

- 周期实信号：正交三角函数 $\{\cos k\omega_0 t, \sin k\omega_0 t\}$ 展开

exponential	triangular
$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$	$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)]$
$C_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt$ <p>($k = 0, \pm 1, \pm 2, \pm 3, \dots$)</p>	$a_0 = \frac{2}{T} \int_0^T x(t) dt$ $a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t) dt$ $b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$ <p>($k = 1, 2, 3, \dots$)</p>

• 周期实信号

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \Rightarrow x^*(t) = \sum_{k=-\infty}^{\infty} C_k^* e^{-jk\omega_0 t}$$

以 $-k$ 代替 k ： $x^*(t) = \sum_{k=-\infty}^{\infty} C_{-k}^* e^{jk\omega_0 t}$

对实信号： $x(t) = x^*(t) \Rightarrow x(t) = \sum_{k=-\infty}^{\infty} C_{-k}^* e^{jk\omega_0 t}$

比较 $x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \Rightarrow C_k = C_{-k}^* \text{ or } C_{-k} = C_k^*$

$$\begin{aligned} \therefore x(t) &= \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{-1} C_k e^{jk\omega_0 t} + C_0 + \sum_{k=1}^{\infty} C_k e^{jk\omega_0 t} \\ &= \sum_{k=-\infty}^{-1} C_{-k}^* e^{jk\omega_0 t} + C_0 + \sum_{k=1}^{\infty} C_k e^{jk\omega_0 t} = \sum_{k=1}^{\infty} C_k^* e^{-jk\omega_0 t} + C_0 + \sum_{k=1}^{\infty} C_k e^{jk\omega_0 t} \\ &= C_0 + \sum_{k=1}^{\infty} (C_k e^{jk\omega_0 t} + C_k^* e^{-jk\omega_0 t}) \end{aligned}$$

$$\text{令} : C_k = |C_k| e^{j\theta_k}$$

$$x(t) = C_0 + \sum_{k=1}^{\infty} (C_k e^{jk\omega_0 t} + C_k^* e^{-jk\omega_0 t})$$

$$= C_0 + \sum_{k=1}^{\infty} (|C_k| e^{j\theta_k} e^{jk\omega_0 t} + |C_k| e^{-j\theta_k} e^{-jk\omega_0 t})$$

$$= C_0 + \sum_{k=1}^{\infty} |C_k| [e^{j(k\omega_0 t + \theta_k)} + e^{-j(k\omega_0 t + \theta_k)}]$$

$$= C_0 + \sum_{k=1}^{\infty} 2|C_k| \cos(k\omega_0 t + \theta_k)$$

C_0 是信号的**直流分量**

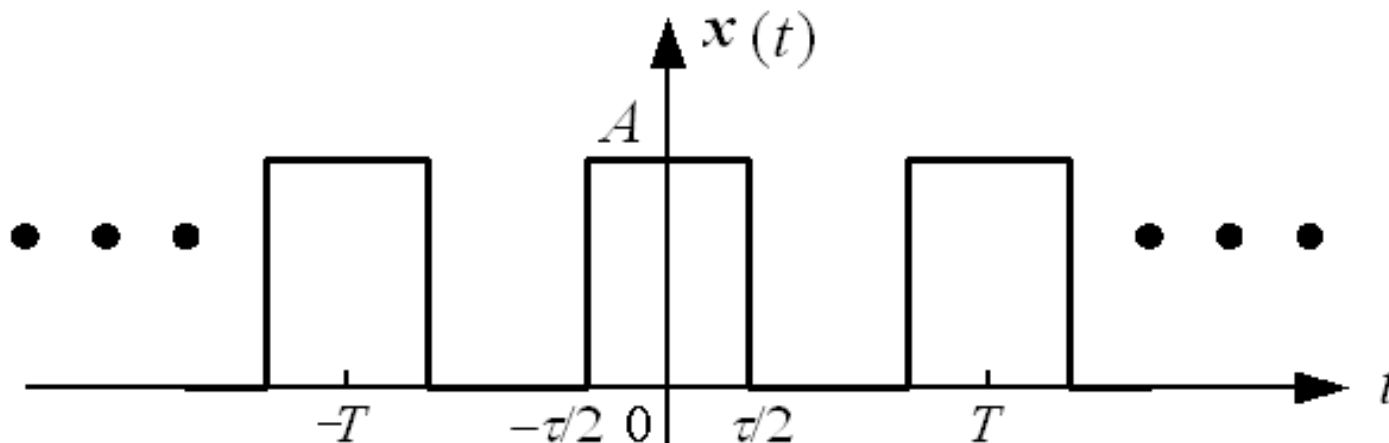
$2|C_k|$ 是信号第 **k 次谐波的振幅**， θ_k 是信号第 **k 次谐波的相位**

● **任意周期实信号可由其直流分量和各次谐波分量合成**

二、离散频谱

$$C_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \quad x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}, \quad |t| \leq T/2$$

例1 计算图示周期矩形脉冲信号的傅里叶级数



解:

$$C_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A e^{-jk\omega_0 t} dt \quad \text{其中 } \omega_0 = \frac{2\pi}{T}$$

当 $k=0$ 时 ,

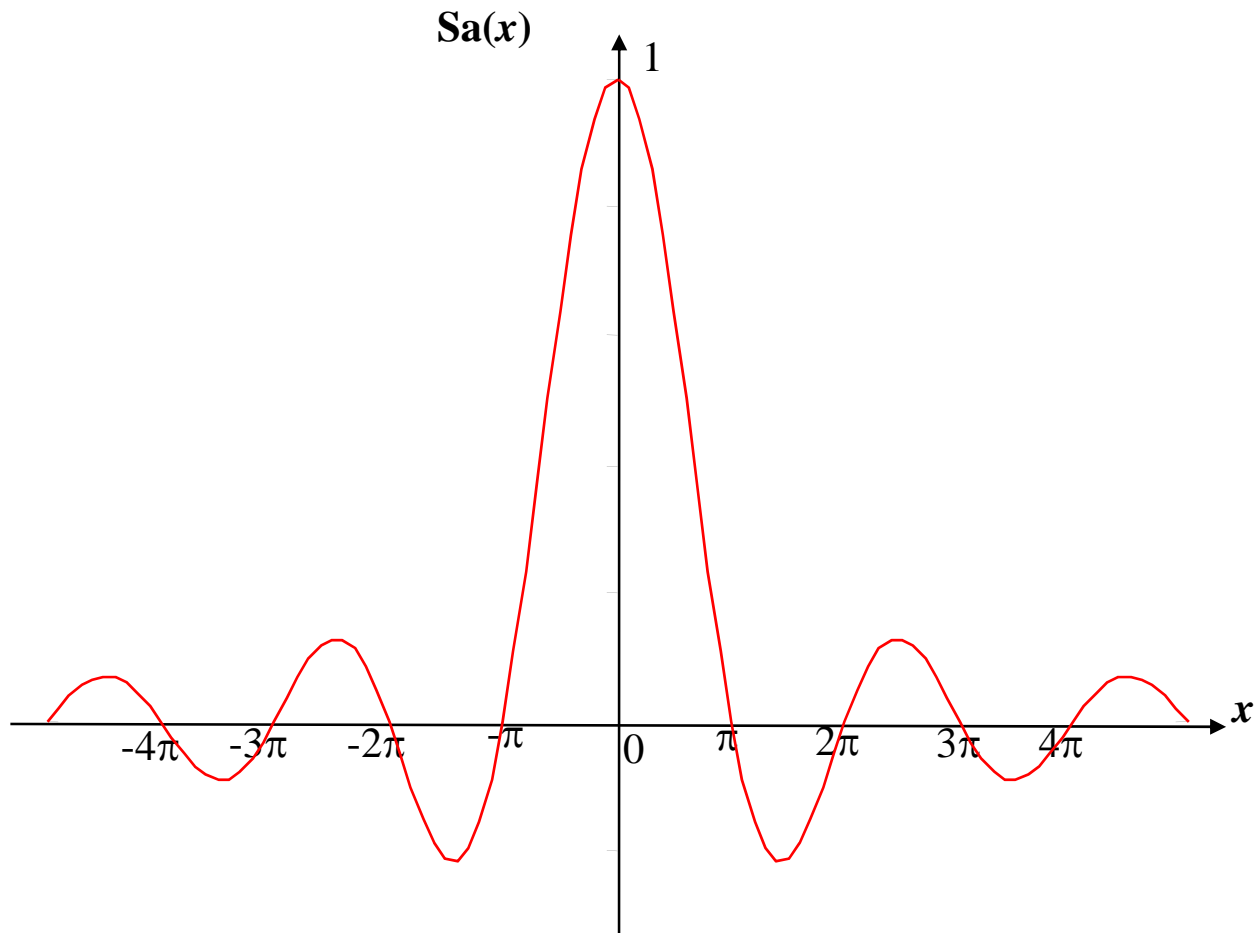
$$C_0 = \frac{1}{T} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A e^0 dt = \frac{1}{T} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A dt = \frac{A\tau}{T}$$

当 $k \neq 0$ 时 ,

$$\begin{aligned} C_k &= \frac{1}{T} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A e^{-jk\omega_0 t} dt = \frac{A}{T} \frac{1}{-jk\omega_0} e^{-jk\omega_0 t} \Bigg|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{A}{T} \frac{1}{-jk\omega_0} \left(e^{-jk\omega_0 \frac{\tau}{2}} - e^{jk\omega_0 \frac{\tau}{2}} \right) \\ &= \frac{A}{T} \frac{1}{-jk\omega_0} \left[-2j \sin\left(\frac{k\omega_0 \tau}{2}\right) \right] = \frac{A}{T} \frac{2 \sin(k\omega_0 \tau / 2)}{k\omega_0} = \frac{A\tau}{T} \frac{\sin(k\omega_0 \tau / 2)}{k\omega_0 \tau / 2} \end{aligned}$$

定义 : 抽样函数

$$Sa(x) = \frac{\sin x}{x}$$



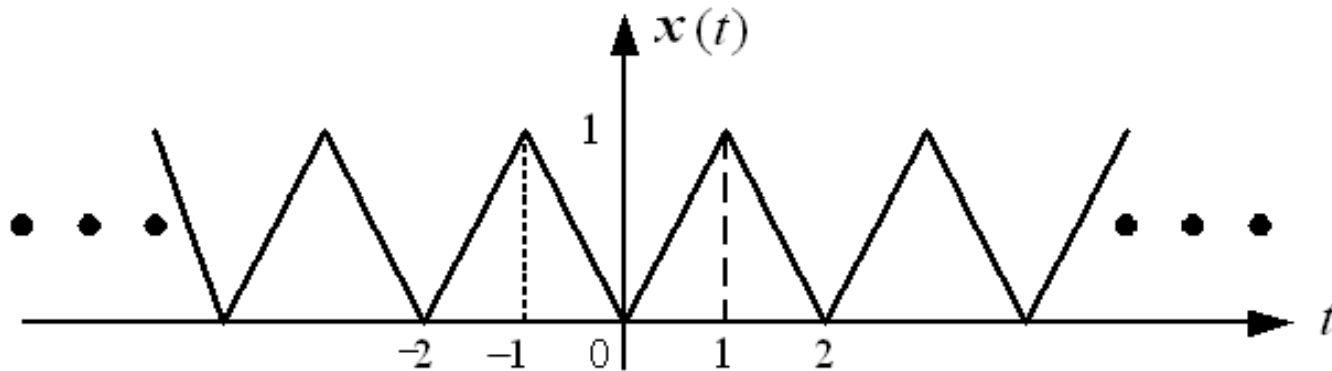
$$C_0 = \frac{A\tau}{T}; \quad C_k = \frac{A\tau \sin(k\omega_0\tau/2)}{T k\omega_0\tau/2} = \frac{A\tau}{T} Sa(k\omega_0\tau/2), \quad k \neq 0$$

$$\Rightarrow C_k = \frac{A\tau}{T} Sa(k\omega_0\tau/2)$$

周期方波信号的傅里叶级数展开式为：

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} = \frac{A\tau}{T} \sum_{k=-\infty}^{\infty} \text{Sa}(k\omega_0\tau/2) e^{jk\omega_0 t}, \quad |t| \leq \frac{T}{2}$$

例2 计算图示周期三角脉冲信号的傅里叶级数



解：

$$C_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt = \frac{1}{2} \left(\int_{-1}^0 -te^{-jk\omega_0 t} dt + \int_0^1 te^{-jk\omega_0 t} dt \right)$$

当k=0时，

$$C_0 = \frac{1}{2} \left(\int_{-1}^0 -te^0 dt + \int_0^1 te^0 dt \right) = \frac{1}{4} \left(-t^2 \Big|_{-1}^0 + t^2 \Big|_0^1 \right) = \frac{1}{2}$$

当 $k \neq 0$ 时, $C_k = \frac{1}{2jk\omega_0} (te^{-jk\omega_0 t} \Big|_{-1}^0 - \int_{-1}^0 e^{-jk\omega_0 t} dt - te^{-jk\omega_0 t} \Big|_0^1 + \int_0^1 e^{-jk\omega_0 t} dt)$

$$\because \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

$$\therefore C_k = \frac{1}{2jk\pi} (0 + e^{jk\pi} - \int_{-1}^0 e^{-jk\pi t} dt - e^{-jk\pi} + 0 + \int_0^1 e^{-jk\pi t} dt)$$

$$= \frac{1}{2jk\pi} (2j \sin k\pi - \frac{1}{-jk\pi} e^{-jk\pi} \Big|_{-1}^0 + \frac{1}{-jk\pi} e^{-jk\pi} \Big|_0^1)$$

$$= \frac{1}{2jk\pi} (-\frac{1}{-jk\pi} + \frac{1}{-jk\pi} e^{jk\pi} + \frac{1}{-jk\pi} e^{-jk\pi} - \frac{1}{-jk\pi})$$

$$= \frac{1}{2(k\pi)^2} (-1 + e^{jk\pi} + e^{-jk\pi} - 1) = \frac{1}{2(k\pi)^2} (2 \cos k\pi - 2)$$

$$= \frac{1}{(k\pi)^2} (\cos k\pi - 1) = \begin{cases} 0 & k \text{为偶数} \\ -\frac{2}{(k\pi)^2} & k \text{为奇数} \end{cases}$$

$$\therefore C_k = \begin{cases} -2/(k\pi)^2 & k \text{ 为奇数} \\ 1/2 & k = 0 \\ 0 & k \text{ 为非0偶数} \end{cases}$$

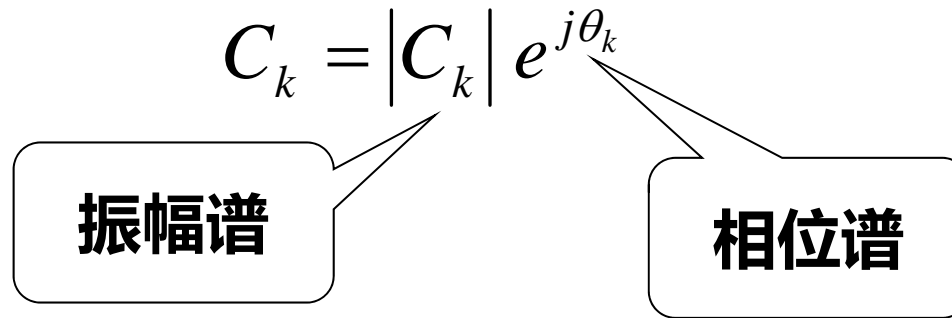
周期三角脉冲信号的傅里叶级数展开式为：

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} = \frac{1}{2} - \sum_{m=-\infty}^{\infty} \frac{2}{[(2m-1)\pi]^2} e^{j(2m-1)\pi t}, \quad |t| \leq T/2 = 1$$

- **周期信号 $x(t)$ 可以分解为不同频率虚指数信号之和**

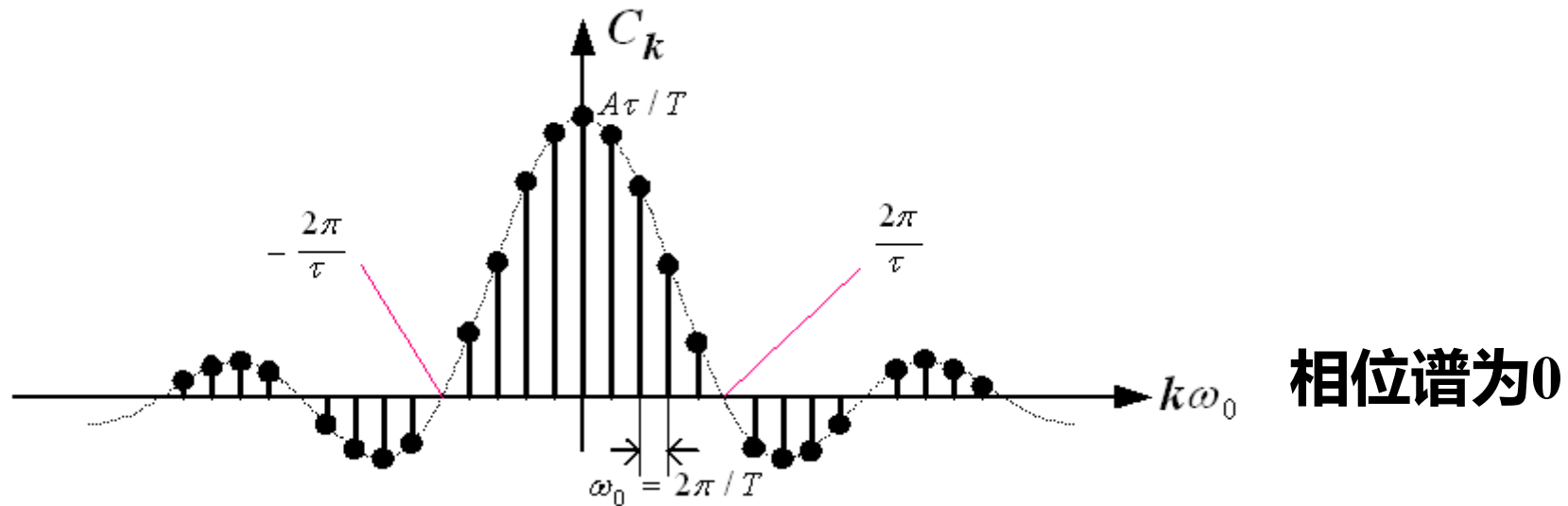
$$x_T(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

C_k 是频率的函数，反映了组成信号各谐波的幅度和相位随频率变化的规律，称为**频谱函数**



• 周期信号的**频谱图**：信号各次谐波对应的 C_k 线状分布图

• 周期矩形脉冲信号的频谱图 $C_k = \frac{A\tau}{T} \text{Sa}(k\omega_0\tau/2)$

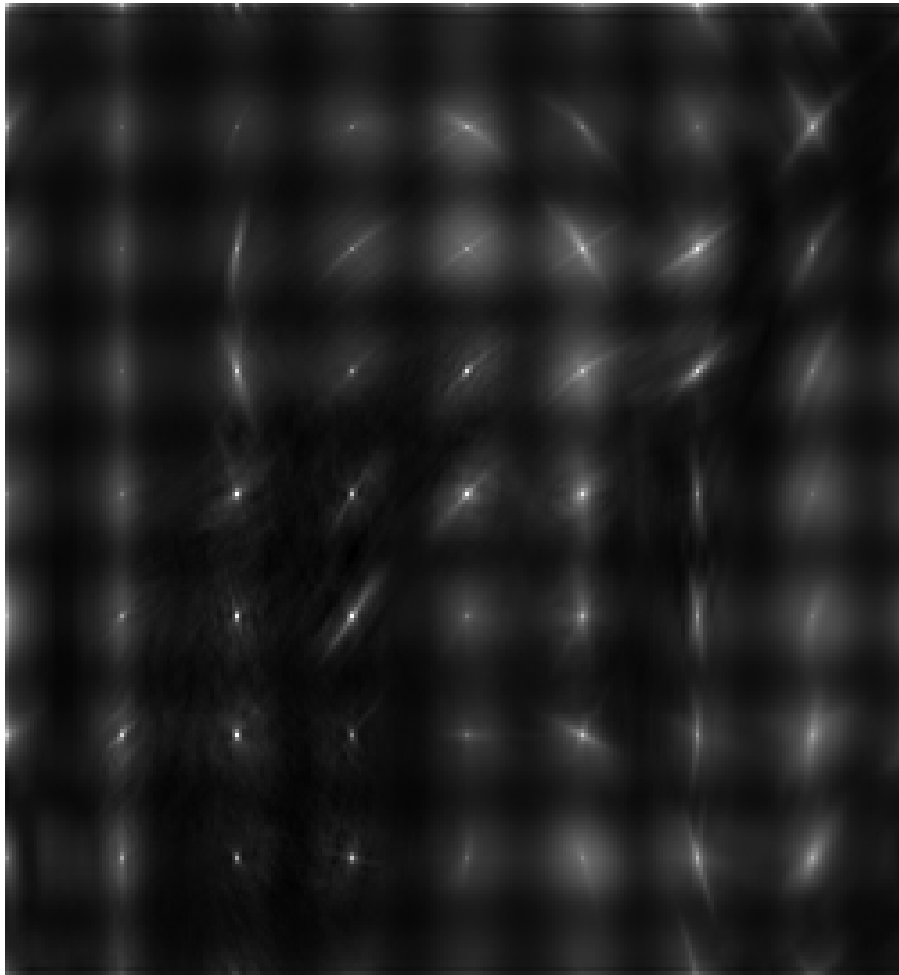


- 相位谱的作用



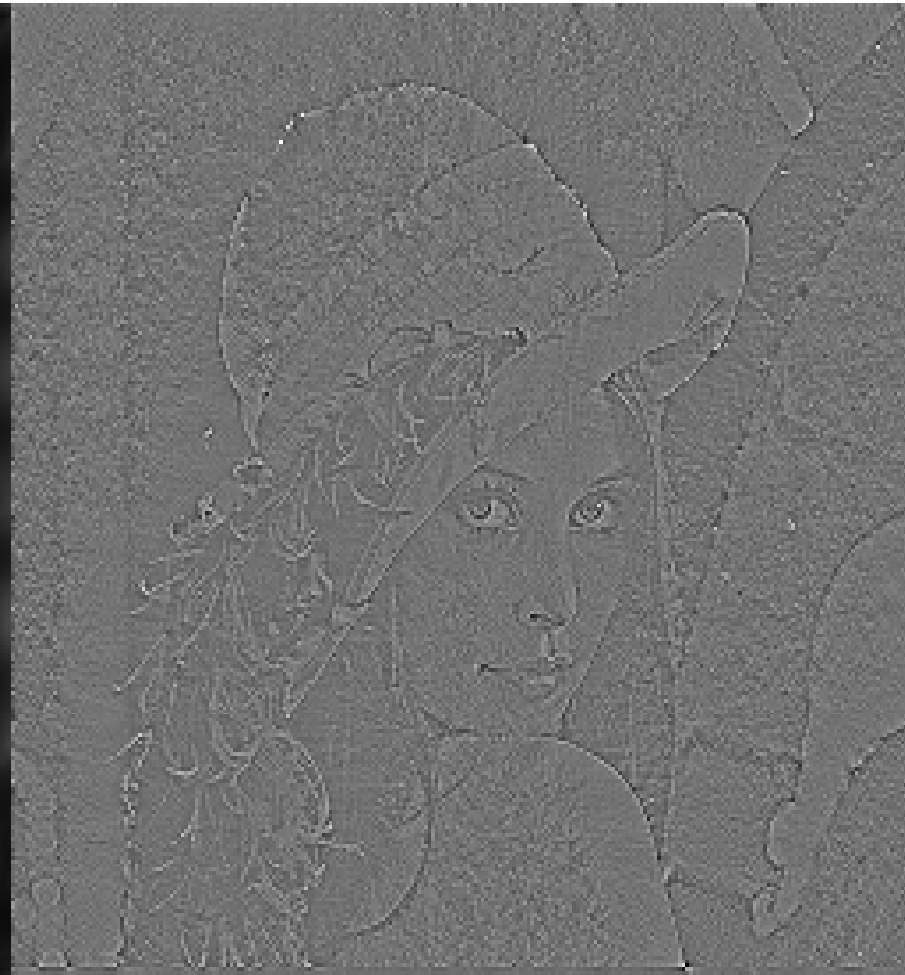
两维傅里叶级数

幅频不变，零相位



02

幅频为常数，相位不变



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- 周期信号频谱的三个特点：**离散性、谐波性、收敛性**
 - ◆ 频谱由不连续的线条组成，每一条线代表一个谐波分量，称为**不连续频谱或离散频谱**
 - ◆ 每条线谱只出现在基波频率 ω_0 的整数倍的频率上，频谱中不可能存在任何具有频率为基波频率非整数倍的分量
 - ◆ 各条线谱的高度(即各次谐波的幅值)，随着谐波次数 k 的增大而逐渐减少的，直至零

信号周期 T 越大， ω_0 就越小，则谱线越密

信号周期 T 越小， ω_0 就越大，则谱线越疏

信号时域波形变化越平缓，高次谐波成分就越少，幅度频谱衰减越快

信号时域波形跳变越多，高次谐波成分就越多，幅度频谱衰减越慢

三、离散频谱的基本性质

1. 线性特性

$$\text{若 } x_1(t) \rightarrow C_{1k}, \quad x_2(t) \rightarrow C_{2k}$$

$$\text{则有 } a_1 \cdot x_1(t) + a_2 \cdot x_2(t) \rightarrow a_1 \cdot C_{1k} + a_2 \cdot C_{2k}$$

证:

$$\begin{aligned} C_k &= \frac{1}{T} \int_T [a_1 \cdot x_1(t) + a_2 \cdot x_2(t)] e^{-jk\omega_0 t} dt \\ &= a_1 \frac{1}{T} \int_T x_1(t) e^{-jk\omega_0 t} dt + a_2 \frac{1}{T} \int_T x_2(t) e^{-jk\omega_0 t} dt \\ &= a_1 \cdot C_{1k} + a_2 \cdot C_{2k} \end{aligned}$$

2. 对称特性

62 (1) 若 $x(t)$ 为实信号, 则 $|C_k| = |C_{-k}|, \quad \theta_k = -\theta_{-k}$

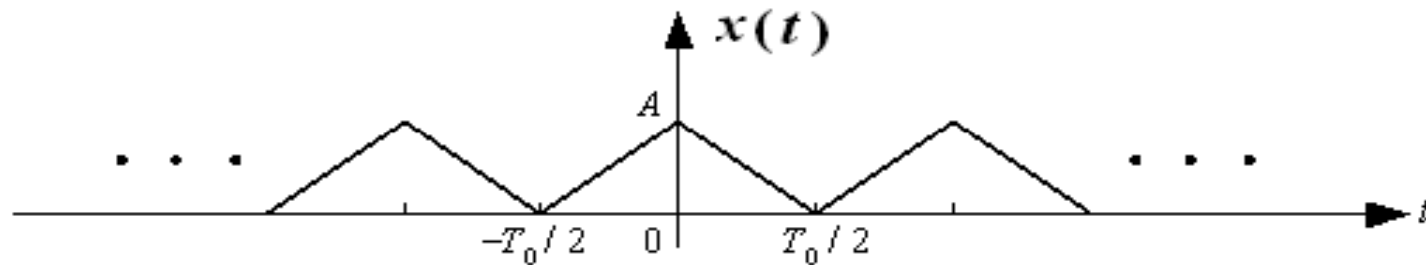
证: $C_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt, \quad C_{-k} = \frac{1}{T} \int_T x(t) e^{jk\omega_0 t} dt$

$x(t)$ 为实信号 $\Rightarrow x(t) = x^*(t)$

$$C_{-k} = \frac{1}{T} \int_T x(t) e^{jk\omega_0 t} dt = \left[\frac{1}{T} \int_T x^*(t) e^{-jk\omega_0 t} dt \right]^* = \left[\frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \right]^* = C_k^*$$

$$\therefore |C_{-k}| = |C_k^*| = |C_k|, \quad \theta_{-k} = \arctan \frac{\text{Im}C_{-k}}{\text{Re}C_{-k}} = \arctan \frac{\text{Im}C_k^*}{\text{Re}C_k^*} = \arctan \frac{-\text{Im}C_k}{\text{Re}C_k} = -\theta_k$$

(2)若 $x(t)$ 为实偶信号: $x(t)=x(-t)$



62 C_k 是 $k\omega_0$ 的实偶函数 \Rightarrow 振幅谱偶对称; 相位谱为0 15

证： $x(t)$ 为实信号，则 $C_k^* = C_{-k}$

$x(t)$ 为偶信号 $\Rightarrow x(t) = x(-t)$

$$\begin{aligned} C_{-k} &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j(-k)\omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0(-t)} dt = \frac{1}{T} \int_{T/2}^{-T/2} x(-t') e^{-jk\omega_0 t'} (-dt') \\ &= \frac{1}{T} \int_{-T/2}^{T/2} x(-t') e^{-jk\omega_0 t'} dt' = \frac{1}{T} \int_{-T/2}^{T/2} x(t') e^{-jk\omega_0 t'} dt' = C_k \end{aligned}$$

令： $C_k = \alpha_k + j\beta_k$

$$\alpha_{-k} + j\beta_{-k} = C_{-k} = C_k = \alpha_k + j\beta_k \Rightarrow \alpha_k = \alpha_{-k}, \beta_k = \beta_{-k}$$

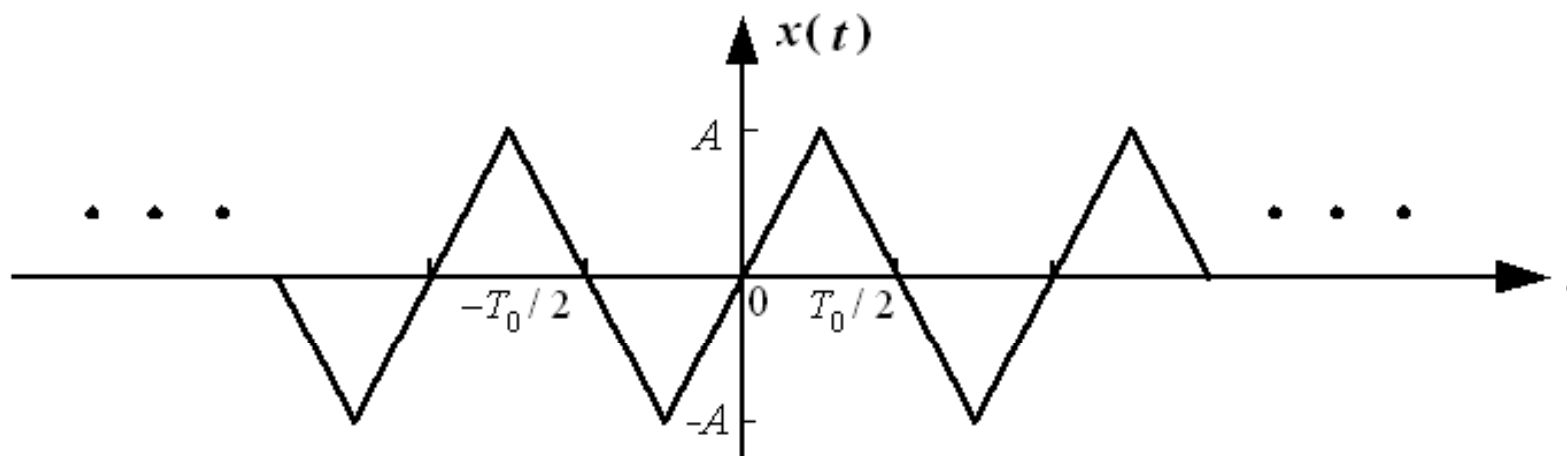
$$\alpha_{-k} + j\beta_{-k} = C_{-k} = C_k^* = \alpha_k - j\beta_k \Rightarrow \alpha_k = \alpha_{-k}, \beta_k = -\beta_{-k}$$

$$\beta_k = \beta_{-k} = -\beta_{-k} \Rightarrow \beta_{-k} = 0 \Rightarrow \beta_k = 0$$

$$\therefore C_k = \alpha_k = \alpha_{-k} = C_{-k}$$

C_k 是 $k\omega_0$ 的实偶函数

(3)若 $x(t)$ 为实奇信号： $x(t)=-x(-t)$



C_k 是 $k\omega_0$ 的纯虚奇函数 \Rightarrow 振幅谱偶对称；相位谱奇对称

证： $x(t)$ 为实信号，则 $C_k^* = C_{-k}$

$x(t)$ 为奇信号 $\Rightarrow x(t) = -x(-t)$ 类似 $\Rightarrow C_{-k} = -C_k$

$$\alpha_{-k} + j\beta_{-k} = C_{-k} = -C_k = -\alpha_k - j\beta_k \Rightarrow \alpha_k = -\alpha_{-k}, \beta_k = -\beta_{-k}$$

$$\alpha_{-k} + j\beta_{-k} = C_{-k} = C_k^* = \alpha_k - j\beta_k \Rightarrow \alpha_k = \alpha_{-k}, \beta_k = -\beta_{-k}$$

$$\alpha_k = \alpha_{-k} = -\alpha_{-k} \Rightarrow \alpha_{-k} = 0 \Rightarrow \alpha_k = 0$$

$$\therefore C_k = j\beta_k = -j\beta_{-k} = -C_{-k}$$

C_k 是 $k\omega_0$ 的纯虚奇函数

$$\Rightarrow |C_k| = |j\beta_k| = |-j\beta_{-k}| = |j\beta_{-k}| = |C_{-k}|$$

if $\beta_k > 0$, then $\beta_{-k} < 0$

$$\Rightarrow C_k = j\beta_k = \beta_k e^{j\frac{\pi}{2}}, C_{-k} = j\beta_{-k} = -j|\beta_{-k}| = |\beta_{-k}| e^{-j\frac{\pi}{2}}$$

if $\beta_k < 0$, then $\beta_{-k} > 0$

$$\Rightarrow C_{-k} = j\beta_{-k} = \beta_{-k} e^{j\frac{\pi}{2}}, C_k = j\beta_k = -j|\beta_k| = |\beta_k| e^{-j\frac{\pi}{2}}$$

3. 周期信号功率与离散频谱的关系

帕塞瓦尔(Parseval)功率守恒定理

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |C_k|^2$$

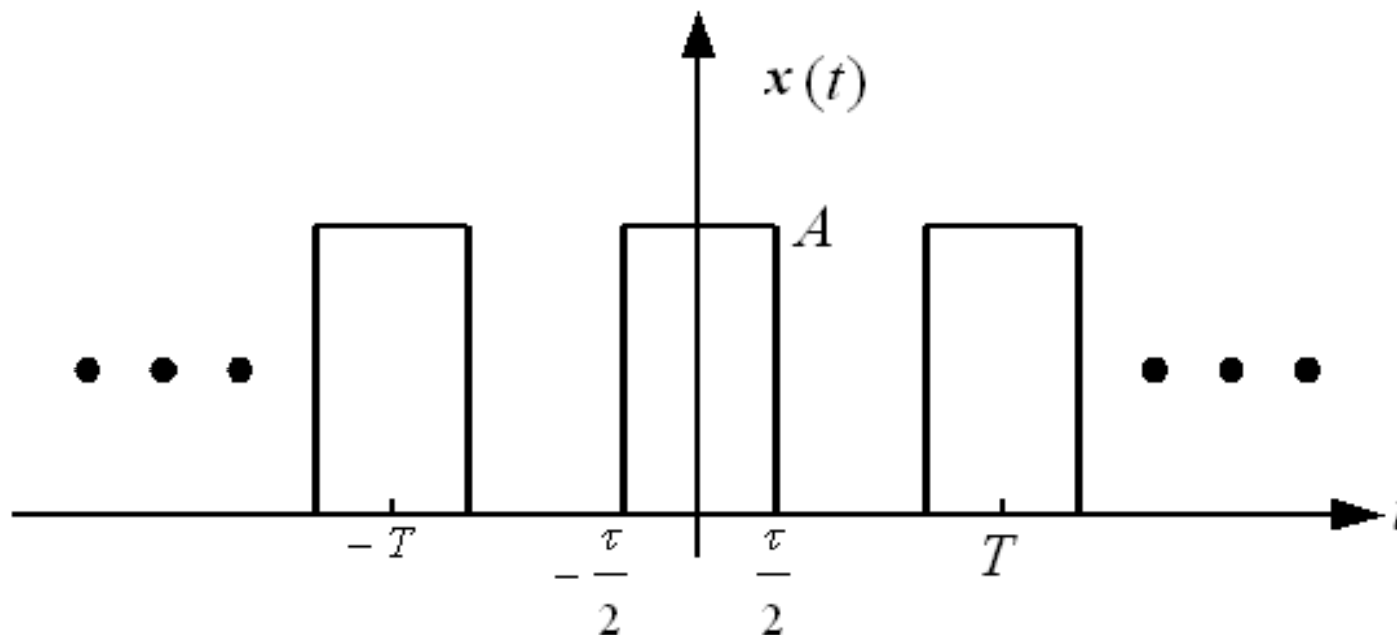
物理意义：

周期信号平均功率按各谐波成分的振幅大小分配给各分量

证： $x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}, \quad |t| \leq T/2$

$$\begin{aligned} P &= \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T} \int_{-T/2}^{T/2} x(t)x^*(t) dt = \frac{1}{T} \int_{-T/2}^{T/2} \left[\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \right] \left[\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \right]^* dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \left[\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \right] \left[\sum_{n=-\infty}^{\infty} C_n^* e^{-jn\omega_0 t} \right] dt = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_n^* C_k \frac{1}{T} \int_{-T/2}^{T/2} e^{jk\omega_0 t} e^{-jn\omega_0 t} dt \\ 62 &= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_n^* C_k \frac{1}{T} \int_{-T/2}^{T/2} e^{j(k-n)\omega_0 t} dt = \sum_{k=-\infty}^{\infty} C_k^* C_k = \sum_{k=-\infty}^{\infty} |C_k|^2 \end{aligned}$$

例3 求周期矩形脉冲信号在其有效带宽($0 \sim 2\pi/\tau$)内谐波分量所具有的平均功率占整个信号平均功率的百分比
(其中 $A=1$, $T=1/4$, $\tau=1/20$)



解： 周期矩形脉冲的傅里叶级数为：

$$C_k = \frac{A\tau}{T} \text{Sa}(k\omega_0\tau/2) = \frac{1 \times 1/20}{1/4} \text{Sa}\left(k \frac{2\pi}{1/4} \cdot \frac{1}{20} \cdot \frac{1}{2}\right) = \frac{1}{5} \text{Sa}\left(\frac{k\pi}{5}\right)$$

信号的平均功率为：

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T} A^2 \tau = \frac{1/20}{1/4} = \frac{1}{5}$$

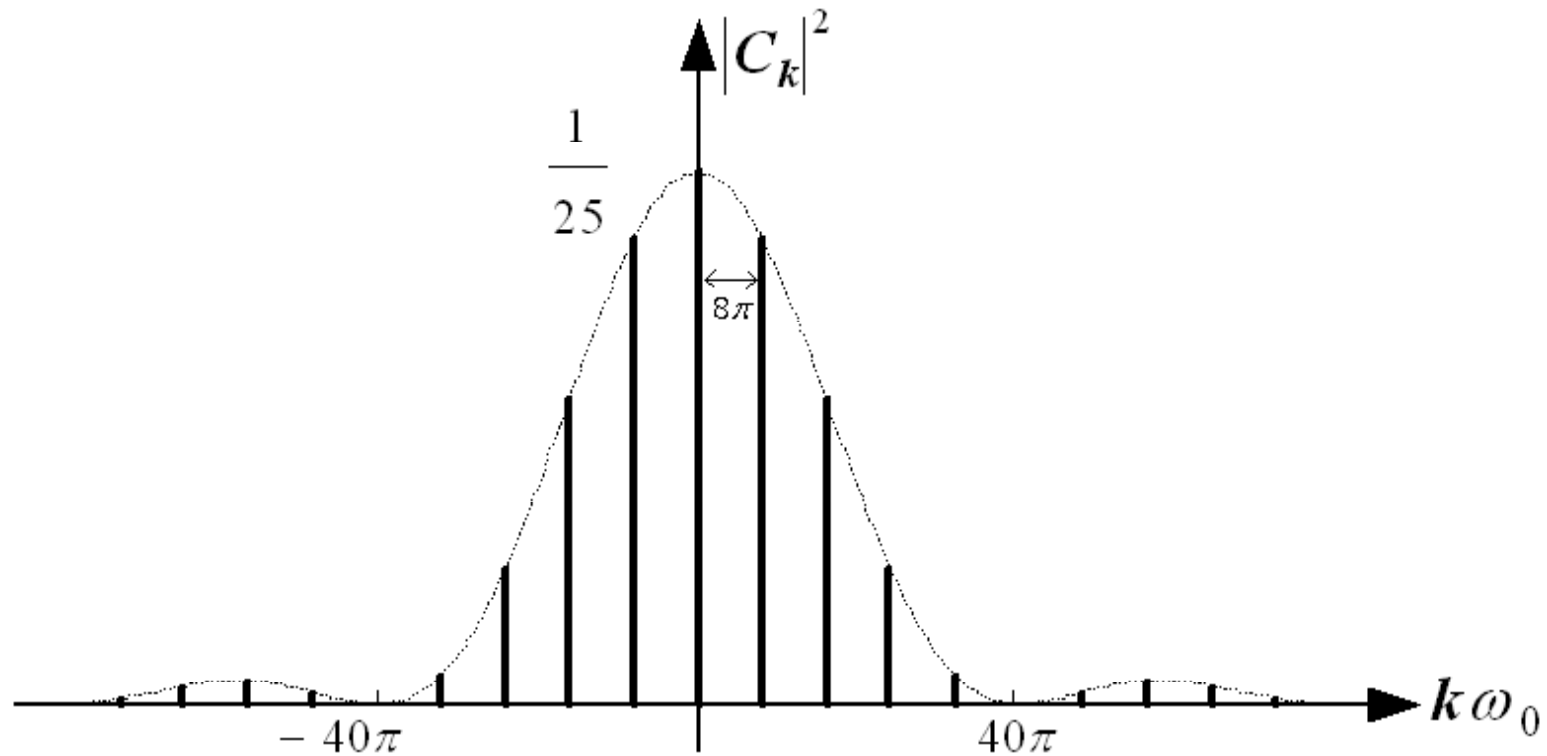
$$k\omega_0 = k \frac{2\pi}{T} \leq \frac{2\pi}{\tau} \Rightarrow k \leq \frac{T}{\tau} = \frac{1/4}{1/20} = 5$$

包含在有效带宽(0~2π/τ)内的各谐波平均功率为：

$$P_1 = \sum_{k=-5}^5 |C_k|^2 = \sum_{k=-5}^5 \left| \frac{1}{5} \text{Sa}\left(\frac{k\pi}{5}\right) \right|^2 \approx 0.1806$$

$$\therefore \frac{P_1}{P} = \frac{0.1806}{0.2000} = 90\%$$

$$|C_k|^2 = \frac{1}{25} [Sa(k\pi/5)]^2, \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1/4} = 8\pi$$



4. 截断离散频谱的高频成分对 $x(t)$ 引入的失真

(1)有限项截断后，傅里叶级数于最小均方误差意义逼近原信号

均方误差

$$\begin{aligned}
 E &= \int_T |e(t)|^2 dt = \int_T [x(t) - x_1(t)][x^*(t) - x_1^*(t)] dt = \int_T x(t)x^*(t) dt - \int_T x^*(t) \sum_{k=-N}^N C_k e^{jk\omega_0 t} dt \\
 &\quad - \int_T x(t) \sum_{k=-N}^N C_k^* e^{-jk\omega_0 t} dt + \int_T \sum_{k=-N}^N \sum_{n=-N}^N C_k^* C_n e^{j(n-k)\omega_0 t} dt = \int_T |x(t)|^2 dt - \sum_{k=-N}^N C_k \int_T x^*(t) e^{jk\omega_0 t} dt \\
 &\quad - \sum_{k=-N}^N C_k^* \int_T x(t) e^{-jk\omega_0 t} dt + \sum_{k=-N}^N \sum_{n=-N}^N C_k^* C_n \int_T e^{j(n-k)\omega_0 t} dt = \int_T |x(t)|^2 dt - \sum_{k=-N}^N C_k \left[\int_T x(t) e^{-jk\omega_0 t} dt \right]^* \\
 &\quad - \sum_{k=-N}^N C_k^* C_k T + \sum_{k=-N}^N C_k^* C_k T = \int_T |x(t)|^2 dt - \sum_{k=-N}^N C_k C_k^* T = \int_T |x(t)|^2 dt - T \sum_{k=-N}^N |C_k|^2 \\
 &= T \sum_{k=-\infty}^{\infty} |C_k|^2 - T \sum_{k=-N}^N |C_k|^2 = T \sum_{k=-\infty}^{-(N+1)} |C_k|^2 + T \sum_{k=N+1}^{\infty} |C_k|^2
 \end{aligned}$$

当 $N \rightarrow \infty$ ， E 单调趋于0

(2) 吉伯斯现象

用有限次谐波分量来近似原信号，在 $x(t)$ 的不连续点将出现过冲，过冲峰值不随 N 增加而减少，约为跳变值的9%

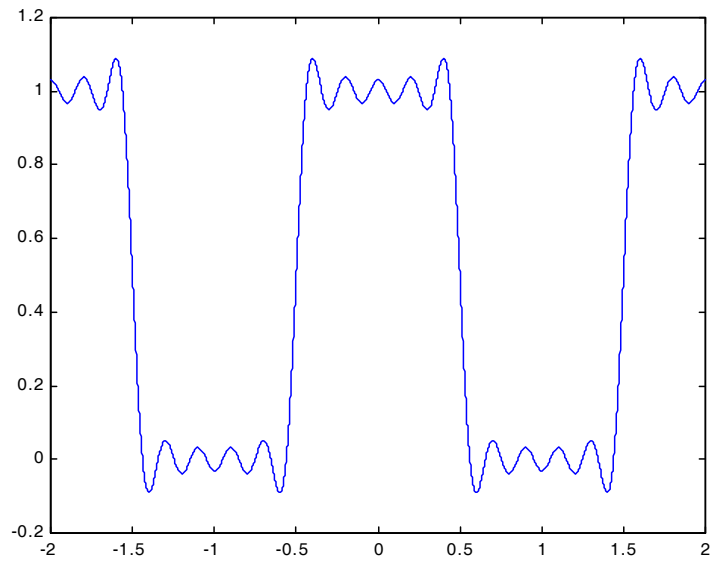
以方波为例： $A=1$, $T=2$, $\tau=T/2=1$

$$C_k = \frac{A\tau}{T} \text{Sa}(k\omega_0\tau/2) = \frac{1 \times 1}{2} \text{Sa}\left(k \frac{2\pi}{2} \times 1/2\right) = \frac{1}{2} \text{Sa}(k\pi/2)$$

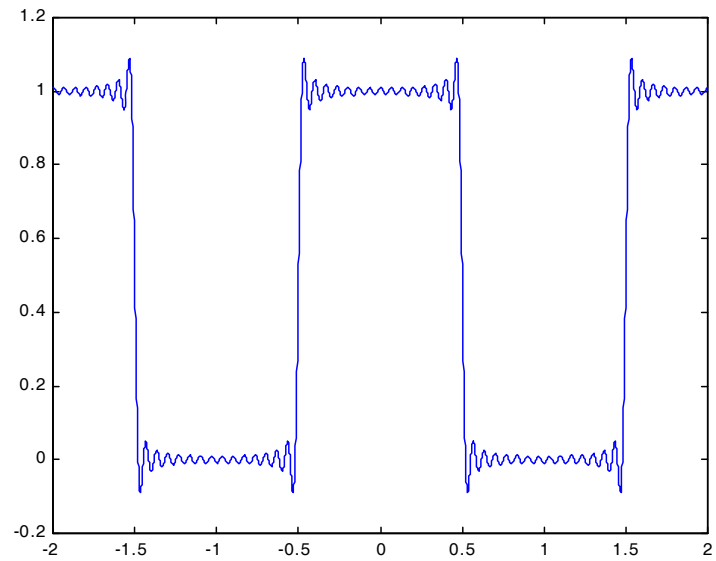
$$C_0 = 1/2; \quad C_{2n} = 0, n \neq 0; \quad C_{2n+1} = \frac{\sin[(2n+1)\pi/2]}{(2n+1)\pi}, \quad C_{2n+1} = C_{-(2n+1)}$$

$$\begin{aligned} \therefore x(t) &= \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} = C_0 + \sum_{n=0}^{\infty} [C_{2n+1} e^{j(2n+1)\pi t} + C_{-(2n+1)} e^{-j(2n+1)\pi t}] \\ &= \frac{1}{2} + \sum_{n=0}^{\infty} C_{2n+1} [e^{j(2n+1)\pi t} + e^{-j(2n+1)\pi t}] = \frac{1}{2} + 2 \sum_{n=0}^{\infty} \frac{\sin[(2n+1)\pi/2]}{(2n+1)\pi} \cos[(2n+1)\pi t] \\ &= \frac{1}{2} + 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\pi} \cos[(2n+1)\pi t] = \frac{1}{2} + \frac{2}{\pi} \cos(\pi t) - \frac{2}{3\pi} \cos(3\pi t) + \dots \end{aligned}$$

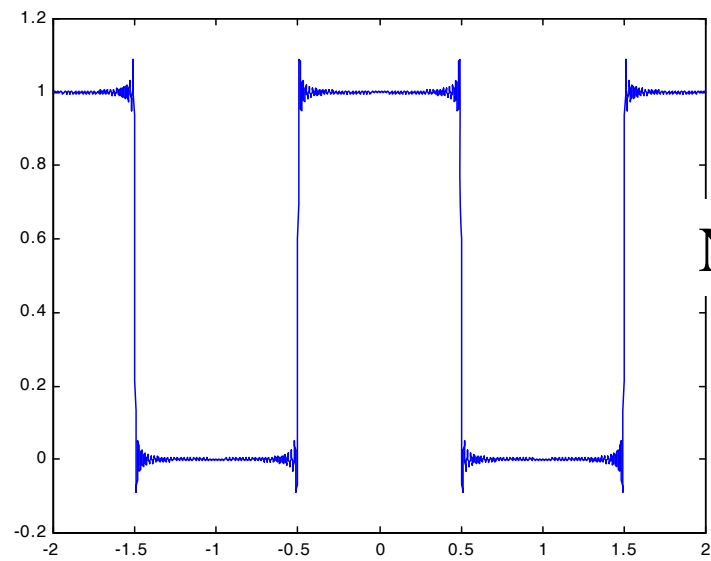
$N=5$



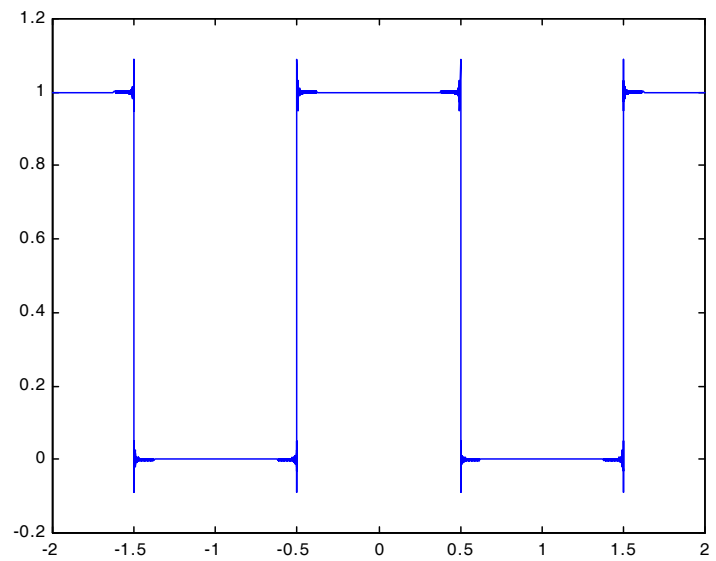
$N=15$



$N=50$



$N=500$



§ 1.2 能量信号的频谱分析

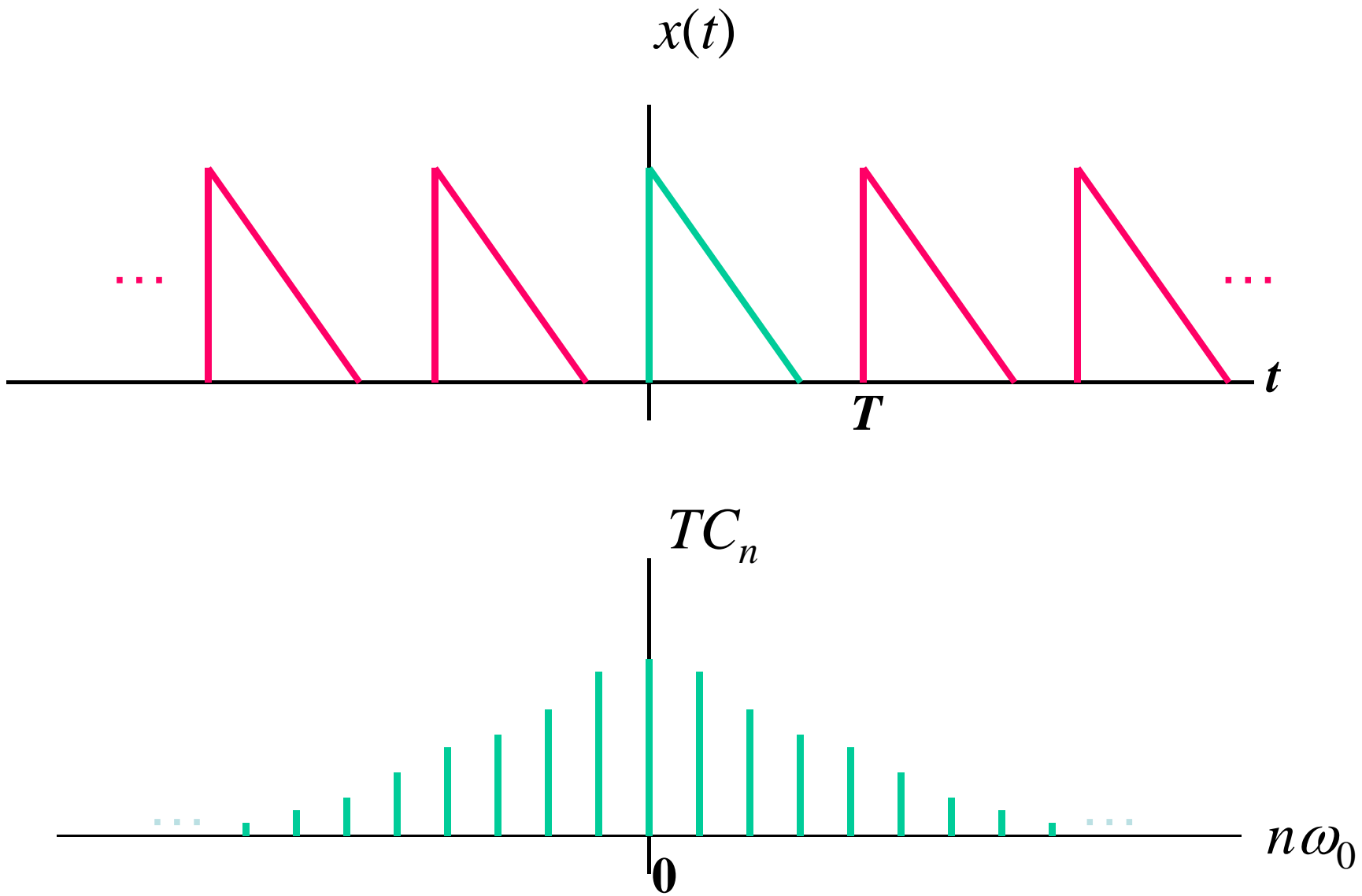
一、傅里叶变换(Fourier transform)

1. 从傅里叶级数到傅里叶变换

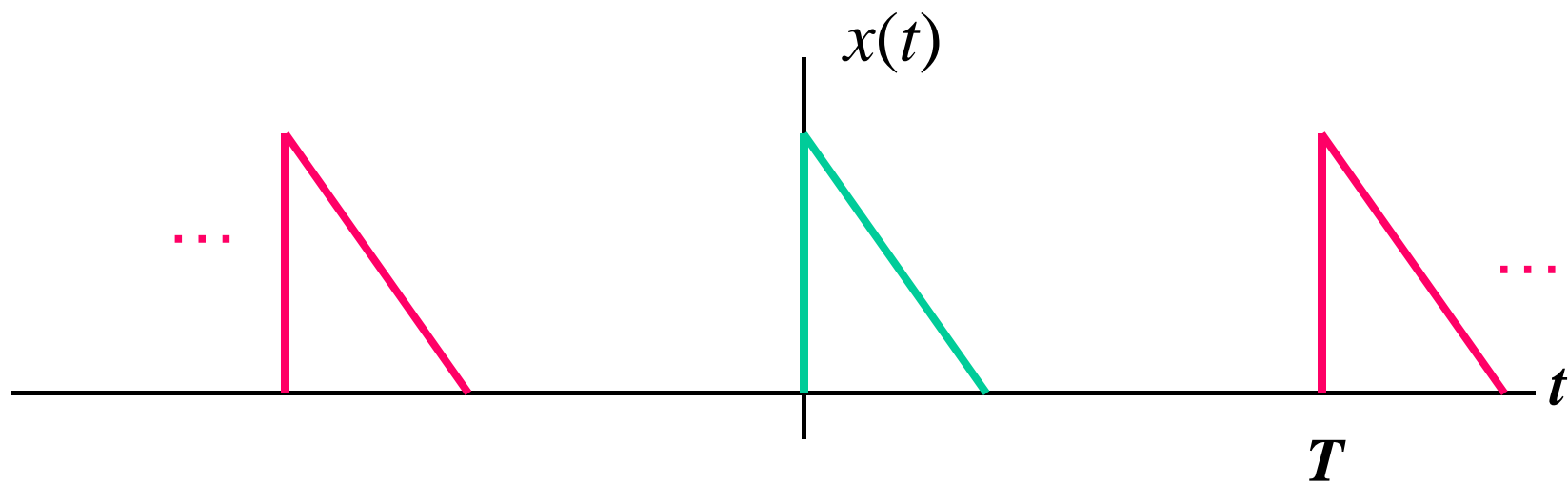
- 讨论周期 T 增加对离散频谱的影响

周期为 T 宽度为 t 的周期矩形脉冲的Fourier级数为：

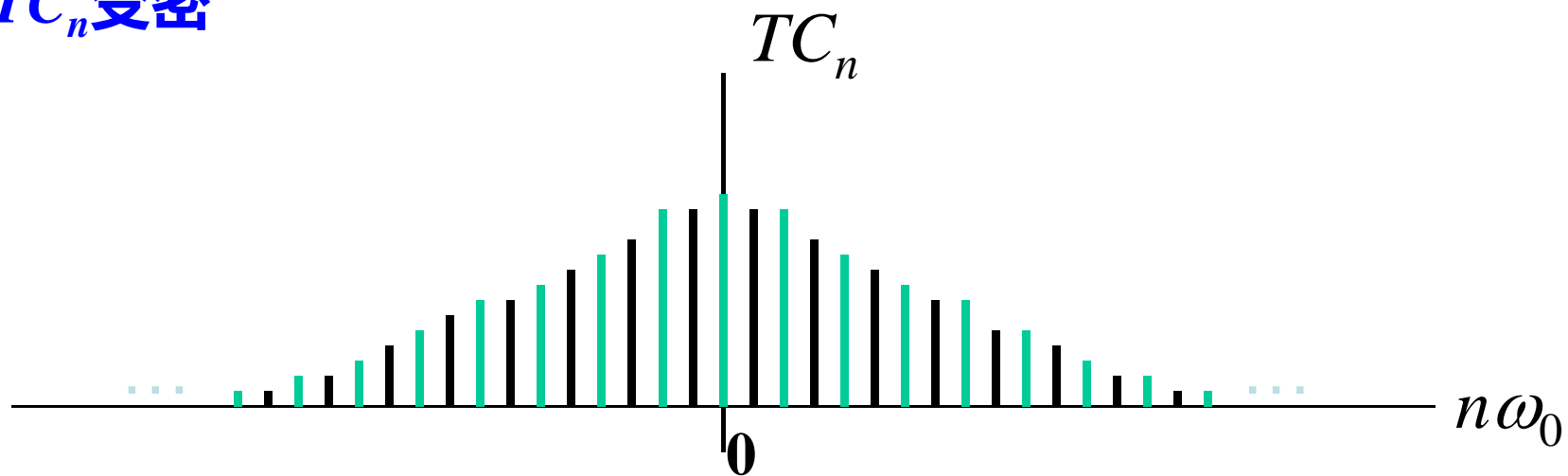
$$C_n = \frac{A\tau}{T} \text{Sa}(n\omega_0\tau/2) \Rightarrow TC_n = A\tau \text{Sa}(n\omega_0\tau/2)$$



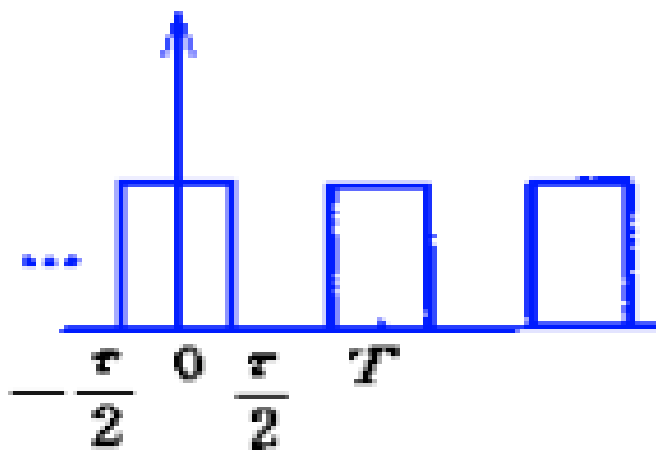
周期 T 增大



TC_n 变密



- 周期矩形脉冲信号的频谱



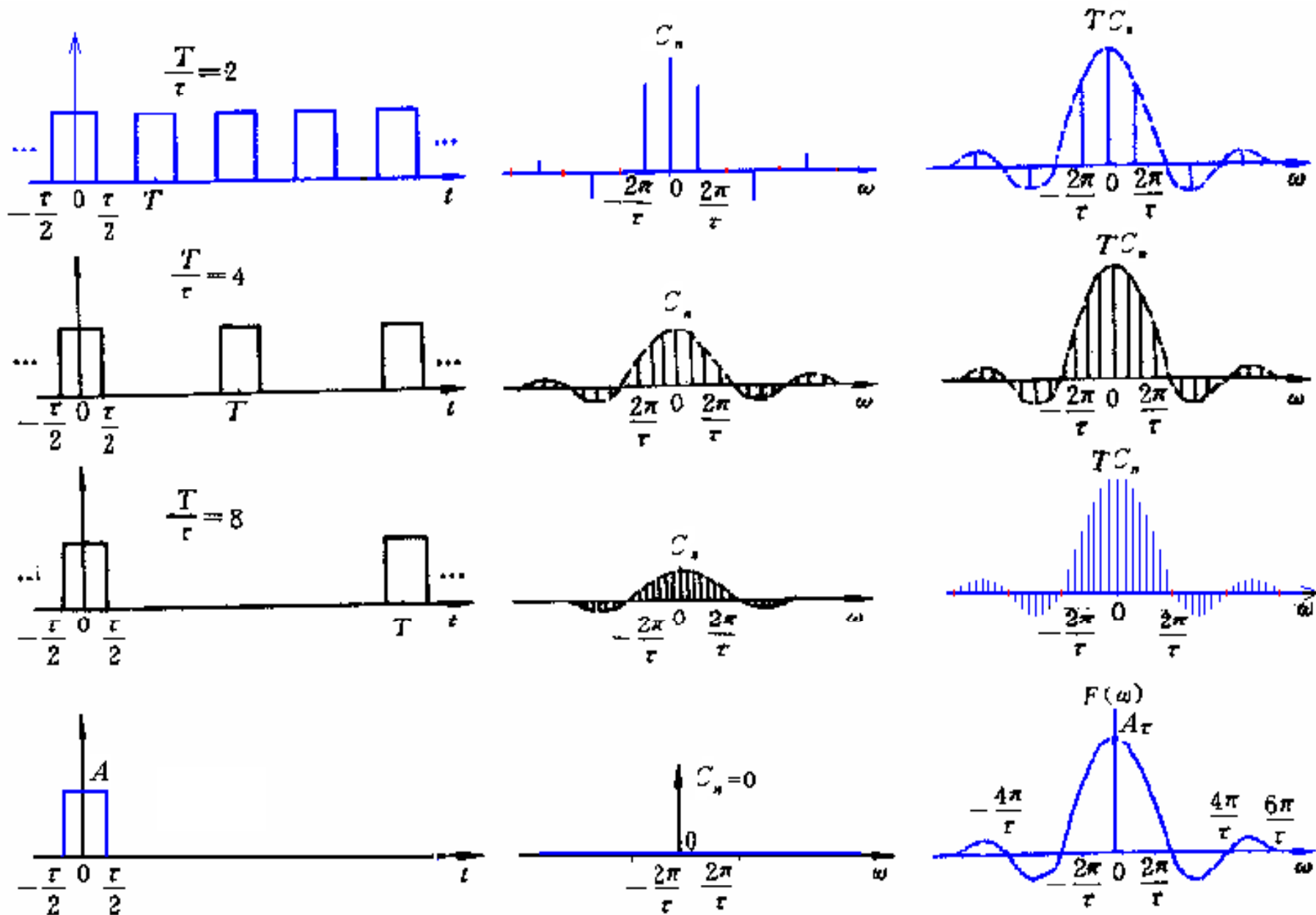
$$x(t) = \begin{cases} A & |t| \leq \frac{\tau}{2} \\ 0 & \frac{\tau}{2} < |t| \leq \frac{T}{2} \end{cases}$$

傅里叶级数展开式为:

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_0 t}, |t| \leq T/2$$

傅里叶级数为:

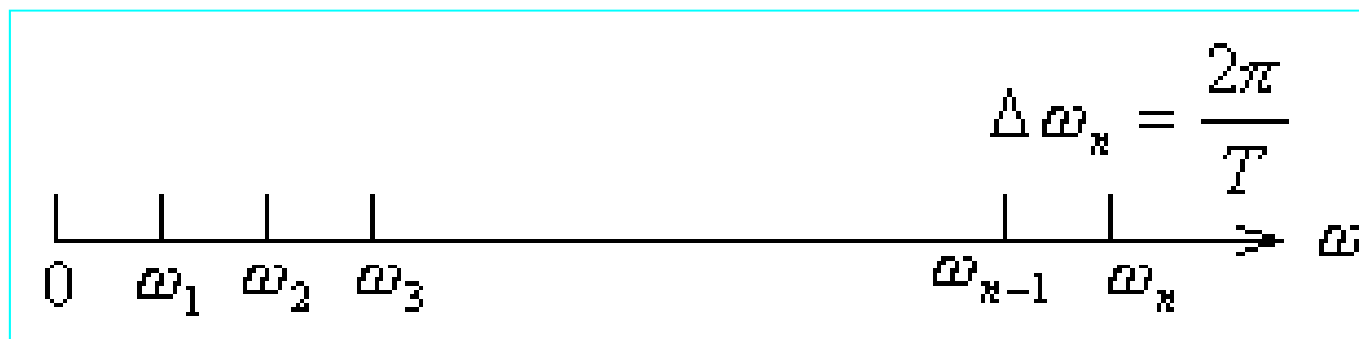
$$C_n = \frac{A\tau}{T} \frac{\sin(n\omega_0\tau/2)}{(n\omega_0\tau/2)} = \frac{A\tau}{T} \text{Sa}(n\omega_0\tau/2), \quad n = 0, \pm 1, \pm 2, \dots$$



当 $T \rightarrow +\infty, \omega_0 = \frac{2\pi}{T} \rightarrow 0$

信号 $x(t)$ 从周期信号变成非周期信号

令 $\omega_n = n\omega_0, \rightarrow \Delta\omega_n = \omega_n - \omega_{n-1} = \omega_0 \quad \therefore T = \frac{2\pi}{\Delta\omega_n}$



周期信号: $C_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt \quad x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}, |t| \leq T/2$

非周期信号: $x(t) = \lim_{\Delta\omega_n \rightarrow 0} \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} (TC_n) e^{j\omega_n t} \Delta\omega_n$

$$\frac{1}{T} = \frac{\Delta\omega_n}{2\pi}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

$$\lim_{T \rightarrow \infty} TC_n = X(\omega)$$

周期信号: $C_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt$ $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}, \quad |t| \leq T/2$

非周期信号:

$$X(\omega) = \lim_{\Delta\omega_n \rightarrow 0} T C_n = \lim_{\Delta\omega_n \rightarrow 0} T \cdot \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t} dt = \lim_{\Delta\omega_n \rightarrow 0} \int_{-T/2}^{T/2} x(t) e^{-j\omega_n t} dt$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

2. 傅里叶变换对

傅里叶正变换:

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

或 $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$

傅里叶反变换:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

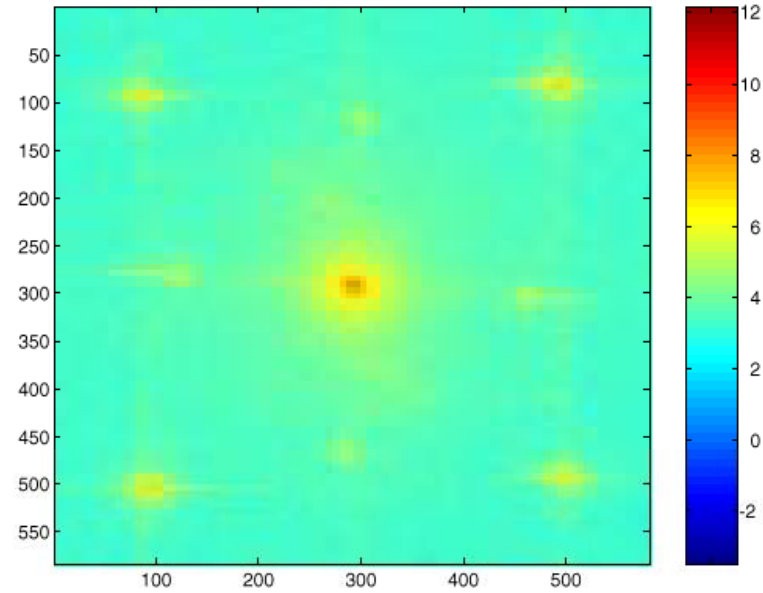
或 $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} d f$

两维Fourier transform



Fourier

582 by 582 pixels
Fourier像



Fourier transformed

图像两维傅里叶变换
幅度的对数

- 符号表示：

$$X(\omega) = F[x(t)], \quad x(t) = F^{-1}[X(\omega)]$$

$$\text{或} \quad x(t) \xleftrightarrow{F} X(\omega)$$

- 傅里叶反变换：

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

物理意义：非周期信号可以分解为无数个频率为 ω 、复振幅为 $[X(\omega)/2\pi]d\omega$ 的复指数信号 $e^{j\omega t}$ 的线性组合

- 称信号 $x(t)$ 的傅里叶变换 $X(\omega)$ 为其**连续频谱**
- 一个信号满足狄里赫利条件，其傅里叶变换存在

狄里赫利条件

(1) 非周期信号在无限区间上绝对可积

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

充分不必要条件

(2) 在任意有限区间内，信号只有有限个最大值和最小值

必要不充分条件

(3) 在任意有限区间内，信号仅有有限个不连续点，且这些点必须是有限值

必要不充分条件

一个连续信号 $x(t)$ 的傅里叶变换 $X(\omega)$ 是唯一的

二、连续频谱

- 周期信号的频谱为**离散频谱**
非周期信号的频谱为**连续频谱**
- 周期信号的频谱为 **C_n** 的分布，表示每个谐波分量的复振幅
非周期信号的频谱为 **TC_n** 的分布，表示每单位带宽内所有谐波分量合成的复振幅(即**频谱密度函数**)

两者关系：

$$X(\omega) = \lim_{T \rightarrow \infty} TC_n$$

$$C_n = \frac{X(\omega)}{T} \Big|_{\omega = n\omega_0}$$

- **连续频谱密度函数(简称连续频谱)：**

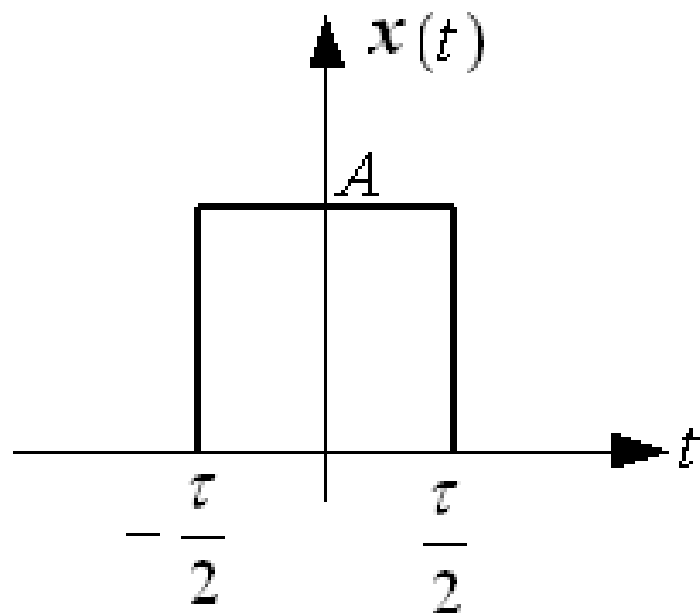
$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) = |X(\omega)| e^{j\varphi(\omega)} = R(\omega) + jI(\omega)$$

振幅密度谱
简称振幅谱

相位密度谱
简称相位谱

例1 计算图示非周期矩形脉冲信号的频谱函数



解：

$$x(t) = \begin{cases} A, & |t| \leq \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

可记作：

$$x(t) = A \operatorname{rect}\left(\frac{t}{\tau}\right)$$

类似，幅度 A 、中心 t_0 、宽度 τ 的矩形脉冲可记作：

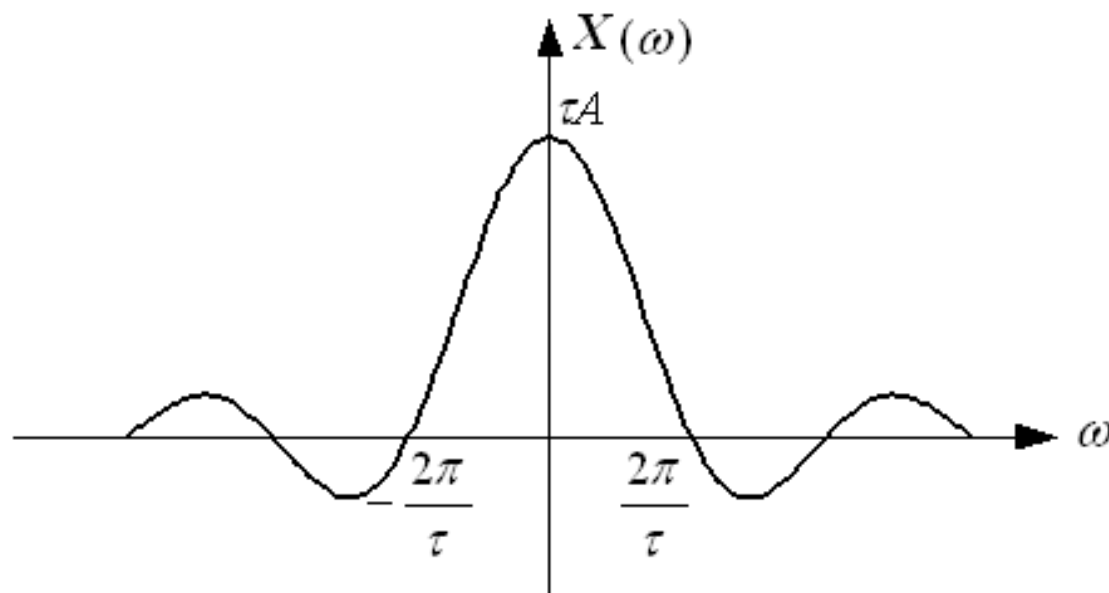
$$x(t) = A \operatorname{rect}\left(\frac{t-t_0}{\tau}\right)$$

由傅里叶正变换定义式，可得：

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} A \cdot e^{-j\omega t} dt = \frac{A}{-j\omega} e^{-j\omega t} \Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{-j\omega} (e^{-j\omega\tau/2} - e^{j\omega\tau/2}) = \frac{A}{-j\omega} [-2j \sin(\omega\tau/2)]$$

$$= \frac{2A \sin(\omega\tau/2)}{\omega} = A\tau \frac{\sin(\omega\tau/2)}{\omega\tau/2} = A\tau \cdot Sa(\omega\tau/2)$$



分析：

1. 非周期矩形脉冲信号的频谱是**连续频谱**，其形状与周期矩形脉冲信号离散频谱的包络线相似
2. 周期信号的**离散频谱**可通过对非周期信号的连续频谱等间隔取样求得
3. 信号在**时域有限**，则在**频域将无限延续**
4. 信号的频谱分量主要集中在**零频到第一个过零点**之间，工程中往往将此宽度作为**有效带宽**
5. 脉冲宽度 τ 越窄，有限带宽越宽，高频分量越多
即信号信息量大、传输速度快，传送信号所占用的频带越宽

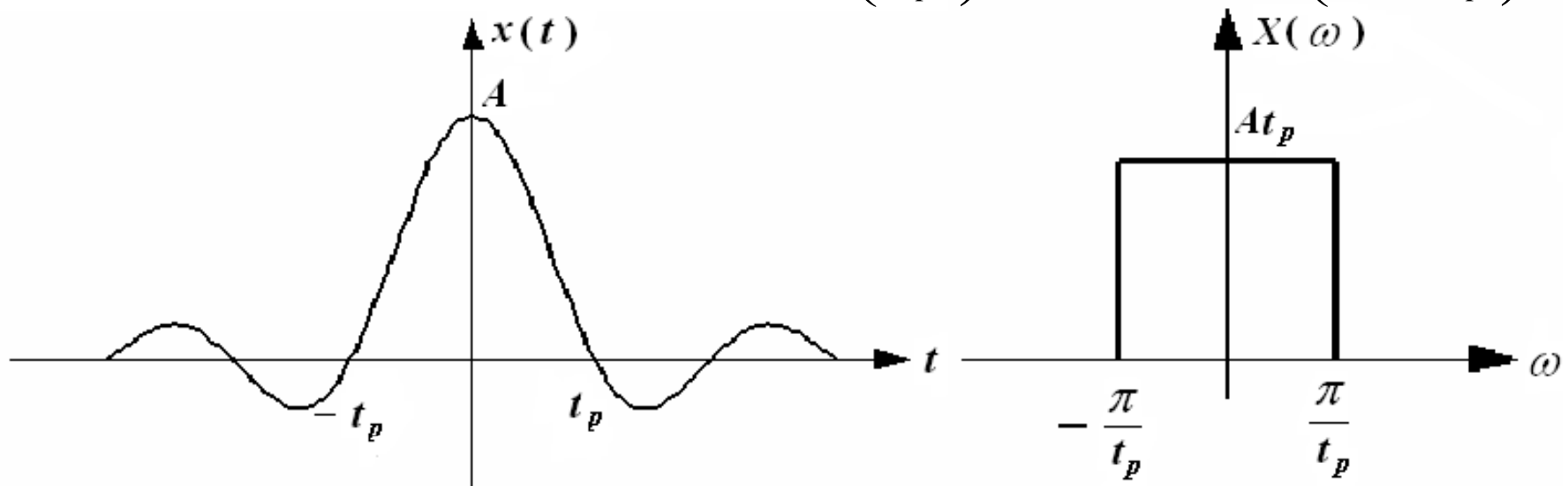
- **时域抽样函数**：主瓣幅度 A 、中心 t_0 、第一零点和中心距离 t_p

$$x(t) = ASa\left[\frac{\pi(t-t_0)}{t_p}\right]$$

若 $t_0=0$ ：

$$x(t) = ASa\left(\frac{\pi t}{t_p}\right)$$

- **利用傅里叶反变换可证明**： $ASa\left(\frac{\pi t}{t_p}\right) \xleftrightarrow{F} At_p \text{rect}\left(\frac{\omega}{2\pi/t_p}\right)$



证：

$$\begin{aligned}
 x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\pi/t_p}^{\pi/t_p} At_p \cdot e^{j\omega t} d\omega = \frac{1}{2\pi} \frac{At_p}{jt} e^{j\omega t} \Big|_{-\pi/t_p}^{\pi/t_p} \\
 &= \frac{At_p}{2\pi jt} (e^{j\pi/t_p} - e^{-j\pi/t_p}) = \frac{At_p}{2\pi jt} [2j \sin(\pi t / t_p)] \\
 &= \frac{At_p \sin(\pi t / t_p)}{\pi t} = A \frac{\sin(\pi t / t_p)}{\pi t / t_p} = A \text{Sa}(\pi t / t_p)
 \end{aligned}$$

例2 计算指数信号 $e^{-a|t|}$ ($a>0$)的频谱函数

解：由傅里叶正变换定义式，可得：

$$\begin{aligned}
 X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-a|t|} \cdot e^{-j\omega t} dt = \int_{-\infty}^0 e^{at} \cdot e^{-j\omega t} dt + \int_0^{\infty} e^{-at} \cdot e^{-j\omega t} dt \\
 &= \frac{1}{a - j\omega} e^{(a-j\omega)t} \Big|_{-\infty}^0 + \frac{1}{-(a+j\omega)} e^{-(a+j\omega)t} \Big|_0^{\infty} = \frac{1}{a - j\omega} + \frac{1}{a + j\omega} = \frac{2a}{a^2 + \omega^2}
 \end{aligned}$$

$$\therefore |X(\omega)| = \frac{2a}{a^2 + \omega^2}, \quad \varphi(\omega) = 0$$

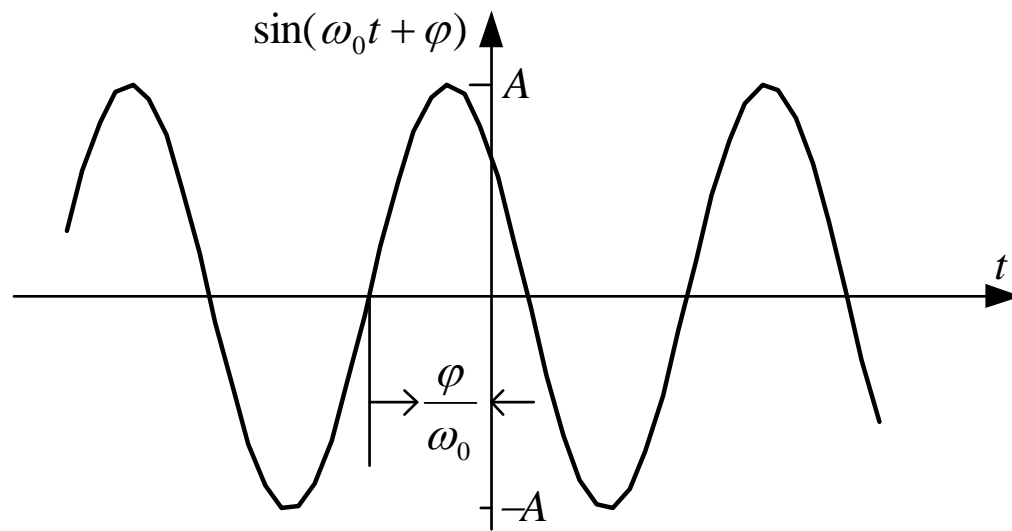
§ 1.3 信号的基本概念

一、典型信号

1. 正弦信号

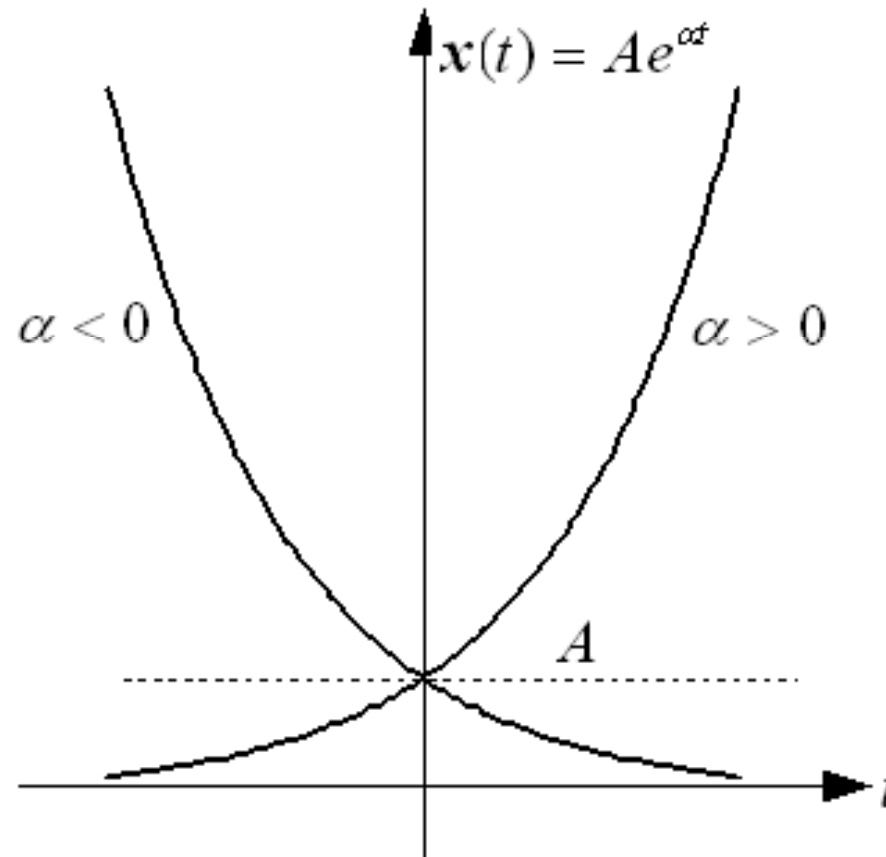
$$x(t) = A \sin(\omega_0 t + \varphi)$$

A: 振幅 **ω_0 : 角频率(弧度/秒)** **φ : 初始相位**



2. 实指数信号

$$x(t) = Ae^{\alpha t}$$



3. 虚指数信号

$$x(t) = e^{j\omega_0 t}$$

虚指数信号是周期信号：

$$x(t + kT) = e^{j\omega_0(t+kT)} = e^{j\omega_0 t} e^{j\omega_0 kT} = e^{j\omega_0 t} e^{j\frac{2\pi}{T}kT} = e^{j\omega_0 t} e^{j2k\pi} = e^{j\omega_0 t} = x(t)$$

其中周期为：

$$T = \frac{2\pi}{\omega_0}$$

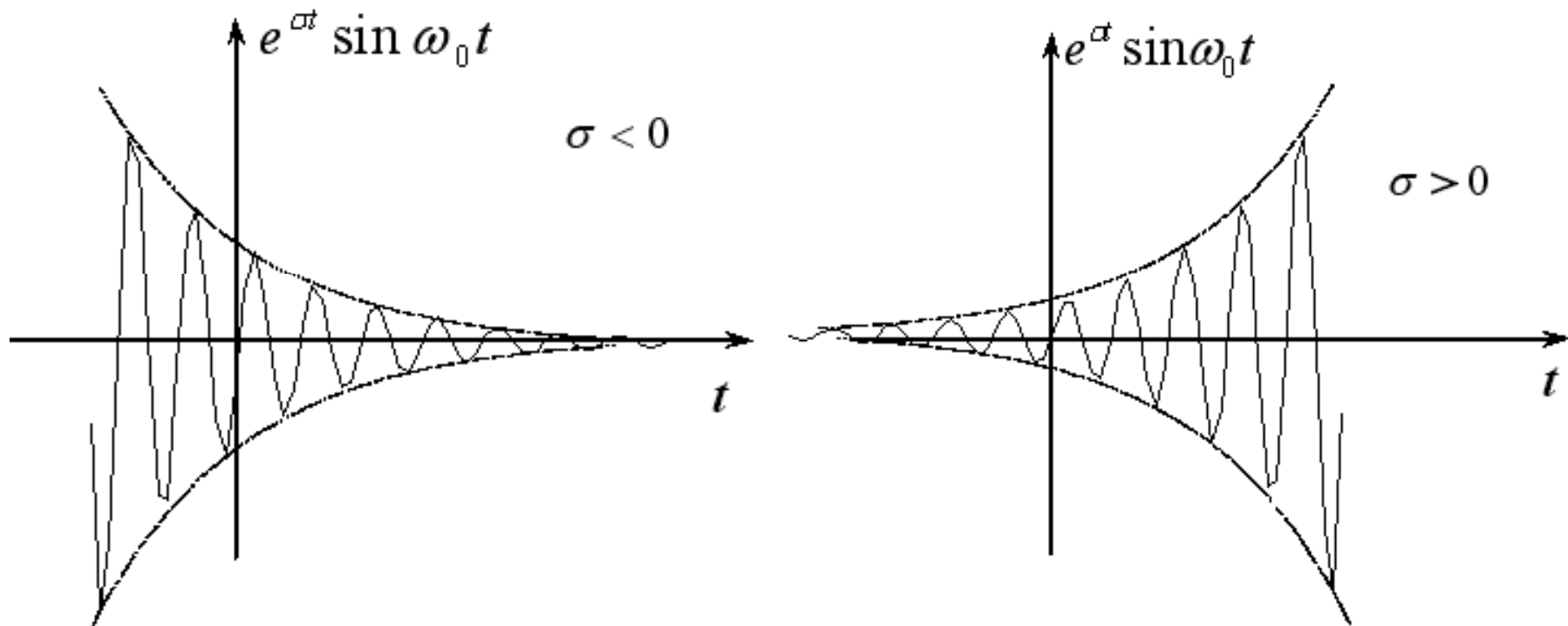
Euler公式：

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t}) \quad \sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

4. 复指数信号

$$x(t) = Ae^{st}, \quad s = \sigma + j\omega_0$$

$$x(t) = Ae^{\sigma t} e^{j\omega_0 t} = Ae^{\sigma t} \cos \omega_0 t + jAe^{\sigma t} \sin \omega_0 t$$



5. 抽样信号

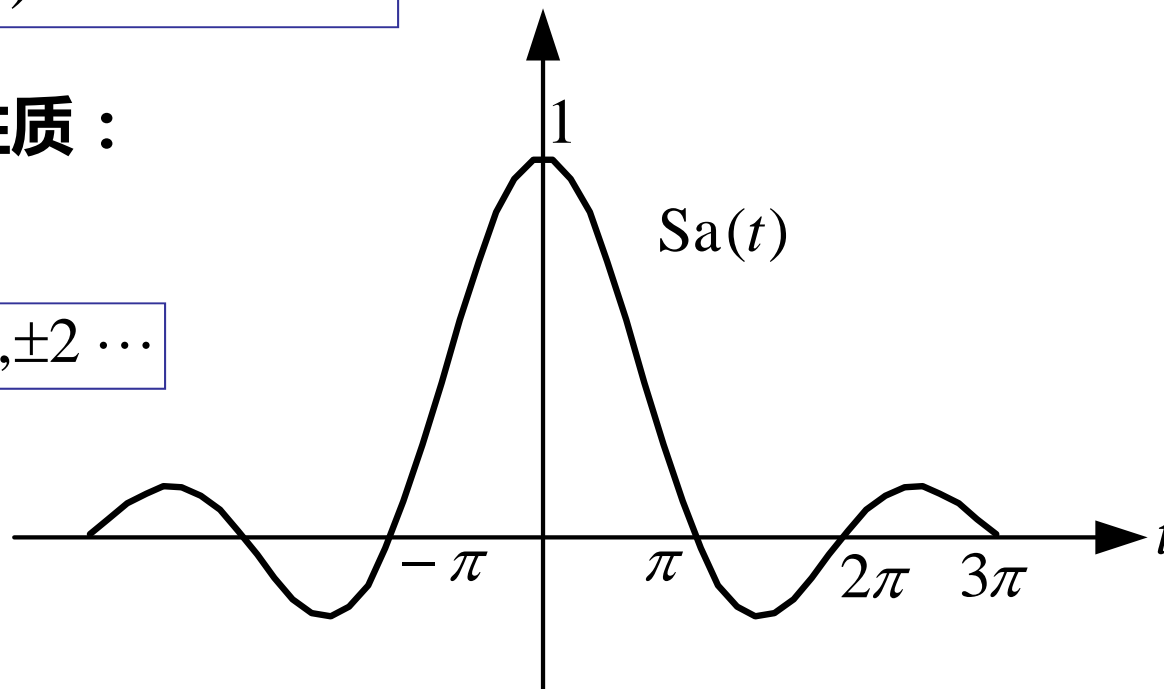
$$Sa(t) = \sin t / t$$

抽样函数具有以下性质：

$$Sa(0) = 1$$

$$Sa(k\pi) = 0, k = \pm 1, \pm 2 \dots$$

$$\int_{-\infty}^{\infty} Sa(t) dt = \pi$$



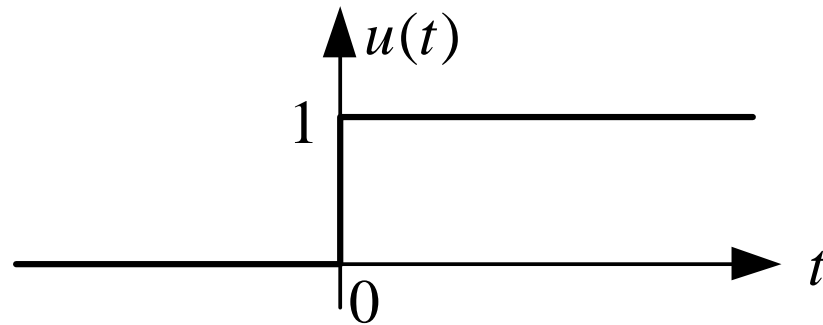
- 与 $Sa(t)$ 函数类似的是 $\text{sinc}(t)$ 函数，其定义为：

$$\text{sinc}(t) = \sin(\pi t) / (\pi t)$$

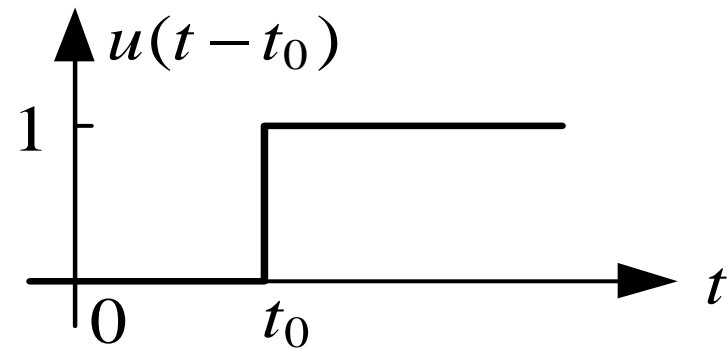
6. 阶跃信号

- 定义:

$$x(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} = u(t)$$

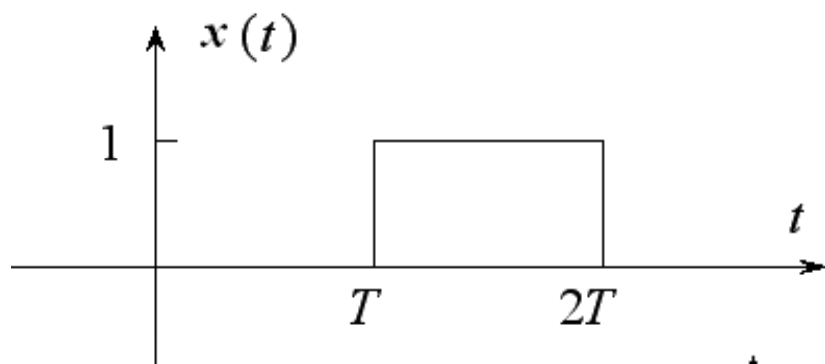


$$u(t - t_0) = \begin{cases} 1 & t > t_0 \\ 0 & t < t_0 \end{cases}$$

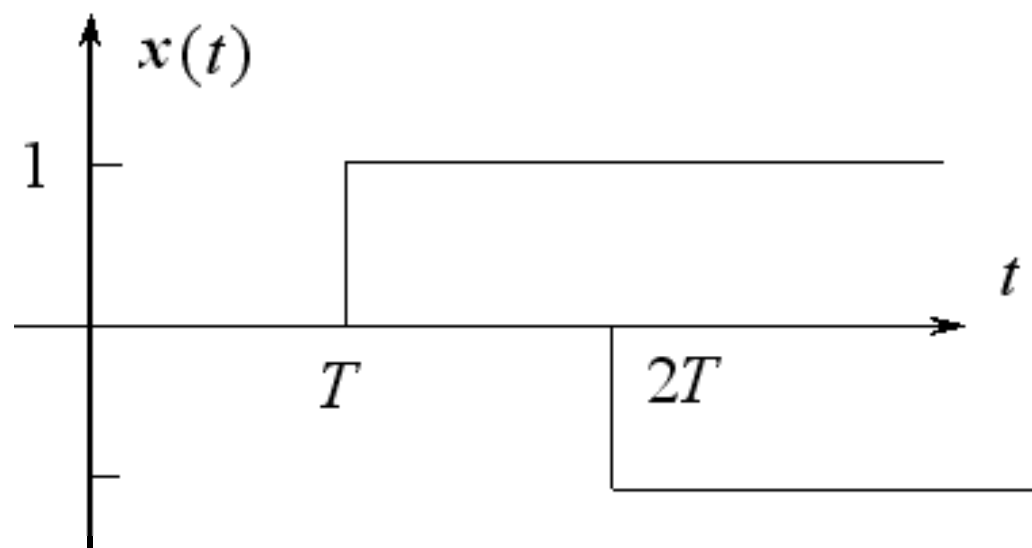


- 作用:

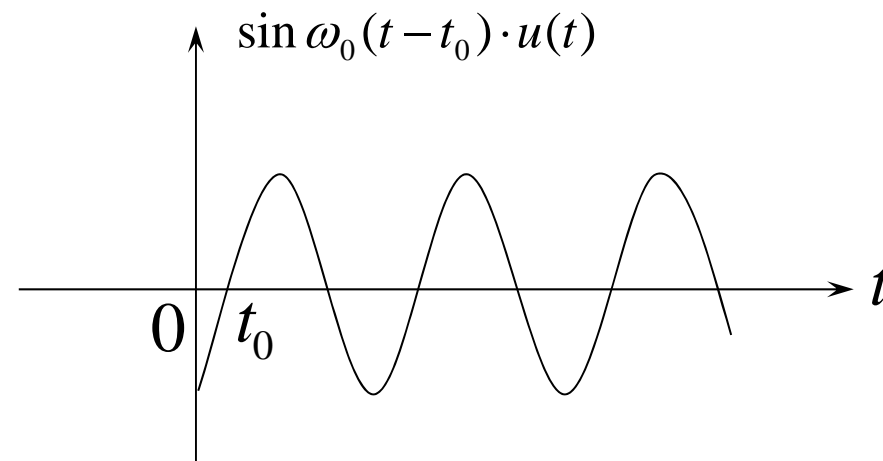
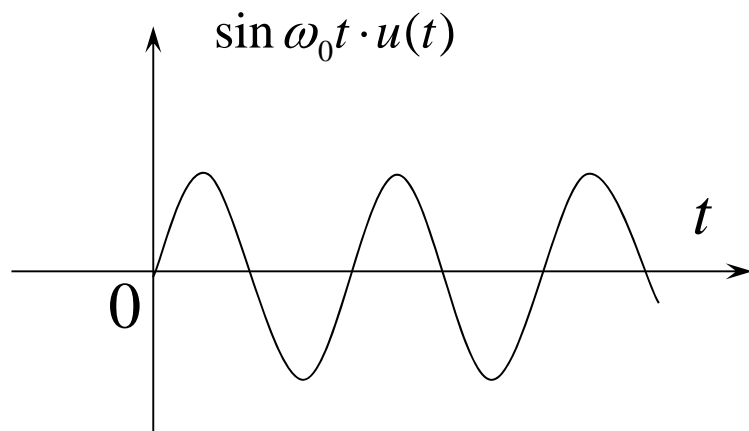
可表示任意的矩形脉冲信号



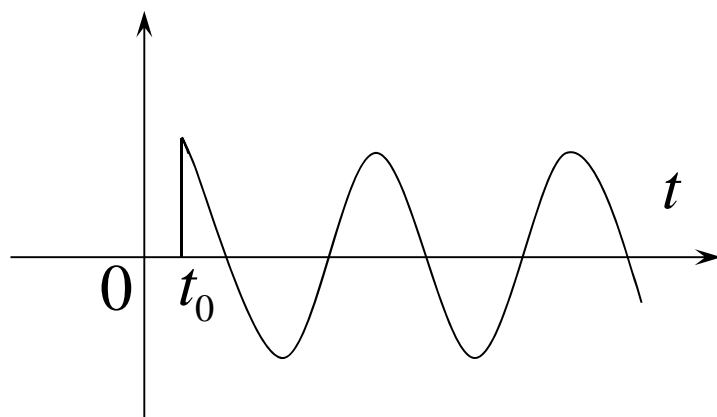
$$x(t) = u(t-T) - u(t-2T)$$



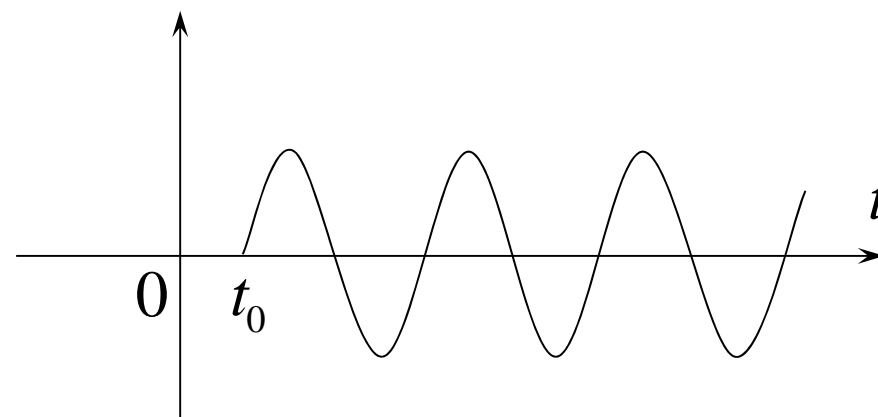
可表示信号的时间范围



$\sin \omega_0 t \cdot u(t - t_0)$



$\sin \omega_0(t - t_0) \cdot u(t - t_0)$

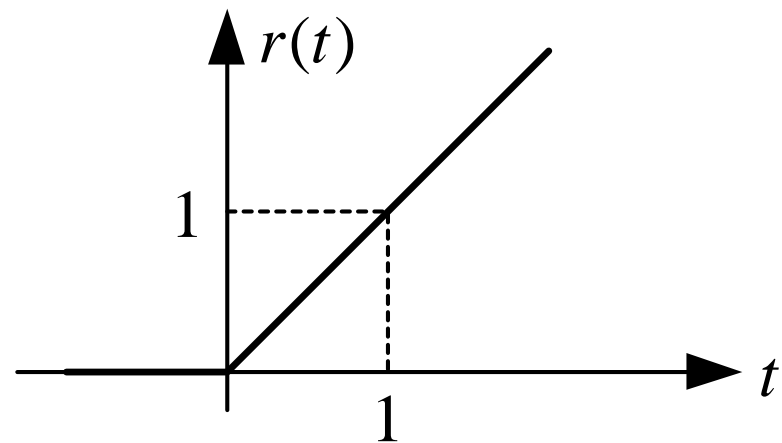


7. 斜坡信号

- 定义：

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

或 $r(t) = t \cdot u(t)$



- 与阶跃信号之间的关系：

$$r(t) = \int_{-\infty}^t u(\tau) \cdot d\tau$$

$$\frac{dr(t)}{dt} = u(t)$$

8. 冲激信号

1) 冲激信号的引出

阶跃信号加在电容两端，流过电容的电流

$i(t)=Cdu(t)/dt$ 可用冲激信号表示

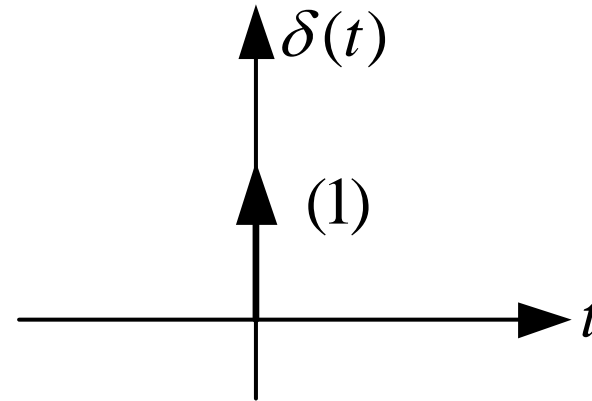
2) 冲激信号的定义

狄拉克定义式：

$$\delta(t)=0, \quad t \neq 0$$

$$\int_{-\infty}^{+\infty} \delta(t) dt = 1$$

3) 冲激信号的图形表示



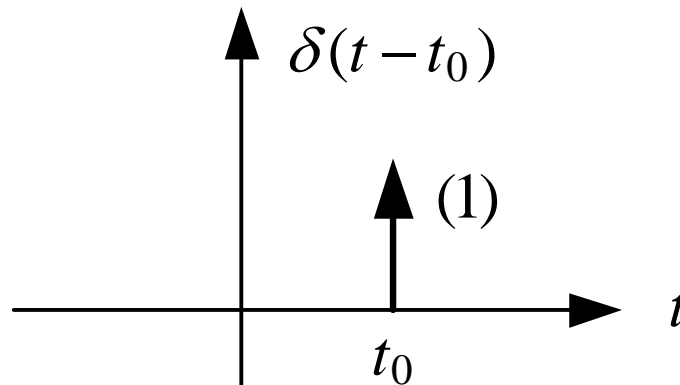
- 说明：

- ◆ 冲激信号可延时至任意时刻 t_0 ，以符号 $\delta(t-t_0)$ 表示

定义式为： $\delta(t-t_0) = 0 \quad t \neq t_0$

$$\int_{-\infty}^{\infty} \delta(t-t_0) dt = \int_{t_0-\Delta}^{t_0+\Delta} \delta(t-t_0) dt = 1$$

波形表示



- ◆ **冲激信号具有强度，强度就是冲激信号对时间的定积分值**

图中用括号注明，以区分信号的幅值

- ◆ **物理意义：作用时间极短，作用值很大的物理现象的数学模型**

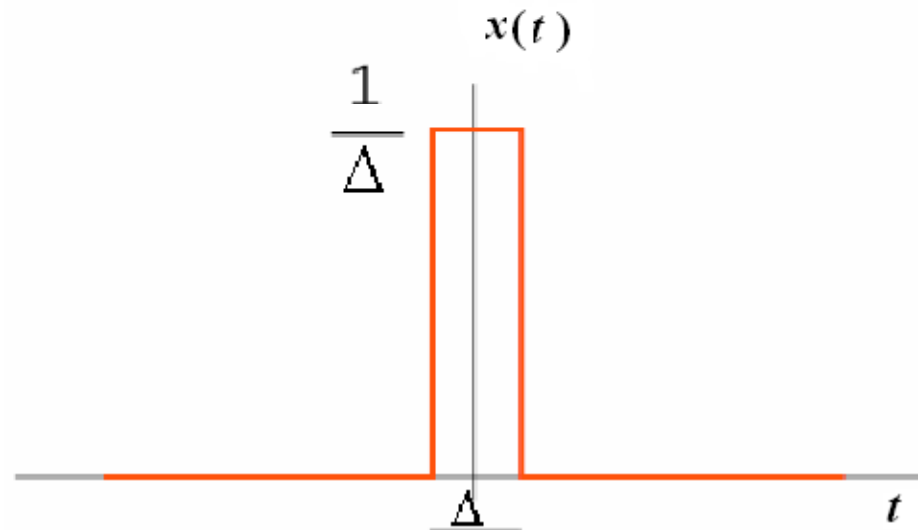
- ◆ **冲激信号的作用：**

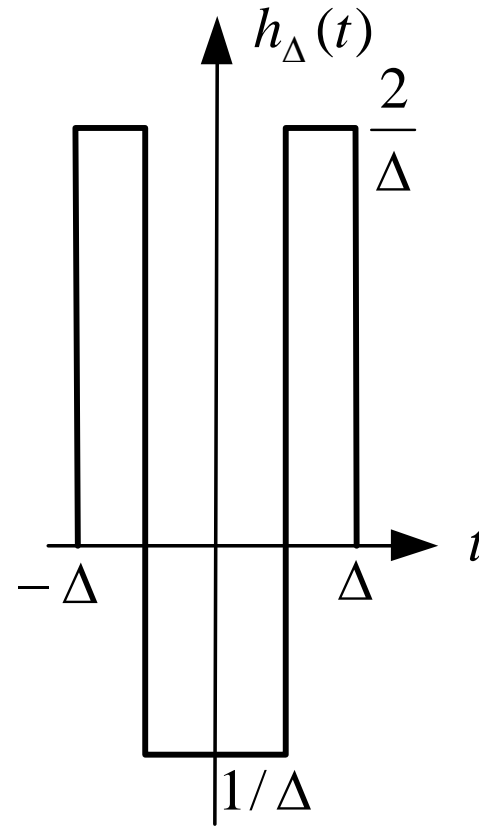
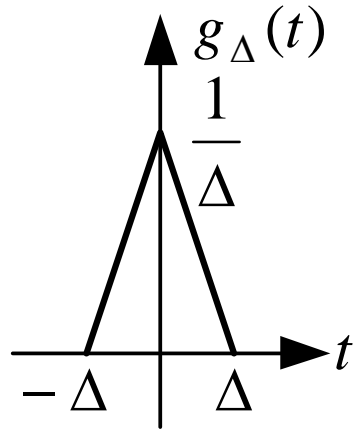
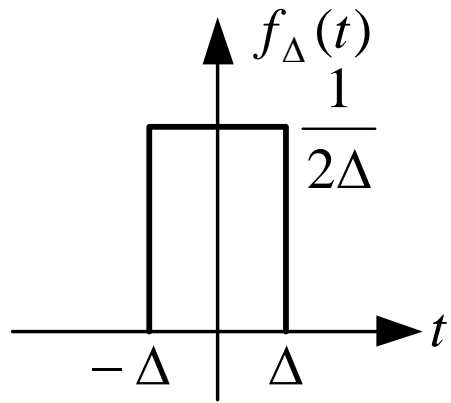
表示其他任意信号

表示信号间断点的导数

4) 冲激信号的极限模型

$$\delta(t) = \lim_{\Delta \rightarrow 0} x(t)$$

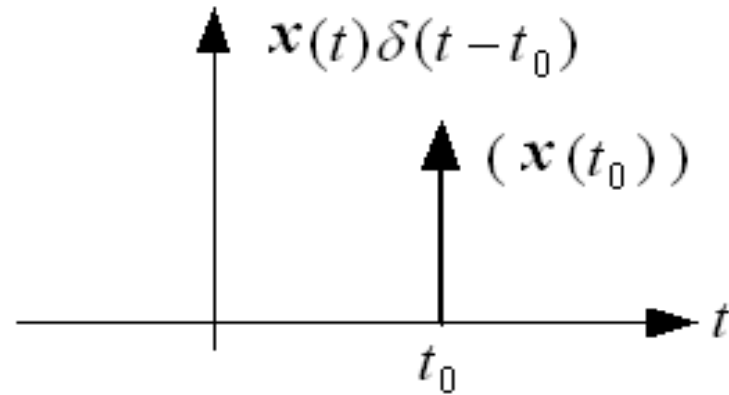
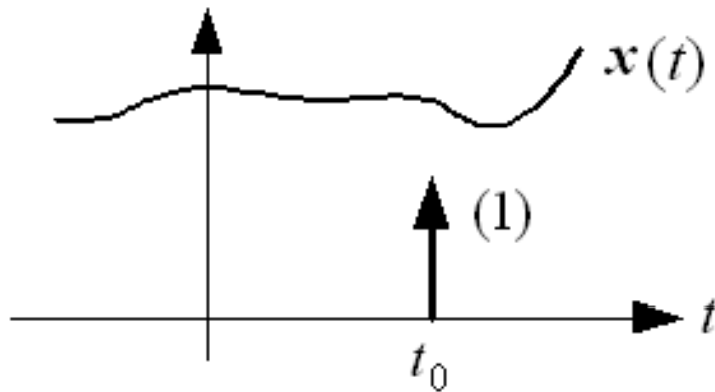




$$\delta(t) = \lim_{\Delta \rightarrow 0} f_{\Delta}(t) = \lim_{\Delta \rightarrow 0} g_{\Delta}(t) = \lim_{\Delta \rightarrow 0} h_{\Delta}(t)$$

5) 冲激信号的性质

- 筛选特性 $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$



- 取样特性 $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$

证:

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = \int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt = x(t_0) \int_{-\infty}^{\infty} \delta(t-t_0)dt = x(t_0)$$

• 展缩特性

$$\delta(at) = \frac{1}{|a|} \delta(t)$$

证：任取一信号 $g(t)$ ，对于 $\int_{-\infty}^{\infty} g(t)\delta(at)dt$

令 $x=at$ ，若 $a>0$ ，则有：

$$\int_{-\infty}^{\infty} g(t)\delta(at)dt = \int_{-\infty}^{\infty} g\left(\frac{x}{a}\right)\delta(x)\frac{dx}{a} = \frac{1}{a} \int_{-\infty}^{\infty} g\left(\frac{x}{a}\right)\delta(x)dx$$

若 $a<0$ ，则有：

$$\int_{-\infty}^{\infty} g(t)\delta(at)dt = \int_{\infty}^{-\infty} g\left(\frac{x}{a}\right)\delta(x)\frac{dx}{a} = \int_{-\infty}^{\infty} g\left(\frac{x}{a}\right)\delta(x)\frac{dx}{-a} = \frac{1}{-a} \int_{-\infty}^{\infty} g\left(\frac{x}{a}\right)\delta(x)dx$$

$$\therefore \int_{-\infty}^{\infty} g(x)\delta(ax)dx = \frac{1}{|a|} \int_{-\infty}^{\infty} g\left(\frac{x}{a}\right)\delta(x)dx = \frac{g(0)}{|a|}$$

$$\int_{-\infty}^{\infty} g(t)\frac{\delta(t)}{|a|}dt = \frac{1}{|a|} \int_{-\infty}^{\infty} g(t)\delta(t)dt = \frac{g(0)}{|a|} \quad \Rightarrow \quad \delta(at) = \frac{1}{|a|} \delta(t)$$

$$\delta(at) = \frac{1}{|a|} \delta(t)$$

$$a = -1 \Rightarrow \delta(t) = \delta(-t)$$

推论：冲激信号是偶函数

• **与阶跃信号的关系**

$$\int_{-\infty}^t \delta(\tau) d\tau = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} = u(t) \qquad \frac{du(t)}{dt} = \delta(t)$$

6) 其他定义形式：

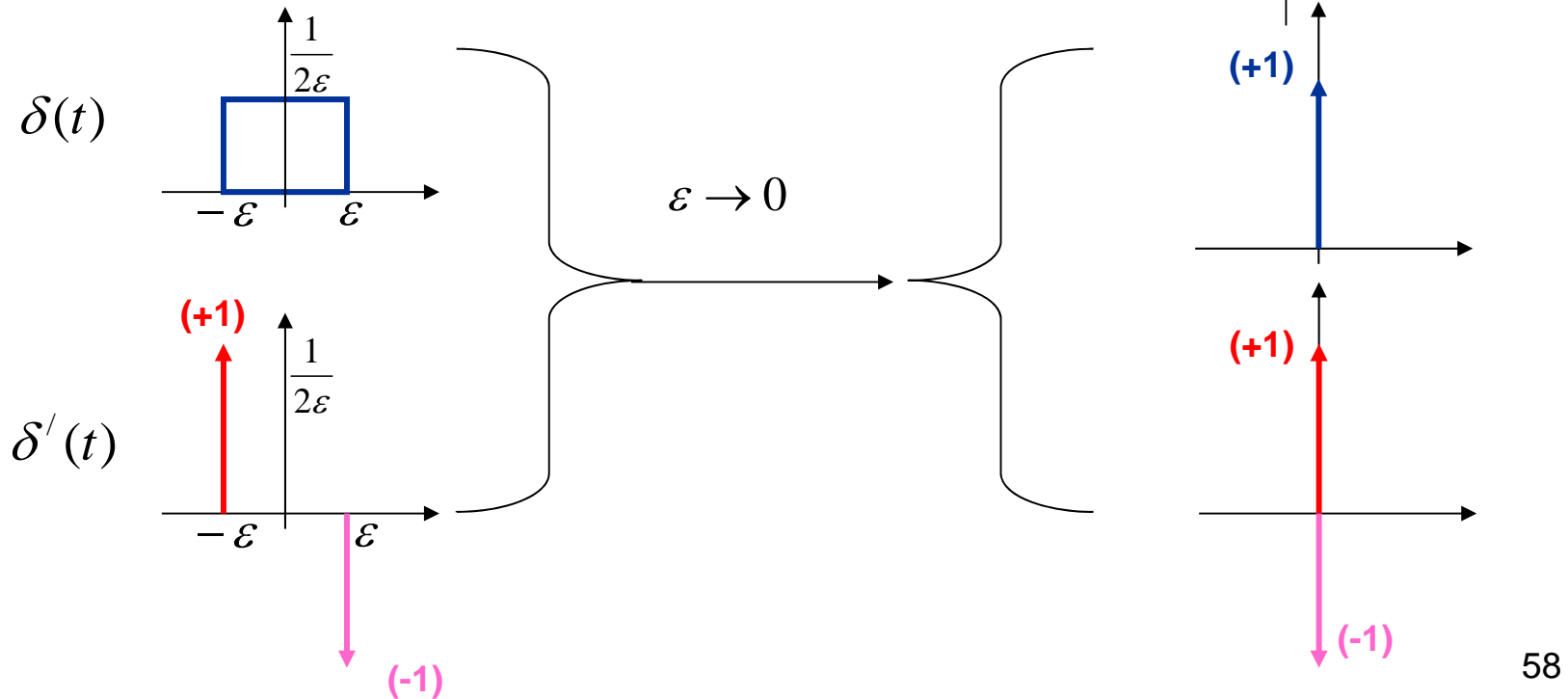
$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm j\omega t} d\omega, \quad \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos \omega t d\omega$$

9. 冲激偶信号

• 定义： $\frac{d\delta(t)}{dt} = \delta'(t)$

$\lim_{\varepsilon \rightarrow 0} x(t) = \frac{d\delta(t)}{dt} = \delta'(t)$

冲激偶信号图形表示



- 性质：

$$\int_{-\infty}^{\infty} \delta'(t) dt = 0$$

$$\int_{-\infty}^t \delta'(\tau) d\tau = \delta(t)$$

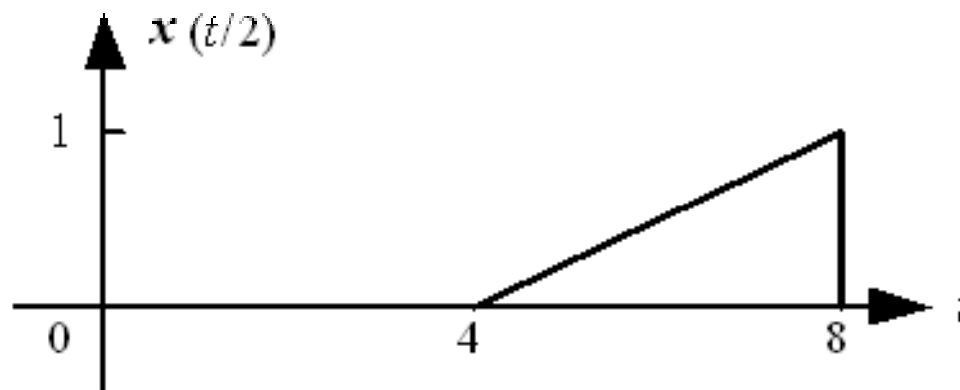
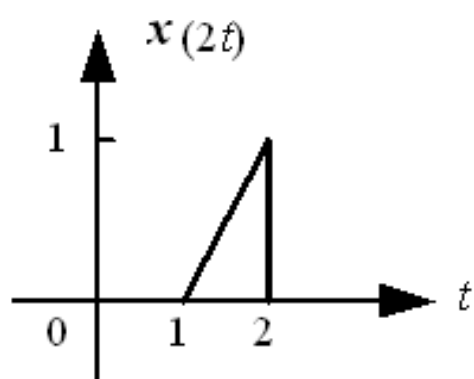
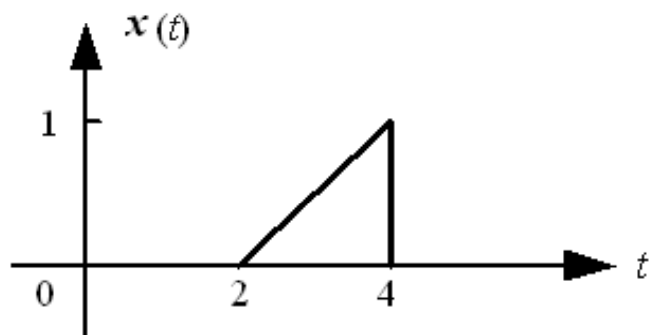
$$\delta(t) = -t\delta'(t) \quad \Rightarrow \quad \delta'(t) = -\delta'(-t)$$

$$x(t)\delta'(t) = x(0)\delta'(t) - x'(0)\delta(t)$$

$$\int_{-\infty}^{\infty} x(t)\delta'(t) dt = -x'(0)$$

二、典型基本运算

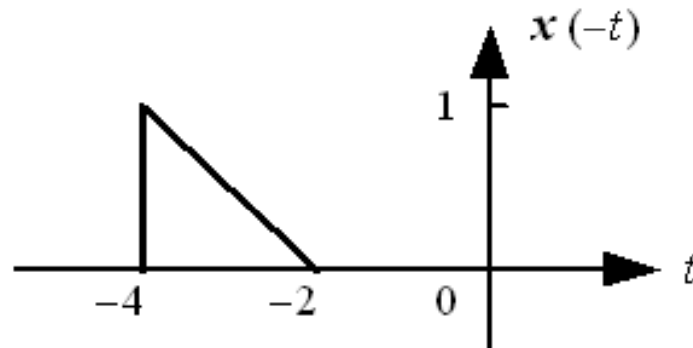
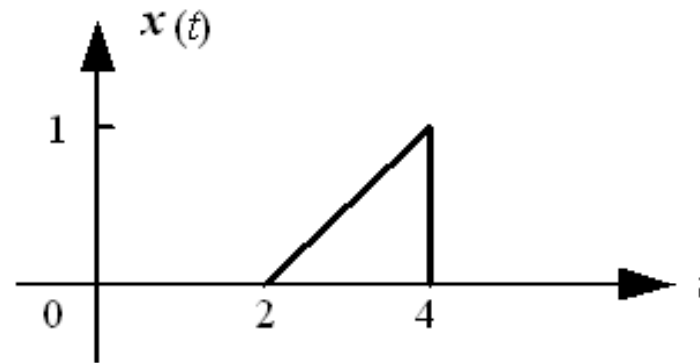
1. 尺度变换 $x(t) \rightarrow x(at)$ $a > 0$



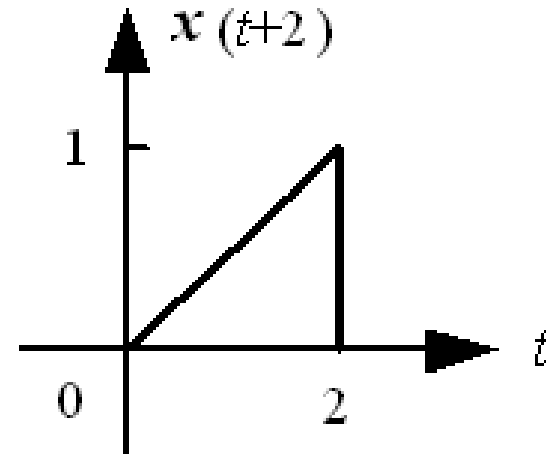
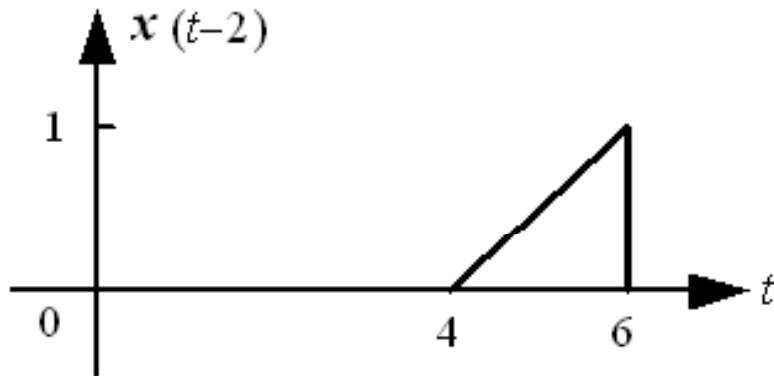
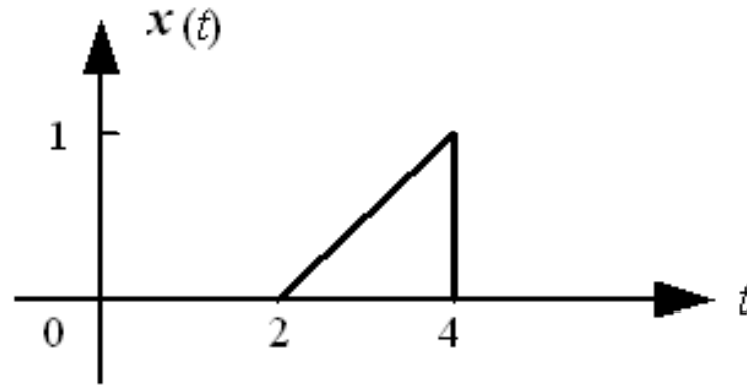
若 $0 < a < 1$, 则 $x(at)$ 是 $x(t)$ 的扩展
若 $a > 1$, 则 $x(at)$ 是 $x(t)$ 的压缩

2. 信号的翻转 $x(t) \rightarrow x(-t)$

将 $x(t)$ 以纵轴为中心作翻转



3. 时移 (平移) $x(t) \rightarrow x(t-t_0)$ $t_0 > 0$



$x(t-t_0)$ 表示信号右移
 $x(t+t_0)$ 表示信号左移