A Three-dimensional Viscoelastic Constitutive Model of Finite Deformation

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Abstract: Hereditary integral formulations and differential formulations identified with spring and dashpot constructions can express the viscoelastic behaviors of polymeric materials at small deformations. In this paper, the models of small deformation serve as a starting point for the development of the viscoelastic constitutive models of finite deformation. A process of finite deformation is decomposed into a series of sub-processes of small deformation. The rotations of stress in sub-processes are determined by elastic constitutive equation. The changes in the principal stress are calculated using the spring and dashpot constructions. Then, a viscoelastic constitutive model, which satisfies the principle of objectivity, is presented. Such form of constitutive model in principle can be suitable for a range of strain-rates, e.g. either for quasi-static loading or for impact loading, with different material parameters in different strain-rates. As an application example, the simple shear deformation is computed to show that the proposed model can adequately well describe the viscoelastic behavior for polymers at finite deformations.

Key words: viscoelastic; finite deformation; the principle of objectivity

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The constitutive theories of viscoelastic material have attracted substantial attention during the past half century. The viscoelastic constitutive models at small deformations can be described with hereditary integral formulation or differential formulations[1-2]. The rate-form rheological models are commonly used. However, much less progress has been made for the constitutive models at finite deformations. Several authors have developed the viscoelastic constitutive model of finite deformation on the basis of the second law of thermodynamics (e.g. [3]). It is known that the constitutive models for finite deformations must satisfy the principle of objectivity[4]. By replacing the strain rate with the stretching (the deformation rate) and the material time derivative of Cauchy's stress with an objective derivative of Cauchy's stress, many authors developed the rate-form rheological models for finite deformations (e.g. [5]). However, there are infinite kinds of objective rates of stress. We don't know which kind of objective rate is suitable for the constitutive equations. Many authors chose an objective rate for the rate-form constitutive equation without any suitable reason. There are many disputes on how to choose the objective rate. The definition of elastic and inelastic strain rates is another issue in developing the constitutive equations of finite deformation. Several authors have defined elastic and inelastic deformation rates by using the decomposition of the deformation gradient as a product of elastic and inelastic parts[6]. However, the multiplicative decomposition of deformation gradient is not consistent with

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the additive decomposition of strain rate into the elastic and inelastic parts. Lion[7] pushed forward the Green's strain to intermediate configuration and separated the strain into elastic and inelastic parts to develop the finite deformation constitutive relation consistent with the second law of thermodynamics. Huber and Tsakmakis[8] separated the transformed Almansi strain into elastic and inelastic parts and defined the associated stress to extend constitutive laws of viscoelastic materials for small deformations to these for finite deformations. Drozdov et al[9] made the objective inelastic stretching by means of a rotation transformation and proposed that the inelastic stretching is proportional to the total stretching. Shen[10] presented a decomposition of deformation into elastic and inelastic parts and proposed a constitutive model for elastic-plastic materials at finite deformations.

This paper will develop a constitutive model for viscoelastic materials at finite deformations through a new approach. Also it will calculate the stress of simple shear to compare the constitutive equations proposed here with the existing constitutive equations.

1 The constitutive models for finite deformations

1.1 The Kelvin model for finite deformations

The Kelvin model for small deformation is expressed by

$$\boldsymbol{\sigma} = \lambda \boldsymbol{I}\,\mathrm{tr}(\varepsilon) + 2\mu\varepsilon + \eta(\dot{\varepsilon} - (1/3)\lambda\,\mathrm{tr}(\dot{\varepsilon})\boldsymbol{I}) + \eta'\,(1/3)\lambda\,\mathrm{tr}(\dot{\varepsilon})\boldsymbol{I}, \tag{1}$$

where super dot denotes the material time derivative, λ and μ are Lame's constants, η and η' are the deviatoric and volumetric viscosities respectively, may be taken as a function of stress or strain rate. For simplification, let η' equal η. In the case of finite deformation, the model may be expressed by

$$\boldsymbol{\sigma} = \lambda \boldsymbol{I}\,\mathrm{tr}(\ln \boldsymbol{V}) + 2\mu \ln \boldsymbol{V} + \eta \boldsymbol{D}, \tag{2}$$

where $\boldsymbol{D}$ is the stretching (the deformation rate), $\boldsymbol{V}$ is the left stretch tensor. In extending the limits of equation (1), the strain is defined by the logarithmic of the left stretch tensor, the strain rate is replaced by the stretching. equation (2) satisfies the principle of objectivity.

1.2 The Maxwell model for finite deformations

The Maxwell model consists of a linear spring and a dashpot in series. For small deformation, the linear spring and the dashpot are expressed by

$$\dot{\boldsymbol{\varepsilon}}_e = ((1+v)/E)\dot{\boldsymbol{\sigma}} - (v/E)\mathrm{tr}(\dot{\boldsymbol{\sigma}})\boldsymbol{I}, \tag{3a}$$

$$\dot{\boldsymbol{\varepsilon}}_v = \boldsymbol{\sigma}/\eta, \tag{3b}$$

respectively, where the sub indeces e and v imply elastic and viscoelastic respectively, E and v are the Young's modulus and the Poisson's ratio. Assume that the strain rate is equal to the sum of elastic and viscoelastic strain rates, i.e.,

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}_e + \dot{\boldsymbol{\varepsilon}}_v, \tag{4}$$

thus, we have

$$\dot{\boldsymbol{\varepsilon}} = ((1+v)/E)\dot{\boldsymbol{\sigma}} - (v/E)\mathrm{tr}(\dot{\boldsymbol{\sigma}})\boldsymbol{I} + \boldsymbol{\sigma}/\eta. \tag{5}$$

For finite deformations, the strain rate may be replaced by the deformation rate, the deformation rate is expressed as the sum of elastic and viscoelastic parts

$$\boldsymbol{D} = \boldsymbol{D}_e + \boldsymbol{D}_v\,. \tag{6}$$

The material time derivative is replaced by the objective derivative. Thus, we obtain

$$\boldsymbol{D} = ((1+v)/E)\overset{o}{\boldsymbol{\sigma}} - (v/E)\mathrm{tr}(\overset{o}{\boldsymbol{\sigma}})\boldsymbol{I} + \boldsymbol{\sigma}/\eta, \tag{7}$$

where $\overset{o}{\boldsymbol{\sigma}}$ is the objective derivative of the Cauchy's stress, which is commonly expressed in the form

$$\overset{o}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega}_*\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{\Omega}_*, \tag{8}$$

where $\boldsymbol{\Omega}_*$ is a rotational rate (spin). However, it is difficult to choose the spin in Formulate (8). The rotational rate is commonly obtained from the total deformation gradient and is believed to be a geometric quantity. The material rotational rate, the relative rotational rate and the Euler frame rotational rate are obtained from the total deformation gradient. We think that the rotational rate applicable to the formulate (8) is dependent on not only the total deformation but also the viscoelastic part of the deformation. In addition, the additive decomposition of deformation rate (6) is not consistent with the multiplicative decomposition of deformation gradient. Shen[10] presented an approach through which the constitutive relations are extended to include the case of finite deformation. In this study, we develop the Maxwell model for finite using this approach. Consider a deformation from time t_0, t_1 up to t_n : $\boldsymbol{I} \to \boldsymbol{F}_1 \to \boldsymbol{F}_2 \to \cdots \boldsymbol{F}_n$. The deformation gradient at time t_1 is expressed in the forms

$$\boldsymbol{F}_1 = \boldsymbol{R}_{E1}\boldsymbol{V}_{\lambda 1}\boldsymbol{R}_{L1}^{\mathrm{T}} = \boldsymbol{R}_{E1}\boldsymbol{V}_{\lambda 1e}\boldsymbol{V}_{\lambda 1v}\boldsymbol{R}_{L1}^{\mathrm{T}}. \quad (9)$$

In Cartesien coordinate system, $\boldsymbol{R}_E$ and $\boldsymbol{R}_L$ are orthotropic matrices, $\boldsymbol{V}_\lambda$ is diagram matrix. Sub indecs e and v implies elastic and viscoelastic. The process from t_0 to t_1 is a small deformation. When the configuration at time t_0 undergoes the rotation $\boldsymbol{R}_{L1}^{\mathrm{T}}$, the deformation gradient at time t_1 becomes $\boldsymbol{R}_{E1}\boldsymbol{V}_{\lambda 1}$. Hence, for isotropic materials, $\boldsymbol{R}_{L1}$ does not affect the stress at the time t_1, and the principal direction of this stress may be expressed by $\boldsymbol{R}_{E1}$. The stress at the time t_1 is expressed in the form

$$\boldsymbol{\sigma}_{t1} = \boldsymbol{R}_{E1}\boldsymbol{\sigma}_{\lambda 1}\boldsymbol{R}_{E1}^{\mathrm{T}}. \quad (10)$$

From the decomposition (9), we have the decomposition of logarithmic strain rate

$$\overset{\bullet}{\overline{\ln \boldsymbol{V}_{\lambda 1}}} = \overset{\bullet}{\overline{\ln \boldsymbol{V}_{\lambda 1e}}} + \overset{\bullet}{\overline{\ln \boldsymbol{V}_{\lambda 1v}}}. \quad (11)$$

The principal values of the stress at t_1 can be obtain as follows

$$\overset{\bullet}{\overline{\ln \boldsymbol{V}_{\lambda 1}}} = ((1+v)/E)\dot{\boldsymbol{\sigma}}_{\lambda 1} - (v/E)\mathrm{tr}(\dot{\boldsymbol{\sigma}}_{\lambda 1})\boldsymbol{I} + \boldsymbol{\sigma}_{\lambda 1}/\eta, \quad (12)$$

where, $\overset{\bullet}{\overline{\ln \boldsymbol{V}_{\lambda 1}}} = (\ln \boldsymbol{V}_{\lambda 1} - \ln \boldsymbol{I})/(t_1 - 0)$, $\dot{\boldsymbol{\sigma}}_{\lambda 1} = (\boldsymbol{\sigma}_{\lambda 1} - \boldsymbol{O})/(t_1 - 0)$. The elastic deformation $\boldsymbol{V}_{\lambda 1e}$ may be obtained as follows

$$\ln \boldsymbol{V}_{\lambda 1e} = ((1+v)/E)\boldsymbol{\sigma}_{\lambda 1} - (v/E)\mathrm{tr}(\boldsymbol{\sigma}_{\lambda 1})\boldsymbol{I}. \quad (13)$$

Taking the free-stress intermediate configuration as reference configuration, the deformation gradients at t_1 and t_2 are respectively expressed by

$$\boldsymbol{F}_{1b} = \boldsymbol{R}_{E1}\boldsymbol{V}_{\lambda 1e}, \quad (14a)$$

and

$$\boldsymbol{F}_{2b} = \boldsymbol{F}_2(\boldsymbol{V}_{\lambda 1v}\boldsymbol{R}_{L1}^{\mathrm{T}})^{-1} = \boldsymbol{R}_{E2b}\boldsymbol{V}_{\lambda 2b}\boldsymbol{R}_{L2b}^{\mathrm{T}}. \quad (14b)$$

The deformation rate is decomposed in the following form

$$\boldsymbol{D} = \boldsymbol{R}_E(\boldsymbol{D}_a + \boldsymbol{D}_b)\boldsymbol{R}_E^{\mathrm{T}}, \quad (15a)$$

where

$$\boldsymbol{D}_a = \dot{\boldsymbol{V}}_\lambda \boldsymbol{V}_\lambda^{-1}, \quad (15b)$$

$$\boldsymbol{D}_b = (1/2)(\boldsymbol{V}_\lambda^{-1}\boldsymbol{R}_L^{\mathrm{T}}\dot{\boldsymbol{R}}_L\boldsymbol{V}_\lambda + \boldsymbol{V}_\lambda\dot{\boldsymbol{R}}_L^{\mathrm{T}}\boldsymbol{R}_L\boldsymbol{V}_\lambda^{-1}), \quad (15c)$$

where $\boldsymbol{D}_a$ is a diagonal matrix, $\boldsymbol{D}_b$ is a symmetric matrix whose diagonal components are all zero. The stress $\boldsymbol{\sigma} = \boldsymbol{R}_E\boldsymbol{\sigma}_\lambda\boldsymbol{R}_E^{\mathrm{T}}$ yields the viscous-strain rate $\boldsymbol{R}_E\boldsymbol{D}_a\boldsymbol{R}_E^{\mathrm{T}}$. Hence, if the elastic deformation vanishes, the strain rate $\boldsymbol{R}_E\boldsymbol{D}_b\boldsymbol{R}_E^{\mathrm{T}}$ consequent upon the increase of $\boldsymbol{R}_L$ is independent of the stress $\boldsymbol{\sigma} = \boldsymbol{R}_E\boldsymbol{\sigma}_\lambda\boldsymbol{R}_E^{\mathrm{T}}$. If the deformation is elastic, $\boldsymbol{R}_L$ does not affect the stress either. Hence, we assume that in the process from t_1 to t_2, $\boldsymbol{R}_{L2b}$ does not affect the stress at time t_2. Thus, the principal direction of the stress is $\boldsymbol{R}_{E2b}$ and the stress may be expressed by

$$\boldsymbol{\sigma}_{t2} = \boldsymbol{R}_{E2b}\boldsymbol{\sigma}_{\lambda 2}\boldsymbol{R}_{E2b}^{\mathrm{T}}. \quad (16)$$

The principal stresses may be obtained as follows

$$\overset{\bullet}{\overline{\ln \boldsymbol{V}_{\lambda 2b}}} = ((1+v)/E\dot{\boldsymbol{\sigma}}_{\lambda 2}) - (v/E)\mathrm{tr}(\dot{\boldsymbol{\sigma}}_{\lambda 2})\boldsymbol{I} + \boldsymbol{\sigma}_{\lambda 2}/\eta, \quad (17)$$

where

$$\overset{\bullet}{\overline{\ln \boldsymbol{V}_{\lambda 2b}}} = (\ln \boldsymbol{V}_{\lambda 2b} - \ln \boldsymbol{V}_{\lambda 1e})/(t_2 - t_1),$$
$$\dot{\boldsymbol{\sigma}}_{\lambda 2} = (\boldsymbol{\sigma}_{\lambda 2} - \boldsymbol{\sigma}_{\lambda 1})/(t_2 - t_1).$$

In a similar way, we can obtain the stresses from the time t_3 up to the time t_n. In the case of pure elastic deformation, the stress-free intermediate configurations at current times are the initial configuration. We may obtain the constitutive equation of elastic deformation

$$\boldsymbol{\sigma}_{ti} = \boldsymbol{R}_{Ei}\boldsymbol{\sigma}_{\lambda i}\boldsymbol{R}_{Ei}^{\mathrm{T}}, \ (i = 1, 2, 3, \cdots n), \quad (18a)$$

$$\overset{\bullet}{\overline{\ln \boldsymbol{V}_{\lambda i}}} = ((1+\mathrm{v})/E)\dot{\boldsymbol{\sigma}}_{\lambda i} - (\mathrm{v}/E)\mathrm{tr}(\dot{\boldsymbol{\sigma}}_{\lambda i})\boldsymbol{I}. \quad (18b)$$

The equation (18) may be rewritten as

$$\boldsymbol{R}_E \overset{\bullet}{\overline{\ln \boldsymbol{V}_\lambda}} \boldsymbol{R}_E^{\mathrm{T}} = ((1+v)/E)\overset{o}{\boldsymbol{\sigma}} - (v/E)\mathrm{tr}(\overset{o}{\boldsymbol{\sigma}})\boldsymbol{I}, \quad (19)$$

where the objective rate $\overset{\mathrm{o}}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - (\dot{\boldsymbol{R}}_E\boldsymbol{R}_E^{\mathrm{T}})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\dot{\boldsymbol{R}}_E\boldsymbol{R}_E^{\mathrm{T}})$ and is Euler frame corotational rate. The constitutive equation (19) may be expressed by

$$\boldsymbol{D} = ((1+v)/E)\overset{o}{\boldsymbol{\sigma}} - (v/E)\mathrm{tr}(\overset{o}{\boldsymbol{\sigma}})\boldsymbol{I}, \quad (20)$$

where the objective rate $\overset{o}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega}^{\log}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{\Omega}^{\log}$ and is the logarithmic corotational rate[11]. If the elastic deformation vanishes, the stress-free intermediate configurations are the current configurations, the constitutive relation becomes

$$\boldsymbol{D} = \boldsymbol{\sigma}/\eta. \quad (21)$$

The constitutive equations (19 and 21) are consistent with the existing constitutive equations.

1.3 The Burgers constitutive model for finite deformations

Burgers model can describe the main characters of some viscoelastic materials. Burgers model implies that the deformation can be decomposed into three parts. For small deformations, the constitutive equation is expressed by

$$\boldsymbol{e}=\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_3,\ \ \theta=\theta_1+\theta_2+\theta_3,$$
$$\boldsymbol{s}=2\mu_1\boldsymbol{e}_1,\ \ \boldsymbol{s}=2\mu_2\boldsymbol{e}_2+\eta_2\dot{\boldsymbol{e}}_2,\ \ \ \boldsymbol{s}=\eta_3\dot{\boldsymbol{e}}_3,$$
$$p=-K_1\theta_1,\ p=-K_2\theta_2-(1/3)\eta_2\dot{\theta}_2,$$
$$p=-(1/3)\eta_3\dot{\theta}_3, \tag{22a-h}$$

where $\boldsymbol{s}$ is the deviatoric stress, $\boldsymbol{e}$ the deviatoric strain, p Hydrostatic stress, θ the volumetric strain and, K is bulk modulus. We easily obtain the constitutive equation of p and θ. From equations (22a) – (22e), we obtain the rate-form constitutive equation of deviatoric stress and deviatoric strain

$$p_0\boldsymbol{s}+p_1\dot{\boldsymbol{s}}+p_2\ddot{\boldsymbol{s}}=q_1\dot{\boldsymbol{e}}+q_2\ddot{\boldsymbol{e}}. \tag{23}$$

The constitutive equation of the p and the θ is easily obtained. By replacing the material time derivative $\dot{\boldsymbol{s}}$ by the objective derivative $\overset{o}{\boldsymbol{s}}$, the two-order derivative $\ddot{\boldsymbol{s}}$ by the two-order objective derivative $\overset{oo}{\boldsymbol{s}}$, the deviatoric strain rate $\dot{\boldsymbol{e}}$ by the deviatoric deformation rate $\boldsymbol{D}'$, the two-order derivative $\ddot{\boldsymbol{e}}$ by the objective derivative $\overset{o}{\boldsymbol{D}}{}'$ of the deformation rate, we can obtain a rate-form constitutive equation for finite deformations

$$p_0\boldsymbol{s}+p_1\overset{o}{\boldsymbol{s}}+p_2\overset{oo}{\boldsymbol{s}}=q_1\boldsymbol{D}'+q_2\overset{o}{\boldsymbol{D}}{}'. \tag{24}$$

However, it is difficult to choose the objective derivatives for the constitutive equation. As in the case of the Maxwell model, the equation (23) is extended to include the case of finite deformation. Consider the deformation from t_0 to t_1 up to t_n : $\boldsymbol{I}\to\boldsymbol{F}_1\to\boldsymbol{F}_2\to\cdots\to\boldsymbol{F}_n$. The deformation gradient at time t_1 is expressed in the forms

$$\boldsymbol{F}_1=\boldsymbol{R}_{E1}\boldsymbol{V}_{\lambda1}\boldsymbol{R}_{L1}^{\mathrm{T}}. \tag{25}$$

We have the decomposition

$$\boldsymbol{F}_1=\boldsymbol{R}_{E1}\boldsymbol{V}_{\lambda1e(1)}\boldsymbol{V}_{\lambda Ie(2)}\boldsymbol{V}_{\lambda1v}\boldsymbol{R}_{L1}^{\mathrm{T}}, \tag{26}$$

where sub indexes $e(1)$, $e(2)$ and v imply the elastic element, the Kelvin element and the dashpot element respectively. Like the Maxwell model, the principal direction of the stress at time t_1 is $\boldsymbol{R}_{E1}$. For the small deformation from time t_0 to t_1, we obtain the deviatoric principal stress from the following equations

$$\boldsymbol{s}_{\lambda1}=2\mu_1(\ln\boldsymbol{V}_{\lambda Ie(1)}-(1/3)\boldsymbol{I}\,\mathrm{tr}(\ln\boldsymbol{V}_{\lambda Ie(1)})), \tag{27a}$$
$$\boldsymbol{s}_{\lambda1}=2\mu_2(\ln\boldsymbol{V}_{\lambda Ie(2)}-(1/3)\boldsymbol{I}\,\mathrm{tr}(\ln\boldsymbol{V}_{\lambda Ie(2)}))+$$
$$\eta_2(\overline{\ln\boldsymbol{V}_{\lambda Ie(2)}}^{\bullet}-(1/3)\boldsymbol{I}\,\mathrm{tr}(\overline{\ln\boldsymbol{V}_{\lambda Ie(2)}}^{\bullet})), \tag{27b}$$
$$\boldsymbol{s}_{\lambda1}=\eta_3(\overline{\ln\boldsymbol{V}_{\lambda1v}}^{\bullet}-(1/3)I\,\mathrm{tr}(\overline{\ln\boldsymbol{V}_{\lambda1v}}^{\bullet})). \tag{27c}$$

Equations (27a) – (27c) are the elastic element, the Kelvin element and the viscous-element constitutive relations respectively. The equation (27b) implies that the Kelvin element consists of spring parallel with dashpot.

It is supposed that the deforming body intermediately unloaded and again loaded generates the same stress as this continuously loaded body does. Hence, we can take the free-stress intermediate configuration as reference configuration. Thus the deformation gradients at t_1 and t_2 are respectively expressed by

$$\boldsymbol{F}_{1b}=\boldsymbol{R}_{E1}\boldsymbol{V}_{\lambda1e(1)}\boldsymbol{V}_{\lambda1e(2)}, \tag{28a}$$

and

$$\boldsymbol{F}_{2b}=\boldsymbol{F}_2(\boldsymbol{V}_{\lambda1v}\boldsymbol{R}_{L1}^{\mathrm{T}})^{-1}=\boldsymbol{R}_{E2b}\boldsymbol{V}_{\lambda2b}\boldsymbol{R}_{L2b}^{\mathrm{T}}. \tag{28b}$$

As in the process from time t_0 to t_1, the stress at t_2 can be obtained. The $\boldsymbol{R}_{L2b}$ does not influence the stress at t_2. Thus, the principal direction of stress at t_2 is $\boldsymbol{R}_{E2b}$. From the equation similar to (27a) – (27c), we can obtain the principal stress at t_2. Thus, the Burgers model for finite deformations is developed. In this study, we propose an approach through which the constitutive relation of small deformation is extended to include the case of finite deformation.

2 The simple shear deformation

We calculate the stresses of simple shear deformation by using the proposed Burgers constitutive model and the existing Burgers constitutive model (24) respectively. The motion in the simple shear is expressed as

$$x_1=X_1+kX_2,\ \ x_2=X_2,\ \ x_3=X_3, \tag{29}$$

where x_i and X_i are rectangular Cartesian coordinates of the current and the initial configuration respectively. We have the deformation gradient $\boldsymbol{F}$

$$\boldsymbol{F}=\begin{bmatrix}1 & k\\ 0 & 1\end{bmatrix}. \tag{30}$$

Here the matrix of tensor is presented in a 2×2 truncated matrix form with all other components that have an index equal to 3 being identically zero. The deformation rate (stretching) $\boldsymbol{D}$ and the material rotational rate $\boldsymbol{w}$ are respectively

$$\boldsymbol{D}=\frac{\dot{k}}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$

and

$$\boldsymbol{w}=\frac{\dot{k}}{2}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}. \tag{31}$$

The deformation gradient $\boldsymbol{F}$ is expressed in the following form

$$\boldsymbol{F}=\boldsymbol{R}_E\boldsymbol{V}_\lambda\boldsymbol{R}_L^{\mathrm{T}}=\boldsymbol{R}\boldsymbol{R}_L\boldsymbol{V}_\lambda\boldsymbol{R}_L^{\mathrm{T}}. \tag{32}$$

The Euler rotational rate and the relative rotational rate are respectively

$$\boldsymbol{\Omega}_E=\dot{\boldsymbol{R}}_E\boldsymbol{R}_E^{\mathrm{T}}=\frac{\dot{k}}{k^2+4}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \tag{33a}$$

and

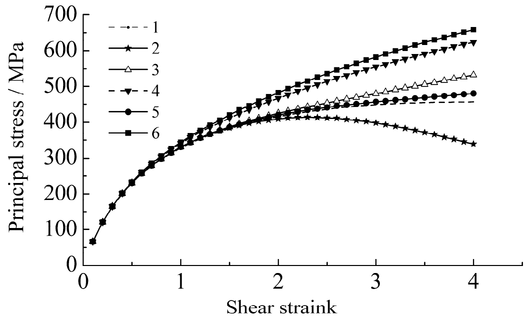
$$\boldsymbol{\Omega}=\dot{\boldsymbol{R}}\boldsymbol{R}^{-1}=\frac{2\dot{k}}{k^2+4}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}. \tag{33b}$$

The logarithmic rotational rate is

$$\boldsymbol{\Omega}_{\mathrm{Log}}=\frac{\dot{k}}{4}\left(\frac{4}{4+k^2}+\frac{k}{\sqrt{4+k^2}\,\mathrm{sh}^{-1}(k/2)}\right)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}. \tag{34}$$

The stresses are shown in Fig. 1 and Fig. 2. For the principal stress, the proposed constitutive model is between the existing constitutive models (24) where the objective rate is the material corotational rate and the relative corotational rate respectively. For the principal direction, the proposed constitutive model is near to the existing constitutive model (24) where the objective rate is the material corotational rate.

Fig. 1 principal stress in simple shear resulting from Burgers constitutive model where the shear strain rate $\dot{k}$=10 s^{-1}, material parameters used μ_1=1 500 MPa, μ_2=1 000 MPa, η_2=100 MPa·s, η_3=180 MPa·s. Curve (1) is based on the proposed Burgers constitutive model. Curves (2 – 6) are based on the existing Burgers constitutive model (24) where the rotational rate is the material rotational rate, the relative rotational rate, the Euler frame rotational rate, the logarithmic rotational rate and zero.



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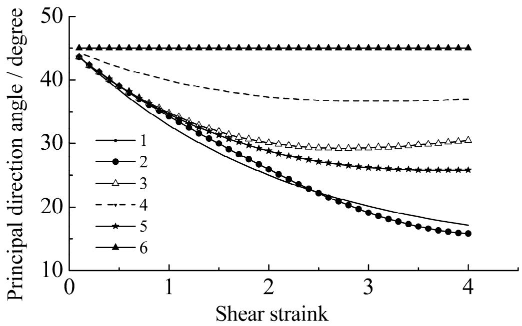

Fig.2 Principal direction angle in simple shear resulting from Burgers constitutive model where the shear strain rate $\dot{k}$ =10 s^{-1}, material parameters used μ_1 = 1 500 MPa, μ_2 = 1 000 MPa, η_2 = 100 MPa·s, η_3 = 180 MPa·s. Curve (1) is based on the proposed Burgers constitutive model. Curves (2—6) are based on the existing Burgers constitutive model (24) where the rotational rate is the material rotational rate, the relative rotational rate, the Euler frame rotational rate, the logarithmic rotational rate and zero

3 Conclusion

The constitutive model of viscoelastic material at small deformation can be expressed by the elastic element and the dashpot in parallel and series. The constitutive model at finite deformation must obey the principle of objectivity. In this study, the constitutive model of small deformation is extended to make the model applicable in finite deformation through a new approach. The deformation gradient is decomposed into elastic and inelastic parts, and thus strain rate can be expressed as the sum of elastic and inelastic parts. The decomposition of deformation gradient is consistent with that of strain rate. The rate-form constitutive relations proposed here obey the principle of objectivity though we need not make the choice of the objective corotational rate of stress which is arbitrarily made by many authors. The simple shear deformation has been worked out by using the existing Burgers model of finite deformation and the proposed model of finite deformation. The proposed model can describe the viscoelastic constitutive relation of finite deformation.

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有限变形的三维粘弹性本构模型

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摘要：遗传积分形式与弹性元件和粘壶串并联表示的微分形式可以描述高聚物小变形情况下的粘弹性特性. 文章以这些小变形的本构模型为起点，推出有限变形下的本构模型. 一个有限变形过程分解成一系列微小变形子过程. 小变形子过程中，应力的转动变化由弹性本构方程确定，主应力的变化由弹性元件和粘壶串并联结构的模型确定，这样提出了一个满足客观性原理的有限变形的粘弹性本构模型. 模型材料参数随应变率而变，所以模型适合于从准静态到冲击载荷的较宽应变率范围. 作为应用，计算了简单剪切有限变形，就建议的模型与现有的模型进行了比较，结果表明建议的模型能够描述高聚物有限变形情况下的粘弹性性质.

关键词：粘弹性；有限变形；客观性原理

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