Non-Lie Point Symmetry and Coherent Soliton Solutions of Nizhnik-Novikov-Veselov Equation

HU Han-wei[1], YU Jun[2*]

(1.Faculty of Science, Ningbo University, Ningbo 315211, China;

2.Department of Physics, Shaoxing University, Shaoxing 312000, China)

Abstract: A new method is presented to find the finite symmetry groups of nonlinear physics systems. For the (2+1)-dimensional Nizhnik-Novikov-Veselov (NNV) equation, both the Lie point symmetry and the non-Lie symmetry group are obtained using the proposed method. In contrast, only the Lie symmetry groups can be found if using the traditional Lie approach. Furthermore, a variety of localized structures of the NNV equation can also be obtained from the non-Lie symmetry group.

Key words: Lax pair; symmetries; symmetry group; exact solution

CLC number: O152.5 **Document code:** A **Article ID:** 1001-5132（2012）02-0083-05

It is getting popular in soliton theory to find symmetries, symmetry groups and symmetry reductions, and to construct group invariant solutions for nonlinear partial differential equations (PDEs)[1-4]. There is a standard method depending on the famous first fundamental theorem of Lie[5] to get the Lie point symmetry group of a nonlinear system, and it had been widely used to obtain Lie point symmetry algebras and groups for almost all the known integrable systems. However it is still quite complicated and difficult to getting non-Lie symmetry groups by this method. Recently, Lou and Ma[6-7] had proposed a general direct method. Using the method, not only the Lie point symmetry groups and the non-Lie symmetry groups can be obtained for some nonlinear partial differential equation (PDEs), but also the expressions of the exact finite transformations of the Lie groups are much simpler than those obtained via the standard approaches for some nonlinear PDEs.

Indeed there have also some researches on the symmetry for an integrable system by means of the general direct method, but for the method concerning the Lax pair is has not been reported. In fact, the Lax pair of an integrable model has been proved to be very important in the study of its various interesting properties. Especially, it is one of the best ways of determining some concrete integrals for a given PDE. It had been also proved that the Lax pairs are very useful in finding infinitely many symmetries.

In this paper, starting from a weak Lax pair, we obtain the general Lie point symmetry group and some localized solutions of the Nizhnik-Novikov-Veselov (NNV) equation by the general direct method. In Sec. 2, we obtain the finite symmetry transformation groups and the related special Lie point symmetries of the NNV equation. Some types of exact solutions obtained from the transformation theorem are given in Sec. 3. The last one is a short summary and discussion.

Received date: 2011−09−15. JOURNAL OF NINGBO UNIVERSITY (NSEE): http://3xb.nbu.edu.cn

Foundation items: Supported by National Natural Science Foundation of China (10875078); Natural Science Foundation of Zhejiang Province (Y7080455).

The first author: HU Han-wei (1987−), male, Jinhua Zhejiang, post of graduate student, research domain: mathematical physics. E-mail: huhanwei5955@sina.com

***Corresponding author:** YU Jun (1959−), female, Shaoxing Zhejiang, professor, research domain: mathematical physics. E-mail: junyu@usx.edu.cn

1 Symmetry groups and symmetries of the NNV equation

The Nizhnik-Novikov-Veselov (NNV) system have the form

$$u_t + u_{xxx} + u_{yyyy} - 3(uv)_x - 3(uw)_y = 0,$$
$$u_x - v_y = 0,$$
$$u_y - w_x = 0, \tag{1}$$

where $u = u(x,y,t)$, $v = v(x,y,t)$, $w = w(x,y,t)$, the subscripts denote partial differential. Eq. (1) is an only known isotropic Lax integrable extension of the well known (1+1)-dimensional KdV equation. It is just the compatibility conditions of the following system:

$$u\varphi - \varphi_{xy} = 0, \tag{2}$$
$$\varphi_t + \varphi_{xxx} - \varphi_{yyy} - 3v\varphi_x - 3w\varphi_y = 0. \tag{3}$$

That is, Eqs. (2)—(3) are Lax pairs of system (1). Some types of the soliton solutions of the NNV equation have been studied by many authors. For example, Boiti solved the NNV equation via the inverse scattering transformation. Hu[8] obtained the soliton-like solutions of the NNV equation by means of the Backlund transformation. Lou[9] got many interesting soliton structures of the NNV equation through the variable separation approach[10]. On the other hand, the nonlinear superposition formula of the NNV equation was given in Ref. [11]. And some special types of multi-dromion solutions were found by Radha and Lakshmanan[12].

In order to obtain the finite symmetry transformation group of Eq. (1), we let

$$\varphi = \beta\psi(\xi,\eta,\tau), \tag{4}$$

where β,ξ,η,τ are all functions of x,y,t, and ψ satisfies the equations

$$\psi_{\xi\eta} = U(\xi,\eta,\tau)\psi, \tag{5}$$
$$\psi_\tau = \psi_{\eta\eta\eta} - \psi_{\xi\xi\xi} + 3W(\xi,\eta,\tau)\psi_\eta + 3V(\xi,\eta,\tau)\psi_\xi, \tag{6}$$

Substituting Eqs. (4)—(6) into Eqs. (2)—(3), we obtain

$$-\beta\tau_x\tau_y\psi_{6\xi} + \cdots = 0, \tag{7}$$
$$\beta(\tau_y^3 - \tau_x^3)\psi_{9\xi} + \cdots = 0. \tag{8}$$

Obviously, β should not be zero and there exists no nontrivial solution for $\tau_x = 0$ and $\tau_y = 0$, so

$$\tau_x = 0,\ \ \tau_y = 0, \ \text{ i.e. } \ \tau = \tau(t). \tag{9}$$

we have

$$-\beta\xi_x\xi_y\psi_{\xi\xi} - \beta\eta_x\eta_y\psi_{\eta\eta} + \cdots = 0. \tag{10}$$

It is easy to see that

$$\xi = \xi(x,t),\ \ \eta = \eta(y,t). \tag{11}$$

Now the substitution of (9) and (11) into Eqs. (7)—(8), by vanishing the coefficients of the polynomials of Ψ and it's derivatives, the remaining determining equations of the functions read

$$\beta_x = 0,\ \ \beta_y = 0,\ \ \beta_t = 0, \tag{12}$$
$$\xi_x^3 - \tau_t = 0,\ \tau_t - \eta_y^3 = 0, \tag{13}$$
$$u - \xi_x\eta_y U = 0,\ \ 3v\xi_x - 3\tau_t V - \xi_t = 0,$$
$$w\eta_y - 3\tau_t W - \eta_t = 0. \tag{14}$$

From Eqs. (12)—(14) we can have the following Theorem.

Theorem 1 If $\{U = U(x,y,t), V = V(x,y,t)\}$ is a solution of the SK equation, then $\{u,v\}$ is given by

$$u = \tau_t^{2/3}U(\xi,\eta,\tau),$$
$$v = \tau_t^{2/3}V(\xi,\eta,\tau) - \tau_{tt}\tau_t^{-1}x/9 - g_t\tau_t^{-1/3}/3,$$
$$v = \tau_t^{2/3}W(\xi,\eta,\tau) + \tau_{tt}\tau_t^{-1}y/9 + f_t\tau_t^{-1/3}/3, \tag{15}$$

where $\xi = \tau_t^{1/3}x + g$, $\eta = \tau_t^{1/3}y + f$ and f, g, τ are arbitrary function of t.

It is obvious that once the symmetry group is obtained, to find Lie point symmetry algebra, the infinitesimal transformations, is quite a straightforward work. The generators of the symmetry algebra related to the symmetry group theorem 1 can be obtained by setting

$$\tau = 1 + \varepsilon h(t),\ g = \varepsilon k(t),\ f = \varepsilon l(t), \tag{16}$$

and ε is an infinitesimal parameter, then we get the Lie point symmetry algebra with the general symmetry elements:

$$\sigma = \sigma_1(h) + \sigma_2(k) + \sigma_3(l) = \begin{pmatrix} hu_t + xh_tu_x/3 + yh_tu_y/3 + 2h_tu/3 \\ hv_t + xh_tv_x/3 + yh_tv_y/3 + 2h_tv/3 + xh_{tt}/9 \\ hw_t + xh_tw_x/3 + yh_tw_y/3 + 2h_tw/3 + yh_{tt}/9 \end{pmatrix} +$$
$$\begin{pmatrix} ku_x \\ kv_x + k_t/3 \\ kw_x \end{pmatrix} + \begin{pmatrix} lu_y \\ lv_y \\ lw_y + l_t/3 \end{pmatrix}, \tag{17}$$

while the related Kac-Moody-Virasoro type symmetry algebra is as follows:

$$[\sigma_1(h_1),\sigma_1(h_2)]=\sigma_1(h_1h_{2t}-h_2h_{1t}),$$
$$[\sigma_1(h),\sigma_2(k)]=\sigma_2(hk_t),$$
$$[\sigma_2(k),\sigma_3(l)]=[\sigma_2(k_1),\sigma_2(k_2)]=[\sigma_3(l_1),\sigma_3(l_2)]=0,$$
$$[\sigma_1(h),\sigma_3(l)]=\sigma_3(hl_t). \quad (18)$$

2 Some special localized solutions of the NNV equation

To find some types of localized excitations in high dimensions is one of the most important and difficult work. In this section, we just write down some types of localized solutions with help of the group transformation theorem and the known solutions for the NNV equation.

It is easy to verify that the NNV system possesses the following single line soliton solution

$$u=\frac{2q_yp_x(a_1a_2-A)}{(1+a_1p+a_2q+Apq)^2}, \quad (19)$$

$$v=\frac{2(a_1+Aq)^2p_x^2}{(1+a_1p+a_2q+Apq)^2}-\frac{2(a_1+Aq)p_{xx}}{(1+a_1p+a_2q+Apq)}+v_0, \quad (20)$$

$$w=\frac{2(a_1+Aq)^2q_y^2}{(1+a_1p+a_2q+Apq)^2}-\frac{2(a_1+Aq)q_{yy}}{(1+a_1p+a_2q+Apq)}+w_0, \quad (21)$$

where $p=p(x,t)$, $q=q(y,t)$, a_1,a_2 and A are arbitrary constants.

Case 1 $p(x,t)=\exp(2x+2t)$, $q(y,t)=\exp(2y)$. In this case, we have a simple straight line soliton solution of NNV equation:

$$u=\frac{8\exp(P)(a_1a_2-A)}{(1+a_1\exp(Q)+a_2\exp(2y)+A\exp(P))^2}, \quad (22)$$

and $P=2(x+y+t)$, $Q=2(x+t)$. Applying Theorem 1 on the solution Eq. (22), the group invariant solution is

$$u=\frac{8\exp(P)(a_1a_2-A)\tau_t^{2/3}}{(1+a_1\exp(Q)+a_2\exp(2\eta)+A\exp(P))^2}, \quad (23)$$

where $P=2(\xi+\eta+\tau)$, $Q=2(\xi+\tau)$, $\xi=\tau_t^{1/3}x+g$, $\eta=\tau_t^{1/3}y+f$, and f, g, τ are arbitrary function of t. It is easy to see that, when we take $\tau=t$, $g=0$ and $f=0$, then the solution Eq. (23) is reduced to Eq. (22).

Case 2 $p(x,t)=\exp(x(\cos(x)+4/3))$, $q(y,t)=\exp(y)$. In this case, a dromion solution oscillating in the x direction will be found:

$$u=\frac{2P\exp(Q+y)(A-a_1a_2)}{3(1+a_1\exp(Q)+a_2\exp(y)+A\exp(Q+y))^2}, \quad (24)$$

and $P=3x\sin x-3\cos x-4$, $Q=x\cos x+4/3$. According to the symmetry transformation group, the oscillating dromion solution of Eq. (24) is changed to

$$u=\frac{2P\exp(Q+\eta)(A-a_1a_2)\tau_t^{2/3}}{3(1+a_1\exp(Q)+a_2\exp(\eta)+A\exp(Q+\eta))^2}, \quad (25)$$

where $P=3\xi\sin\xi-3\cos\xi-4$, $\xi=\tau_t^{1/3}x+g$, $\eta=\tau_t^{1/3}y+f$ and f, g, τ are arbitrary function of t. It is easy to see that, when we take $\tau=t$, $g=0$ and $f=0$, then the solution Eq. (25) is reduced to Eq. (24).

Fig. 1 displays a plot of the oscillating dromion structure given by Eq. (25) and $a_1=A=1$, $a_2=2$, $\tau=2\sin(t)$, $f=\sin(t)$, $g=\exp(t^3)$ at times $t=0$, $t=\pi/3$.

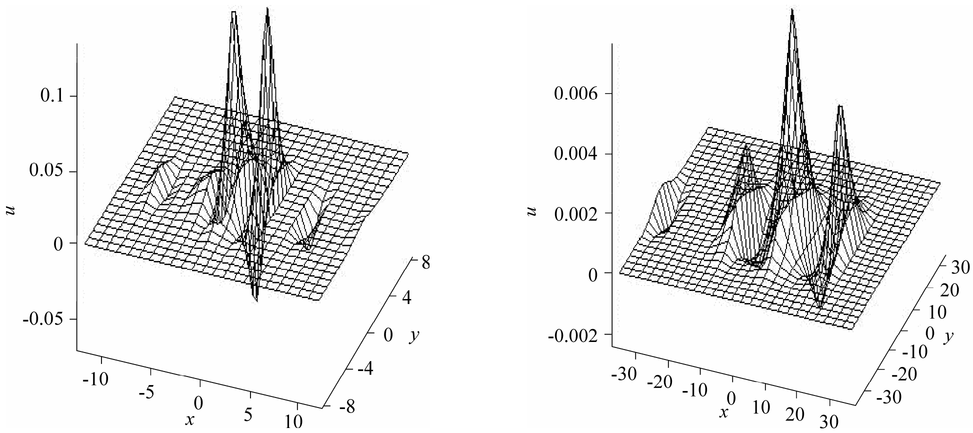
Case 3 $p(x,t)=\exp(x\cos t+20x/19)$, $q(y,t)=\exp(y+\sin t)$. In this case, substituting p and q into Eq. (22), a dromion type of breather solution is given as follow:

$$u=-\frac{2(19\cos t+20)PQ(A-a_1a_2)}{19(1+a_1P+a_2Q+APQ)^2}, \quad (26)$$

where $P=\exp(x\cos t+20x/19)$, $Q=\exp(y+\sin t)$, applying Theorem 1 on the solution Eq. (29), the group invariant solution is

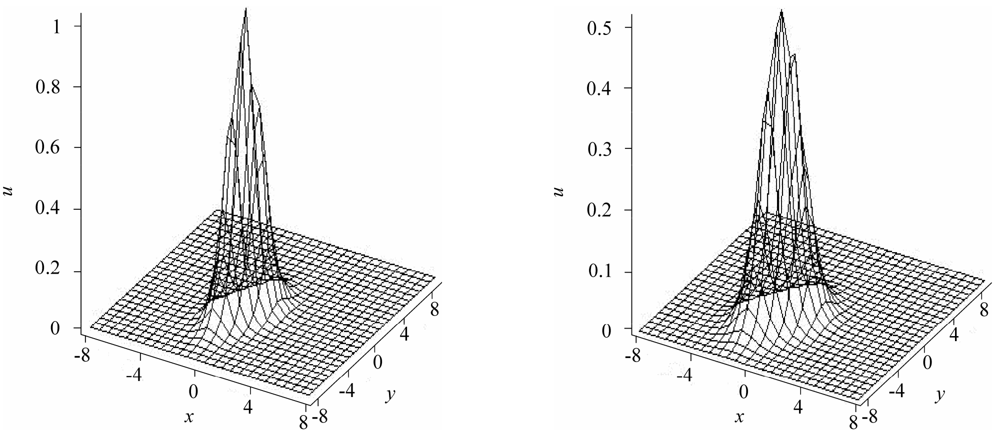
$$u=-\frac{2(19\cos\tau+20)PQ(A-a_1a_2)\tau_t^{2/3}}{19(1+a_1P+a_2Q+APQ)^2}, \quad (27)$$

where $P=\exp(\xi\cos\tau+20\xi/19)$, $Q=\exp(\eta)\exp(\sin\tau)$, $\xi=\tau_t^{1/3}x+g$, $\eta=\tau_t^{1/3}y+f$ and f, g, τ are arbitrary function of t. It is easy to see that, when we take $\tau=t$, $g=0$ and $f=0$, then the solution Eq. (27) is reduced to Eq. (26).



(a) $t=0$ (b) $t=\pi/3$

Fig. 1 Two oscillating dromion solutions for the field u of Eq. (25) with $a_1 = A = 1$, $a_2 = 2$, $\tau = 2\sin t$, $f = \sin t$, $g = \exp(t^3)$ at times



(a) $t=0$ (b) $t=\pi/6$

Fig. 2 A plot of a dromion type of breather solution of Eq. (30) with $a_1 = A = 10$, $a_2 = 20$, $\tau = 2\sin t$, $f = \sin t$, $g = \sin t$ at times

Fig. 2 is a plot of a dromion type of breather solution of Eq. (30) with $a_1 = A = 10$, $a_2 = 20$, $\tau = 2\sin t$, $f = \sin t$, $g = \sin t$ at $t = 0, t = \pi/6$.

Using the obtained symmetry groups and known solutions, one can obtain various interesting localized excitations, and some known solutions in Ref. [11] are only special cases when $\tau = t$, $g = 0$ and $f = 0$. By selecting the arbitrary functions τ, g, f, we can get complicated and abundant structures of the NNV equation.

3 Summary and discussion

In summary, starting from the Lax pair expressions of the NNV equation, the finite symmetry transformation groups theorem 1 is obtained in a very simple way. According to the theorem 1, the related Lie point symmetries which have a Kac-Moody-Virasoro structure can be derived simply by restricting the arbitrary functions in infinitesimal forms. It is necessary to point out that we can obtain all group invariable solutions from known solutions by theorem 1. This method is valid for other types of known (2+1)-dimensional integrable models such as the Kadomtsev-Petviashvili equation, the Davey-Stewartson system and Broer-Kaup equation, Especially, it is important in the (3+1)-dimensional Jimbo-Miwa equation.

References:

[1] Bluman G W, Anco S C. Symmetry and integration methods for differential equations[M]. New York: Springer, 2002:101-171.

[2] Lou S Y. Generalized symmetries and w_∞ algebras in three-dimension toda field theory[J]. Phys Rev lett, 1993, 71:4099-4102.

[3] Clarkson P A, Kruskal M D. New similarity reductions of the Boussinesq equation[J]. J Math Phys, 1989, 30:2201-2213.

[4] Tang X Y, Qian X M, Lin J. Conditional similarity reductions of the (2+1)-dimensional KdV equation via the extended Lie group approach[J]. J Phys Soc Jpn, 2004, 73: 1464-1475.

[5] Olver P G. Application of Lie group to differential equation[M]. New York: Springer, 1986:106-204.

[6] Lou S Y, Ma H C. Non-Lie symmetry groups of (2+1)-dimensional nonlinear systems obtained from a simple direct method[J]. J Phys A, 2005, 38:129-137.

[7] Lou S Y, Ma H C. Finite symmetry transformation groups and exact solutions of Lax integrable systems[J]. Chaos, Solitons and Fractals, 2006, 30:804-821.

[8] Hu X B, Li Y S. Nonlinear superposition formulae of the lto equation and a model equation for shallow water waves[J]. J Phys A, 1991, 24:1979-1986.

[9] Lou S Y. On the coherent structures of the Nizhnik-Novikov-Veselov equation[J]. Phys Lett A, 2000, 277: 94-100.

[10] Lou S Y, Ruan H Y. Revisitation of the localized excitations of the (2+1)-dimensional KdV equation[J]. J Phys A, 2001, 34:305-316.

[11] Hu X B. Nonlinear superposition formulae of Novikov-Veselov equation [J]. J Phys A, 1994, 27:1331-1338.

[12] Radha R, Lakshmanan M. Singularity analysis and localized coherent structures in (2+1) dimensional generalized Kortewegde Vries equations[J]. J Math Phys, 1994, 35:4746-4752.

Nizhnik-Novikov-Veselov 方程的非李点对称及其精确解

胡瀚玮[1]，俞　军[2*]

（1.宁波大学 理学院，浙江 宁波 315211; 2.绍兴文理学院 物理系，浙江 绍兴 312000）

摘要：对于非线性物理系统的有限对称群，一个新的方法被提出．将该方法作用于 Nizhnik-Novikov-Veselov (NNV)方程，李点和非李点对称能同时得到，而使用经典李群法只能得到李点对称．最后，通过对称变化群能得到许多新的孤子解.

关键词：Lax 对；对称；对称群；精确解

（责任编辑　史小丽）