmatrix}$$

$$=\begin{bmatrix} \frac{4}{3}(-0.2)^{k}-\frac{1}{3}(-0.8)^{k} & \frac{5}{3}(-0.2)^{k}-\frac{5}{3}(-0.8)^{k} \\ \frac{0.8}{3}(-0.2)^{k}-\frac{0.8}{3}(-0.8)^{k} & \frac{1}{3}(-0.2)^{k}-\frac{4}{3}(-0.8)^{k} \end{bmatrix}$$

$$u(k)=1 \qquad U(z)=\frac{z}{z-1}$$

$$X(z) = (zI - G)^{-1} zX(0) + (zI - G)^{-1} HU(z)$$

$$= (zI - G)^{-1}\left[zX(0) + HU(z)\right]$$

$$zX(0) + HU(z) = \begin{bmatrix} z \\ -z \end{bmatrix} + \begin{bmatrix} \dfrac{z}{z-1} \\ \dfrac{z}{z-1} \end{bmatrix} = \begin{bmatrix} \dfrac{z^2}{z-1} \\ \dfrac{-z^2+2z}{z-1} \end{bmatrix}$$

$$X(z) = (zI - G)^{-1}\left[zX(0) + HU(z)\right]$$

$$= \begin{bmatrix} \dfrac{z+1}{(z+0.2)(z+0.8)} & \dfrac{1}{(z+0.2)(z+0.8)} \\ \dfrac{-0.16}{(z+0.2)(z+0.8)} & \dfrac{z}{(z+0.2)(z+0.8)} \end{bmatrix} \begin{bmatrix} \dfrac{z^2}{z-1} \\ \dfrac{-z^2+2z}{z-1} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{(z+2)z^2}{(z+0.2)(z+0.8)(z-1)} \\ \dfrac{(-z^2+1.84z)z}{(z+0.2)(z+0.8)(z-1)} \end{bmatrix} = \begin{bmatrix} -\dfrac{\frac{17}{6}z}{z+0.2} + \dfrac{\frac{22}{9}z}{z+0.8} + \dfrac{\frac{25}{18}z}{z-1} \\ \dfrac{\frac{3.4}{6}z}{z+0.2} - \dfrac{\frac{17.6}{9}z}{z+0.8} + \dfrac{\frac{7}{18}z}{z-1} \end{bmatrix}$$

$$X(k)=\begin{bmatrix} -\frac{17}{6}(-0.2)^k+\frac{22}{9}(-0.8)^k+\frac{25}{18}(1)^k \\ \frac{3.4}{6}(-0.2)^k-\frac{17.6}{9}(-0.8)^k+\frac{7}{18} \end{bmatrix}$$

3.6 Discretization of linear continuous system

3.6.1 Hypothesis

(1) Adopting equal time-interval sampling, sampling period T;

neglecting sampling time;

the sampling valve is zero between two samplings.

(2) T satisfy Shanon theorem.

sampling frequency $\geq$2 times upper limit frequency of continu ous function spectrum.

(3) 0 order holder

3.6.2 Linear time variant system discretazation

- Theorem: Linear time variant system equations are

$$\dot{X} = A(t)X(t) + B(t)u(t)$$

$$y = C(t)X(t) + D(t)u(t)$$

its discretazation equations are

$$X(k+1) = G(k)X(k) + H(k)u(k)$$

$$y(k) = C(k)X(k) + D(k)u(k)$$

The coefficient relation are

$$G(k) = G(kT) = \Phi((k+1)T, kT)$$

$$H(k) = H(kT) = \int_{kT}^{(k+1)T} \Phi((k+1)T, \tau)B(\tau)d\tau$$

$$C(k) = [C(t)]_{t=kT}$$

$$D(k) = [D(t)]_{t=kT}$$

- Approximate represents of linear time variant system discretazation state equation

  In general case, The matrix A of time variant system is difficult to be written out, Approximate represents is necessary .

$$\dot{X}(kT) \approx \tfrac{1}{T}\{X[k+1)T] - X(kT)\}$$

$$\tfrac{1}{T}\{X[k+1)T] - X(kT)\}$$
$$= A(kT)X(kT) + B(kT)u(kT)$$

$$X[(k+1)T] = [1 + TA(kT)]X(kT) + TB(kT)u(kT)$$
$$= G(kT)X(kT) + H(kT)\cdot u(kT)$$

$$G(kT) = 1 + TA(kT)$$

$$H(kT) = TB(kT)$$

Example: linear time variant system state equation

$$\dot{X}(t) = A(t)X(t) + B(t)u(t)$$

where $A(t) = \begin{bmatrix} 0 & 5(1-e^{-5t}) \\ 0 & 5e^{-5t} \end{bmatrix}$, $\quad B(t) = \begin{bmatrix} 5 & 5e^{-5t} \\ 0 & 5(1-e^{-5t}) \end{bmatrix}$

Initial condition $\quad u(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\quad X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Find sampling moment approximate valve.

Sol: Suppose T=0.2s

$$G(kT) = 1 + TA(kT) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0.2\begin{bmatrix} 0 & 5(1-e^{-5(kT)}) \\ 0 & 5e^{-5(kT)} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1-e^{-k} \\ 0 & e^{-k} \end{bmatrix} = \begin{bmatrix} 1 & 1-e^{-k} \\ 0 & 1+e^{-k} \end{bmatrix}$$

$$H(kT) = T \cdot B(kT) = 0.2\begin{bmatrix} 5 & 5e^{-5(kT)} \\ 0 & 5(1-e^{-5(kT)}) \end{bmatrix} = \begin{bmatrix} 1 & e^{-k} \\ 0 & 1-e^{-k} \end{bmatrix}$$

Descetazation state equations are

$$\begin{bmatrix} x_1[(k+1)T] \\ x_2[(k+1)T] \end{bmatrix} = \begin{bmatrix} 1 & 1-e^{-k} \\ 0 & 1+e^{-k} \end{bmatrix}\begin{bmatrix} x_1(kT) \\ x_2(kT) \end{bmatrix} + \begin{bmatrix} 1 & e^{-k} \\ 0 & 1-e^{-k} \end{bmatrix}\begin{bmatrix} u_1(kT) \\ u_2(kT) \end{bmatrix}$$

Using iterative method

$$\begin{bmatrix} x_1(0.2) \\ x_2(0.2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0.4) \\ x_2(0.4) \end{bmatrix} = \begin{bmatrix} 1 & 0.632 \\ 0 & 1.368 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0.368 \\ 0 & 0.632 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.368 \\ 0.632 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0.6) \\ x_2(0.6) \end{bmatrix} = \begin{bmatrix} 1 & 0.865 \\ 0 & 1.1353 \end{bmatrix} \begin{bmatrix} 1.368 \\ 0.632 \end{bmatrix} + \begin{bmatrix} 1 & 0.135 \\ 0 & 0.865 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.05 \\ 1.582 \end{bmatrix}$$

$$\vdots$$

3.6.3 Linear time invariant system discretazation

- Theorem: Linear time invariant system equations are

$$\begin{cases} \dot{X}(t) = AX + Bu \\ y = CX + Du \end{cases}$$

its discretazation equations are

$$\begin{cases} X(k+1) = GX(k) + Hu(k) \\ y(k) = CX(k) + Du(k) \end{cases} \qquad k = 0,1,2,\cdots$$

The coefficient relation are

$$G = e^{AT}$$

$$H = \left(\int_0^T e^{At} \cdot dt\right) \cdot B$$

G, H, C and D are constant matrix.

Prove:

$$G = \Phi((k+1)T - kT) = \Phi(T) = e^{AT}$$

$$H = \int_{kT}^{(k+1)T} \Phi((k+1)T - \tau) B d\tau = \left(\int_{kT}^{(k+1)T} \Phi((k+1)T - \tau) d\tau \right) \cdot B$$

Let $t = (k+1)T - \tau,$

then $dt = -d\tau,$

when $\tau = kT, \quad t = T;$

when $\tau = (k+1)T, \quad t = 0;$

$$\therefore \quad H = \left(-\int_{T}^{0} \Phi(t)\ dt \right) B = \left(\int_{0}^{T} e^{At} dt \right) B$$

Example 1: Known $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

Find discretization equations

Sol: $$G = e^{AT} = L^{-1}[(sI - A)^{-1}]_{t=T},$$

$$|sI - A| = \begin{vmatrix} s & -1 \\ 0 & s+2 \end{vmatrix} = s(s+2)$$

$$(sI - A)^{-1} = \frac{1}{s(s+2)} \begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix}$$

$$\therefore \quad G = L^{-1} \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}_{t=T} = \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2T}) \\ 0 & e^{-2T} \end{bmatrix}$$

$$H = \left(\int_0^T e^{At} dt \right) \cdot B$$

$$\int_0^T e^{At} dt = \int_0^T \begin{bmatrix} 1 & \frac{1}{2}(1-e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} dt = \begin{bmatrix} T & \frac{1}{2}T + \frac{1}{4}e^{-2T} - \frac{1}{4} \\ 0 & -\frac{1}{2}e^{-2T} + \frac{1}{2} \end{bmatrix}$$

$$\therefore \quad H = \begin{bmatrix} T & \frac{1}{2}T + \frac{1}{4}e^{-2T} - \frac{1}{4} \\ 0 & -\frac{1}{2}e^{-2T} + \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}T + \frac{1}{4}e^{-2T} - \frac{1}{4} \\ -\frac{1}{2}e^{-2T} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(T + \frac{e^{-2T}-1}{2}) \\ \frac{1}{2}(1-e^{-2T}) \end{bmatrix}$$

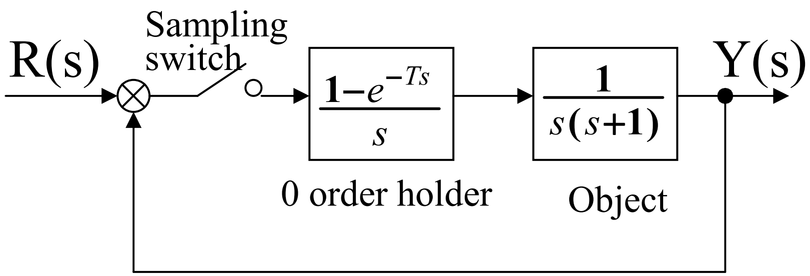
the discretization state equation is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2}T + \frac{1}{4}e^{-2T} - \frac{1}{4} \\ 0 & e^{-2T} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(T + \frac{e^{-2T}-1}{2}) \\ \frac{1}{2}(1 - e^{-2T}) \end{bmatrix} u(k)$$

If T=1s, the discretization state equation can be written as

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.432 \\ 0 & 0.135 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.284 \\ 0.432 \end{bmatrix} u(k)$$

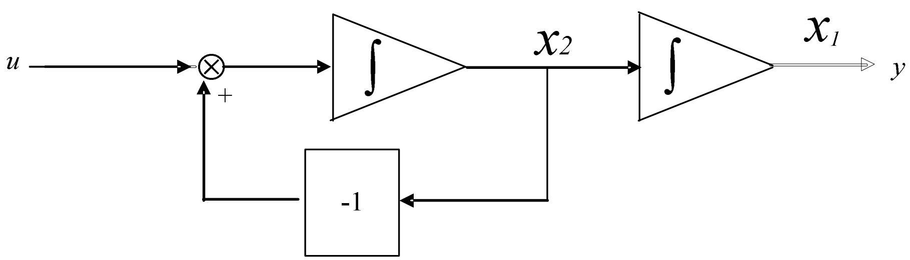
Example 2:



Find discretization equations

Sol: (1) Open-loop system discrtization equation

$$W(s) = \frac{1}{s^2 + s} = \frac{1/s^2}{1 + 1/s}$$



$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 + u \end{cases}$$

$$y = x_1$$

$$\therefore \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

Discretization $e^{AT} = L^{-1}[(sI-A)^{-1}]\big|_{t=T}$

$$(sI-A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix}^{-1} = \frac{1}{s(s+1)}\begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix} \qquad e^{AT} = \begin{bmatrix} 1 & 1-e^{-T} \\ 0 & e^{-T} \end{bmatrix}$$

$$H(T) = \left(\int_0^T e^{At}dt\right)B = \int_0^T \begin{bmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}dt$$

$$= \int_0^T \begin{bmatrix} 1-e^{-t} \\ e^{-t} \end{bmatrix}dt = \begin{bmatrix} T-1+e^{-T} \\ 1-e^{-T} \end{bmatrix}$$

$\therefore$ Discretization state space description of open-loop system is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1-e^{-T} \\ 0 & e^{-T} \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} T-1+e^{-T} \\ 1-e^{-T} \end{bmatrix}u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Let T=1 $G(T)=\begin{bmatrix}0 & 0.632\\ 0 & 0.368\end{bmatrix}$ $H(T)=\begin{bmatrix}0.368\\ 0.632\end{bmatrix}$

$$\begin{bmatrix}x_1(k+1)\\ x_2(k+1)\end{bmatrix}=\begin{bmatrix}1 & 0.632\\ 0 & 0.368\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}+\begin{bmatrix}0.368\\ 0.632\end{bmatrix}u(k)$$

$$y(k)=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(k)\\ x_2(k)\end{bmatrix}$$

(2) close-loop system discrtization equation

For close-loop system

$$u(k)=r(k)-y(k)=r(k)-CX(k)$$

$\therefore$ Discretization state space description of close-loop system is

$$X(k+1)=G(T)X(k)+H(T)u(k)=G(T)X(k)+H(T)[r(k)-CX(k)]$$
$$=[G(T)-H(T)C]X(k)+H(T)r(k)$$

Or

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 2 - T - e^{-T} & 1 - e^{-T} \\ e^{-T} - 1 & e^{-T} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} T - 1 + e^{-T} \\ 1 - e^{-T} \end{bmatrix} r(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

When t=1s

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.632 & 0.632 \\ -0.632 & 0.368 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.368 \\ 0.632 \end{bmatrix} r(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Excise:

1. (P85 3.1) Known system state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Find $x_1(t),\ x_2(t)$

2. (P85 3.6) Known state transition matrix of system $\dot{X} = AX$ is

$$\boldsymbol{\Phi}(t,\mathbf{0}) = \begin{bmatrix} \mathbf{2}e^{-t} - e^{-\mathbf{2}t} & \mathbf{2}(e^{-\mathbf{2}t} - e^{-t}) \\ e^{-t} - e^{-\mathbf{2}t} & \mathbf{2}e^{-\mathbf{2}t} - e^{-t} \end{bmatrix}$$

Find A

3. (P87 3.14)Known continuous system state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} u$$

Find discretization equations

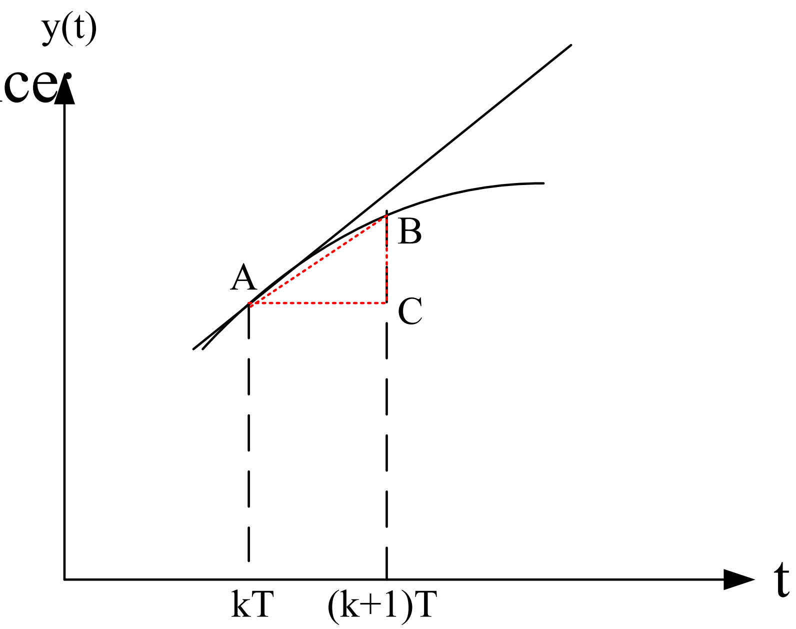
4. (P87 3.12) Known motion equation is

$$y(k+3)+3y(k+2)+2y(k+1)+y(k)=u(k+2)+2u(k+1)+u(k)$$

Find state equations.

The concept of difference equation

The concept of difference:



The slope of curve at point A is $\frac{dy}{dt}$

When the sampling period is short enough, the slope of curve at point A can be substituted by the slope of line AB, that is

$$\frac{y[(k+1)T]-y(kT)}{T}$$

so $$\left.\frac{dy}{dt}\right|_{t=kT} \approx \frac{y[(k+1)T]-y(kT)}{T}$$

This is difference of y at kT.

For forward moment (k+1)T is used, this difference is called forward difference.

Backward difference: $$\left.\frac{dy}{dt}\right|_{t=kT} = \frac{y(kT)-y[(k-1)T]}{T}$$

Center difference: $$\left.\frac{dy}{dt}\right|_{t=kT} = \frac{y[(k+\frac{1}{2})T]-y[(k-\frac{1}{2})T]}{T}$$

Example: Turn $\dot{y} + ay = f(t)$ to difference equation

Sol: t=kT

$$\frac{y[(k+1)T] - y(kT)}{T} + ay(kT) = f(kT)$$

$$y[(k+1)T] + (aT-1)y(kT) = Tf(kT)$$

$$or \quad y(k+1) + (aT-1)y(k) = Tf(k)$$

return

Z-Transform

Definition:

$$X(z)=\sum_{k=0}^{\infty}x(kT)z^{-k}$$

Denoted as:

$$Z(x(kT)]=\sum_{k=0}^{\infty}x(kT)z^{-k}$$

Example 1:

$$x(kT)=1$$

Sol: When $|z|>1$

$$\mathrm{Z}[1]=\sum_{k=0}^{\infty}1\cdot \mathrm{z}^{-k}=\frac{1}{\mathrm{z}}+\frac{1}{\mathrm{z}^2}+\frac{1}{\mathrm{z}^3}+\cdots=\frac{\mathrm{z}}{\mathrm{z}-1}$$

Example 2: $x(kT) = kT$

Sol: When $|z| > 1$

$$Z[kt] = \sum_{k=0}^{\infty} kT \cdot z^{-k} = Tz \sum_{k=0}^{\infty} kz^{-k-1} = -Tz \sum_{k=0}^{\infty} \frac{d}{dz} z^{-k} = -Tz \frac{d}{dz} \sum_{k=0}^{\infty} z^{-k}$$

$$= -Tz \frac{d}{dz}\left(\frac{1}{1-1/z}\right) = -Tz \frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -Tz \frac{-1}{(z-1)^2} = \frac{Tz}{(z-1)^2}$$

Z-transform formulas of common function

$$Z(a) = \frac{z}{z+a}$$

$$Z(kT) = \frac{Tz}{(z-1)^2}$$

$$Z(e^{-akT}) = \frac{z}{z-e^{-aT}}$$

$$Z[\sin(\omega kT)] = \frac{z \sin \omega T}{z^2 - 2z\cos\omega T + 1}$$

$$Z[\sin(\omega kT)] = \frac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}$$

Properties of Z-transform

1. $Z[x(kT)+y(kT)]=Z[x(kT)]+Z[y(kT)]$

2. $Z[ax(kT)]=aZ[x(kT)]$

3. $Z[x(kT-nT)]=z^{-n}Z[x(kT)]$

$$Z[x(kT+nT)]=z^{n}\{Z[x(kT)]-\sum_{k=0}^{n-1}x(kT)z^{-k}\}$$

return