

# Is the Schwarzschild Metric a Vacuum Solution of the Einstein Equation?

Horace Crater\*

The University of Tennessee Space Institute

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## Abstract

This paper examines the inhomogeneous Einstein equation for a static spherically symmetric metric with a source term corresponding to perfect fluid with  $p = -\rho$ . By a careful treatment of the equation near the origin we find an analytic solution for the metric, dependent on a small parameter  $\varepsilon$ , which can be made arbitrarily close to the Schwarzschild solution as  $\varepsilon \rightarrow 0$  and which in that same limit can be viewed as arising from a point-like source structure.

## 1 Introduction

This paper examines solutions of the inhomogeneous Einstein equation for static spherically symmetric metrics corresponding to a perfect fluid at rest with  $p = -\rho$ . Our aim is to develop an analytic solution for the metric that can be made arbitrarily close to the Schwarzschild solution but yet retains a nonvanishing contribution to the source term, unlike the Schwarzschild solution. At the same time we find that the second solution with the same source term is arbitrarily close to the de Sitter solution. The Einstein equations for the metric are second order and highly nonlinear. This implies that if one has two independent solutions, then their linear combination will not be a solution. Nevertheless, as we emphasize in the first section, under those special circumstances for the source term in the Einstein equation, the metric can be written in terms of the logarithm of a potential-like function which satisfies a very simple linear differential equation, with two linear independent solutions. One of the potential-like functions is the Newtonian gravitational potential which through the logarithm we associate with the Schwarzschild solution while the other is a harmonic oscillator potential associated with the de Sitter solution. The contribution of the latter to the source term is a constant while that of the Schwarzschild solution

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\*hcrater@utsi.edu

has a vanishing contribution to the source term. In order to accomplish our aim, we recast the inhomogeneous Einstein equation as the limiting case of a closely related inhomogeneous equation dependent on a small parameter  $\varepsilon$ . A careful treatment of this problem near the origin leads to source terms with two separate non-vanishing contributions

## 2 Solutions of the Inhomogeneous Einstein Equation

### 2.1 Static Spherically Symmetric Solutions of the Inhomogeneous Einstein Equation for Perfect Fluid with $p = -\rho$ .

In this section we first review the static spherically symmetric standard solution of the vacuum Einstein equation. The equation is second order and has two solutions with one being the Schwarzschild solution while the other is a constant. We then remind the reader how if one adds a source term corresponding to a perfect fluid at rest with  $p = -\rho$  a second nonconstant solution emerges in addition to the Schwarzschild one. If the pressure and density are constants the second solution is one found originally by de Sitter for the Einstein equation with a cosmological constant[3]-[6] and is a static form of the time dependent one used in models of inflation and dark energy in modern cosmology.

For a spherically symmetric solution one chooses the coordinates

$$\begin{aligned} x^0 &= t, \\ x^1 &= r, \\ x^2 &= \theta, \\ x^3 &= \phi. \end{aligned} \tag{1}$$

In a vacuum with static conditions as well as spherical symmetry, we use Dirac's exponential parametrization of the metric,[1],

$$\begin{aligned} d\tau^2 &= e^{2\nu(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \\ g_{00} &= -e^{2\nu(r)} = 1/g^{00} \\ g_{11} &= e^{2\lambda(r)} = 1/g^{11}, \\ g_{22} &= r^2 = 1/g^{22}, \\ g_{33} &= r^2 \sin^2\theta = 1/g^{33}. \\ g_{\mu\nu} &= 0, \quad \mu \neq \nu. \end{aligned} \tag{2}$$

With

$$\Gamma_{\mu\nu}^{\kappa} = \frac{g^{\kappa\sigma}}{2}(g_{\nu\sigma,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma}) = \Gamma_{\nu\mu}^{\kappa}, \tag{3}$$

the only nonzero  $\Gamma$ 's are [1]

$$\begin{aligned}
\Gamma_{00}^1 &= \nu' e^{2\nu-2\lambda}, & \Gamma_{10}^0 &= \nu', \\
\Gamma_{11}^1 &= \lambda', & \Gamma_{12}^2 &= \Gamma_{13}^3 = r^{-1}, \\
\Gamma_{22}^1 &= -r e^{-2\lambda}, & \Gamma_{23}^3 &= \cot \theta, \\
\Gamma_{33}^1 &= -r \sin^2 \theta e^{-2\lambda}, & \Gamma_{33}^2 &= -\sin \theta \cos \theta.
\end{aligned} \tag{4}$$

With

$$R_{\nu\sigma} = \Gamma_{\nu\lambda,\sigma}^\lambda - \Gamma_{\nu\sigma,\lambda}^\lambda + \Gamma_{\nu\sigma}^\kappa \Gamma_{\kappa\lambda}^\lambda - \Gamma_{\nu\kappa}^\kappa \Gamma_{\sigma\lambda}^\lambda, \tag{5}$$

the diagonal elements of the Ricci tensor are

$$\begin{aligned}
R_{00} &= \left( -\nu'' + \lambda' \nu' - \nu'^2 - \frac{2\nu'}{r} \right) e^{2\nu-2\lambda}, \\
R_{11} &= \nu'' - \lambda' \nu' + \nu'^2 - \frac{2\lambda'}{r}, \\
R_{22} &= (1 + r\nu' - r\lambda') e^{-2\lambda} - 1, \\
R_{33} &= R_{22} \sin^2 \theta.
\end{aligned} \tag{6}$$

From this we have that the scalar curvature is

$$\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} = - \left( -\nu'' + \lambda' \nu' - \nu'^2 - \frac{2\nu'}{r} \right) e^{-2\lambda} + \left( \nu'' - \lambda' \nu' + \nu'^2 - \frac{2\lambda'}{r} \right) e^{-2\lambda} \\
&\quad + \frac{2(1 + r\nu' - r\lambda') e^{-2\lambda} - 2}{r^2} \\
&= \left( 2\nu'' - 2\lambda' \nu' + 2\nu'^2 - \frac{4\lambda' - 4\nu'}{r} + \frac{2}{r^2} \right) e^{-2\lambda} - \frac{2}{r^2}
\end{aligned} \tag{7}$$

For our model for  $T_{\mu\nu}$  we take that of a perfect fluid, [2]

$$\begin{aligned}
T_{\mu\nu} &= p g_{\mu\nu} + (p + \rho) u_\mu u_\nu, \\
g_{\mu\nu} u^\mu u^\nu &= -1.
\end{aligned} \tag{8}$$

Consider the fluid at rest, for which  $\mathbf{u} = \mathbf{0}$  and so

$$\begin{aligned}
g_{00} u^{02} &= -1, \\
u_0 &= g_{00} u^0 = g_{00} (-g_{00})^{-1/2}, \\
u_0^2 &= -g_{00}.
\end{aligned} \tag{9}$$

Thus, the only nonzero elements of  $T_{\mu\nu}$  are

$$\begin{aligned}
T_{00} &= p g_{00} - g_{00} (p + \rho) = -g_{00} \rho, \\
T_{11} &= p g_{11}, \\
T_{22} &= p g_{22}, \\
T_{33} &= p g_{33}.
\end{aligned} \tag{10}$$

Now, the Einstein equations  $G_{\mu\nu} = T_{\mu\nu}$  become

$$\begin{aligned}
G_{00} &= R_{00} - \frac{1}{2}g_{00}R = -g_{00}\rho \\
G_{00} &= \left(-\nu'' + \lambda'\nu' - \nu'^2 - \frac{2\nu'}{r}\right)e^{2\nu-2\lambda} + \left(\nu'' - \lambda'\nu' + \nu'^2 - \frac{2\lambda' - 2\nu'}{r} + \frac{1}{r^2}\right)e^{2\nu-2\lambda} - \frac{e^{2\nu}}{r^2} \\
&= \left(-\frac{2\lambda'}{r} + \frac{1}{r^2}\right)e^{2\nu-2\lambda} - \frac{e^{2\nu}}{r^2} = e^{2\nu}\rho, \\
\rho &= e^{-2\lambda}\left(-\frac{2\lambda'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \tag{11}
\end{aligned}$$

and

$$\begin{aligned}
G_{11} &= R_{11} - \frac{1}{2}g_{11}R = pg_{11}, \\
pe^{2\lambda} &= \nu'' - \lambda'\nu' + \nu'^2 - \frac{2\lambda'}{r} - \frac{1}{2}e^{2\lambda}\left(2\nu'' - 2\lambda'\nu' + 2\nu'^2 - \frac{4\lambda' - 4\nu'}{r} + \frac{2}{r^2}\right)e^{-2\lambda} - \frac{2}{r^2} \\
&= -\frac{1}{2}e^{2\lambda}\left(\left(\frac{4\nu'}{r} + \frac{2}{r^2}\right)e^{-2\lambda} - \frac{2}{r^2}\right), \\
p &= e^{-2\lambda}\left(-\frac{2\nu'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \tag{12}
\end{aligned}$$

and

$$\begin{aligned}
G_{22} &= R_{22} - \frac{1}{2}g_{22}R = pg_{22}, \\
pr^2 &= (1 + r\nu' - r\lambda')e^{-2\lambda} - 1 - \frac{1}{2}r^2\left(2\nu'' - 2\lambda'\nu' + 2\nu'^2 - \frac{4\lambda' - 4\nu'}{r} + \frac{2}{r^2}\right)e^{-2\lambda} - \frac{2}{r^2} \\
p &= \left(\frac{\nu'}{r} - \frac{\lambda'}{r}\right)e^{-2\lambda} - \left(\nu'' - \lambda'\nu' + \nu'^2 - \frac{2\lambda' - 2\nu'}{r}\right)e^{-2\lambda} \\
&= -\left(\nu'' - \lambda'\nu' + \nu'^2 - \frac{\lambda' - \nu'}{r}\right)e^{-2\lambda}, \tag{13}
\end{aligned}$$

and the fourth equation, the one for  $G_{33}$ , gives nothing new beyond that for  $G_{22}$ . Hence, the above three simultaneous equations become

$$\begin{aligned}
\rho &= e^{-2\lambda}\left(-\frac{2\lambda'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}, \\
p &= e^{-2\lambda}\left(-\frac{2\nu'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}, \\
p &= -\left(\nu'' - \lambda'\nu' + \nu'^2 - \frac{\lambda' - \nu'}{r}\right)e^{-2\lambda}. \tag{14}
\end{aligned}$$

These are three nonlinear inhomogeneous equations for two unknown functions of  $r$ .

For empty space ( $\rho = p = 0$ ) these equations become those originally solved by Schwarzschild, that is.

$$\begin{aligned}
e^{-2\lambda} \left( -\frac{2\lambda'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} &= 0 \\
e^{-2\lambda} \left( -\frac{2\nu'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} &= 0, \\
-\left( \nu'' - \lambda'\nu' + \nu'^2 - \frac{\lambda' - \nu'}{r} \right) e^{-2\lambda} &= 0.
\end{aligned} \tag{15}$$

Combining the first two equations implies that

$$\begin{aligned}
\lambda' &= -\nu', \\
\lambda &= -\nu + \lambda_0(t).
\end{aligned} \tag{16}$$

The third equation then leads

$$\nu'' + 2\nu'^2 + \frac{2\nu'}{r} = 0. \tag{17}$$

We parametrize the exponential metric function  $\nu$  by introducing a potential-like function  $V$ ,

$$\begin{aligned}
\nu &= \frac{1}{2} \ln(1 + V), \\
\nu' &= \frac{1}{2} V' \frac{1}{1 + V} \\
\nu'' &= \frac{1}{2} V'' \frac{1}{1 + V} - \frac{1}{2} V'^2 \frac{1}{(1 + V)^2} = \frac{1}{2} V'' \frac{1}{1 + V} - 2\nu'^2.
\end{aligned} \tag{18}$$

Thus, using

$$e^{2\nu} = (1 + V),$$

Eq. (17) becomes

$$V'' + \frac{2V'}{r} = 0. \tag{19}$$

This second order equation is an equidimensional one and has the general solution of

$$V = \frac{k_1}{r} + k_2. \tag{20}$$

Our metric is thus

$$\begin{aligned}
e^{2\nu} &= 1 + \frac{k_1}{r} + k_2, \\
e^{2\lambda} &= e^{-2\nu + 2\lambda_0} = \frac{e^{-2\nu + 2\lambda_0}}{1 + \frac{k_1}{r} + k_2}.
\end{aligned} \tag{21}$$

In order for the metric to become Minkowskian at  $r \rightarrow \infty$  we must have

$$k_2 = 0 = \lambda_0. \quad (22)$$

Matching  $g_{00}$  to  $-1 - 2\phi$  where for large  $r$  where  $\phi$  is the Newtonian potential  $-MG/r$  gives

$$k_1 = -2MG \equiv -r_s, \quad (23)$$

the Schwarzschild radius.

Another set of exact solutions is found by assuming that

$$p = -\rho \neq 0. \quad (24)$$

Just as with the Schwarzschild solution with  $\rho = p = 0$ , combining the first two equations of (14) implies that

$$\begin{aligned} \nu' &= -\lambda', \\ \nu &= -\lambda + \nu_0(t). \end{aligned} \quad (25)$$

In this case we absorb the factor  $\nu_0(t)$  into a redefinition of the time scale used in the metric. Thus we have  $\nu = -\lambda$  and the last two equations of (14) become

$$\begin{aligned} p &= -e^{2\nu} \left( \frac{2\nu'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} \\ p &= - \left( \nu'' + 2\nu'^2 + \frac{2\nu'}{r} \right) e^{2\nu}. \end{aligned} \quad (26)$$

Note that these two equations both determine the metric function  $\nu(r)$  and the pressure (and density) so that one does not have a freedom of choice for the pressure.

We parametrize the exponential metric function  $\nu$  by introducing a potential-like function  $V$  just as in Eq. (18) and in that case the last of the two crucial Einstein equations in (26) become

$$p = - \left( \nu'' + 2\nu'^2 + \frac{2\nu'}{r} \right) e^{2\nu} = p = -\frac{1}{2} \left( V'' + \frac{2V'}{r} \right). \quad (27)$$

Substituting this in the first of Eqs. (26) we obtain

$$\begin{aligned} -\frac{1}{2} \left( V'' + \frac{2V'}{r} \right) &= p = -(1+V) \left( \frac{2(1/2)V'}{(1+V)r} + \frac{1}{r^2} \right) + \frac{1}{r^2} \\ &= -\frac{V'}{r} - \frac{V}{r^2} \end{aligned} \quad (28)$$

This leads to the linear equation

$$V'' = \frac{2V}{r^2}. \quad (29)$$

Note the difference between this equidimensional equation and (19) for the homogeneous case. This one has the general solution of a linear combination of a harmonic oscillator with a Newtonian potential,

$$V(r) = kr^2 - \frac{r_s}{r}. \quad (30)$$

Use of the Schwarzschild radius in this case would imply that we are at distances at which the harmonic term is completely negligible. Our metric is thus

$$\begin{aligned} g_{00} &= -e^{2\nu} = -\frac{1}{1 + kr^2 - r_s/r} = 1/g^{00} \\ g_{11} &= e^{2\lambda} = e^{-2\nu} = 1 + kr^2 - r_s/r = 1/g^{11}, \\ g_{22} &= r^2 = 1/g^{22}, \\ g_{33} &= r^2 \sin^2 \theta = 1/g^{33}. \\ g_{\mu\nu} &= 0, \quad \mu \neq \nu. \end{aligned} \quad (31)$$

This corresponds, without the Newtonian term, to the solution obtained by de Sitter for the Einstein equation with a cosmological constant[3]-[6].

The pressure and density terms are thus required to be by the equations respectively

$$\begin{aligned} p &= -\frac{1}{2}(V'' + \frac{2V'}{r}) = -3k, \\ \rho &= -p = 3k. \end{aligned} \quad (32)$$

Note that the contributions to the energy density and pressure from the Newtonian part of the potential vanishes.

It is of interest that in the nonrelativistic limit this vanishing of the source term for that portion of the metric is contrary to what occurs in the Poisson equation for a point mass density

$$\begin{aligned} \nabla^2 \Phi &= 4\pi GM \delta^3(\mathbf{r}), \\ \Phi &= -\frac{GM}{r}. \end{aligned} \quad (33)$$

Is there a point mass at the origin in the case of the Einstein equation? This would seem to be implied by the Newtonian-Poisson connection. One may be tempted to replace  $V'' + \frac{2V'}{r}$  with  $\nabla^2 V$  and proclaim that  $\rho = -p = -\frac{1}{2}\nabla^2 V = 2\pi r_s \delta^3(\mathbf{r}) + 3k$  but this is not consistent with the other expression for the pressure of  $\rho = -p = \frac{V'}{r} + \frac{V}{r^2} = 3k$ . This calls for a more careful treatment of the Einstein equation near the origin.

## 2.2 The Schwarzschild Solution as a Valid Approximation for a Nonlinear Solution of the Full Non-homogeneous Einstein Equation

In order to treat the problem at the origin more carefully and achieve the aim of this paper, we view the Einstein equations in their reduced forms given in

(26) as the limit for small  $\varepsilon$  of the following

$$\begin{aligned} p &= -e^{2\nu} \left( \frac{2\nu'}{\bar{r}} + \frac{1}{\bar{r}^2} \right) + \frac{1}{\bar{r}^2}, \\ p &= - \left( \nu'' + 2\nu'^2 + \frac{2\nu'}{\bar{r}} \right) e^{2\nu}. \end{aligned} \quad (34)$$

where

$$\bar{r} \equiv (r^2 + \varepsilon^2)^{1/2}. \quad (35)$$

We shall solve these equations instead of the reduced forms (26) of the Einstein equations and view the proper solutions of the Einstein equation as the limit of small  $\varepsilon$  of the modified equations. Thus, as before, using Eq. (??) the two crucial Einstein equations in (26) become

$$\begin{aligned} p &= -\frac{1}{2}(V'' + \frac{2V'}{\bar{r}}) = -e^{2\nu} \left( \frac{V'}{(1+V)\bar{r}} + \frac{1}{\bar{r}^2} \right) + \frac{1}{\bar{r}^2} \\ &= -\frac{V'}{\bar{r}} - \frac{V}{\bar{r}^2}, \end{aligned} \quad (36)$$

and this leads to

$$V'' = \frac{2V}{\bar{r}^2} = \frac{2V}{(r^2 + \varepsilon^2)}. \quad (37)$$

Clearly, one solution is

$$V_1(r, \varepsilon) = k_1(r^2 + \varepsilon^2). \quad (38)$$

Using the connection

$$V_2 = V_1 \int \frac{dr}{V_1^2}, \quad (39)$$

between the first and second solution of a homogeneous second order differential equation, the second solution is

$$V_2(r, \varepsilon) = k_2(r^2 + \varepsilon^2) \int^r \frac{dr'}{(r'^2 + \varepsilon^2)^2}. \quad (40)$$

We choose the lower limit to be  $r = \infty$  so that (see Appendix)

$$V_2(r, \varepsilon) = -k_2 \frac{(r^2 + \varepsilon^2)}{2\varepsilon^3} \left[ \arctan \varepsilon/r - \frac{\varepsilon/r}{(1 + (\varepsilon/r)^2)} \right]. \quad (41)$$

If we let  $\varepsilon \rightarrow 0$ , we should get (to match with the Newtonian solution for large  $r$ )

$$V_2(r, 0) = -k_2 r^2 \int_r^\infty \frac{dr'}{r'^4} = -\frac{k_2 r^2}{3r^3} = -\frac{k_2}{3r} = -\frac{r_s}{r}, \quad (42)$$

so we take

$$k_2 = 3r_s. \quad (43)$$



As a check, let us check that our integrated result (41) before we take  $\varepsilon \rightarrow 0$  has this same limit

$$\begin{aligned}
& -3r_s \frac{(r^2 + \varepsilon^2)}{2\varepsilon^3} \left[ \arctan \varepsilon/r - \frac{\varepsilon/r}{(1 + (\varepsilon/r)^2)} \right] \\
\rightarrow & -3r_s \left( \frac{r^2}{2\varepsilon^3} + \frac{1}{2\varepsilon} \right) \left( \frac{\varepsilon}{r} - \frac{1}{3} \left( \frac{\varepsilon}{r} \right)^3 - \frac{\varepsilon}{r} (1 - (\varepsilon/r)^2) \right) \\
\rightarrow & -3r_s \left( \frac{r^2}{2\varepsilon^3} + \frac{1}{2\varepsilon} \right) \left( \frac{\varepsilon}{r} - \frac{\varepsilon^3}{3r^3} - \frac{\varepsilon}{r} + \frac{\varepsilon^3}{r^3} \right) = -3r_s \left( \frac{r^2}{2\varepsilon^3} + \frac{1}{2\varepsilon} \right) \left( \frac{2\varepsilon^3}{3r^3} \right) \\
\rightarrow & -\frac{r_s}{r}.
\end{aligned} \tag{44}$$

Thus, our general solution to Eq. (37) is

$$V(r, \varepsilon) = k_1(r^2 + \varepsilon^2) + 3r_s \frac{(r^2 + \varepsilon^2)}{2\varepsilon^3} \left[ \frac{\varepsilon r}{(r^2 + \varepsilon^2)} - \arctan \varepsilon/r \right]. \tag{45}$$

with the corresponding metric given by

$$\begin{aligned}
g_{00} &= -e^{2\nu} = -\frac{1}{1 + V(r, \varepsilon)} = 1/g^{00} \\
g_{11} &= e^{2\lambda} = e^{-2\nu} = 1 + V(r, \varepsilon) = 1/g^{11}, \\
g_{22} &= r^2 = 1/g^{22}, \\
g_{33} &= r^2 \sin^2 \theta = 1/g^{33}. \\
g_{\mu\nu} &= 0, \quad \mu \neq \nu.
\end{aligned} \tag{46}$$

Now, let us determine what the density and pressure are for the limit of small  $\varepsilon$ . This will provide us with insight into the nature of the source terms. The simplest way is to evaluate  $p = -\frac{V'}{r} - \frac{V}{r^2}$ . Details omitted here are given in the Appendix. There we obtain

$$\begin{aligned}
& p(r, \varepsilon) \\
&= p_1(r, \varepsilon) + p_2(r, \varepsilon) \\
&= -k_1 \left( \frac{2r}{(r^2 + \varepsilon^2)^{1/2}} + 1 \right) - 3r_s \left[ \frac{1}{(r^2 + \varepsilon^2)^{1/2}} \frac{r}{\varepsilon^3} \left( \frac{\varepsilon}{r} - \arctan \varepsilon/r \right) + \frac{1}{2\varepsilon^3} \left( \frac{\varepsilon r}{(r^2 + \varepsilon^2)} - \arctan \varepsilon/r \right) \right]
\end{aligned}$$

where  $p_1$  is the pressure term that arises from the oscillator-like part of the solution while  $p_2$  is the pressure term that arises from the Newtonian-like part of the solution. For  $r \neq 0$  and  $\varepsilon \rightarrow 0$

$$\begin{aligned}
p_2 &\rightarrow -(1 - \varepsilon^2/2r^2) \frac{k_2}{3r^3} + 3r_s \frac{1}{3r^3} \\
&\rightarrow 0,
\end{aligned} \tag{48}$$

while for  $r = 0$  and  $\varepsilon \rightarrow 0$  this becomes

$$\begin{aligned}
p_2 &\rightarrow -\frac{1}{\varepsilon} \frac{k_2}{\varepsilon^3} (\varepsilon - 0) - 3r_s \frac{1}{2\varepsilon^3} \left[ 0 - \frac{\pi}{2} \right] \\
&= \frac{3r_s}{\varepsilon^3} \left( \frac{\pi}{4} - 1 \right).
\end{aligned} \tag{49}$$

These two limits taken together have the appearance an attractive delta function. To complete the verification let us check by integrating the Newtonian term for the density, using the divergence theorem, that we obtain the appropriate constant (independent of  $\varepsilon$ )

$$\begin{aligned}
\int d^3r \rho_2(r, \varepsilon) &= - \int d^3r p_2(r, \varepsilon) \\
&= 3r_s \int d^3r \left[ \frac{1}{(r^2 + \varepsilon^2)^{1/2}} \frac{r}{\varepsilon^3} \left( \frac{\varepsilon}{r} - \arctan \varepsilon/r \right) + \frac{1}{2\varepsilon^3} \left[ \frac{\varepsilon r}{(r^2 + \varepsilon^2)} - \arctan \varepsilon/r \right] \right] \\
&= - \int d^3r \left[ -\frac{V'}{\bar{r}} - \frac{V}{\bar{r}^2} \right] = \frac{1}{2} \int d^3r \left( V'' + \frac{2V'}{\bar{r}} \right) \\
&\rightarrow \frac{1}{2} \int d^3r \left( V'' + \frac{2V'}{r} \right) = \frac{1}{2} \int d^3r \nabla^2 V = \frac{1}{2} \lim_{R \rightarrow \infty} R^2 4\pi V'(R) \\
&= 2\pi \lim_{R \rightarrow \infty} R^2 \frac{R 3r_s}{\varepsilon^3} \left( \frac{\varepsilon}{R} - \arctan \varepsilon/R \right) = 2\pi \lim_{R \rightarrow \infty} \frac{R^3 3r_s}{\varepsilon^3} \left( \frac{\varepsilon}{R} - \frac{\varepsilon}{R} + \frac{1}{3} \left( \frac{\varepsilon}{R} \right)^3 \right) \\
&= 2\pi r_s = 4\pi GM. \tag{50}
\end{aligned}$$

Such would not be the case for the homogeneous equation which would give zero for the integrated density (see below Eq. (32)). Thus, in the limit  $\varepsilon \rightarrow 0$  where our  $\varepsilon$ -modified Einstein equations become the actual Einstein equation, the integral of the density over an arbitrarily small volume remains a constant independent of  $\varepsilon$ . So, defining (rearranged)

$$\delta^3(\mathbf{r}, \varepsilon) \equiv \frac{\rho_2(r, \varepsilon)}{4\pi GM} = -\frac{3}{2\pi\varepsilon^3} \left[ \frac{\varepsilon/r}{(1 + (\varepsilon/r)^2)^{1/2}} \left( 1 + \frac{1}{2} \frac{1}{(1 + (\varepsilon/r)^2)^{1/2}} \right) - \arctan \varepsilon/r \left( \frac{1}{(1 + (\varepsilon/r)^2)^{1/2}} + \frac{1}{2} \right) \right], \tag{51}$$

we have

$$\int d^3r \delta^3(\mathbf{r}, \varepsilon) = 1. \tag{52}$$

Our  $\delta^3(\mathbf{r}, \varepsilon)$  therefore has the requisite properties for a distribution that in the limit represents a Dirac delta function. Its value for  $\mathbf{r} \neq \mathbf{0}$  tends to zero as  $\varepsilon \rightarrow 0$  and its integral over all space is unity. This establishes that for  $k_1 = 0$ , the source term is non-zero and has the property of a sharply confined distribution. What makes this source distinct from others[7], beginning with the matching solution of Schwarzschild[8] to an incompressible fluid confined within a finite spherical surface, is that within this sharply confined region, the pressure is negative.

### 2.2.1 The Schwarzschild Limit

We consider in this section the potential-like function  $V(r, \varepsilon)$  given in Eq. (45). We wish to determine in what sense that, if we choose  $k_1 = 0$ , the second portion of  $V(r, \varepsilon)$  for  $\varepsilon > 0$  sufficiently small agrees with the Schwarzschild solution  $V_s(r) = -r_s/r$  for a given range of  $r$ . Let us make this statement precise. We show that for a positive  $\delta > 0$  that if,

$$\frac{\varepsilon}{r_s} \equiv \epsilon < \delta. \tag{53}$$

then

$$\left(\frac{r}{r_s}\right)^3 |V(r, \varepsilon) - V_s(r)| < \frac{\delta^2}{5}. \quad (54)$$

for  $r$  in the range  $\varepsilon < r < \infty$ . If  $\delta < 1$ , then the range for  $r$  of agreement between the potentials in the above sense would extend down below the Schwarzschild radius with no upper bound.

To show this we consider the Taylor series for  $V(r, \varepsilon) - V_s(r)$  in  $\varepsilon$  about  $\varepsilon = 0$ .

$$V(r, \varepsilon) - (-)\frac{r_s}{r} = \frac{3r_s}{2\varepsilon^3}[\varepsilon r - (r^2 + \varepsilon^2) \arctan \varepsilon/r] + \frac{r_s}{r} \quad (55)$$

The series for  $\arctan \varepsilon/r$  converges for  $r > \varepsilon$ . Expanding we find that

$$\begin{aligned} V(r, \varepsilon) - (-)\frac{r_s}{r} &= \frac{3r_s}{2\varepsilon^3}[\varepsilon r - r^2(1 + \frac{\varepsilon^2}{r^2}) \sum_{n=0}^{\infty} \frac{(-)^n}{2n+1} \left(\frac{\varepsilon}{r}\right)^{2n+1}] + \frac{r_s}{r} \\ &= \frac{3r_s}{2\varepsilon^3}[\varepsilon r - r^2(\sum_{n=0}^{\infty} \frac{(-)^n}{2n+1} \left(\frac{\varepsilon}{r}\right)^{2n+1} + \sum_{n=0}^{\infty} \frac{(-)^n}{2n+1} \left(\frac{\varepsilon}{r}\right)^{2n+3})] + \frac{r_s}{r} \\ &= \frac{3r_s}{2\varepsilon^3}[-r^2(\sum_{n=2}^{\infty} \frac{(-)^n}{2n+1} \left(\frac{\varepsilon}{r}\right)^{2n+1} - \sum_{n=2}^{\infty} \frac{(-)^n}{2n-1} \left(\frac{\varepsilon}{r}\right)^{2n+1})] \\ &= \frac{3r_s}{r^3} \left[\left(\frac{\varepsilon}{r}\right)^2 \left(\sum_{n=0}^{\infty} \frac{1}{(2n+5)(2n+3)}\right) (-)^n \left(\frac{\varepsilon}{r}\right)^{2n}\right] \end{aligned}$$

Thus, with  $\frac{\varepsilon}{r_s} = \epsilon < \delta$ , we have since  $\epsilon \frac{r_s}{r} < 1$

$$\begin{aligned} \left(\frac{r}{r_s}\right)^3 \left| \frac{r}{r_s} V(r, \varepsilon) + 1 \right| &= \frac{3\varepsilon^2}{r_s^2} \left[ \left(\sum_{n=0}^{\infty} \frac{1}{(2n+5)(2n+3)}\right) (-)^n \left(\frac{\varepsilon}{r}\right)^{2n} \right] \\ &= \epsilon^2 \left[ \left(\sum_{n=0}^{\infty} \frac{3}{(2n+5)(2n+3)}\right) (-)^n \left(\epsilon \frac{r_s}{r}\right)^{2n} \right] \\ &= \frac{\epsilon^2}{5} \left( 1 + \sum_{n=1}^{\infty} \frac{15}{(2n+5)(2n+3)} (-)^n \left(\epsilon \frac{r_s}{r}\right)^{2n} \right) \\ &< \frac{\epsilon^2}{5} < \frac{\delta^2}{5}. \end{aligned}$$

So, in the above sense, we have demonstrated how the Schwarzschild solution can be viewed as a valid approximation for a bona fide nonlinear solution of the full non-homogeneous Einstein equation.

### 3 Discussion

We have found that the general solution of the Einstein equation for the special case of  $p = -\rho$  to yield a metric governed by a linear combination of a Newtonian and simple harmonic oscillator potential, two independent solution of a

linear second order differential equation for the potential-like function  $V$ . If the behavior about the origin is not handled carefully, the simple harmonic contribution gives a constant density and pressure while the Newtonian contribution gives rise to no point-like (delta function) source. With  $k_1 = 0$ , this is the usual no source or vacuum solution. Handled more carefully by the method indicated we obtain a density and pressure terms sharply peaked about the origin, with a unit volume integral. The density corresponding to the elastic portion of the metric ranges from  $k_1$  for to  $3k_1$ .

Do our mathematical solutions of the inhomogeneous Einstein equation have any physical significance? Superimposed on our point-like density is a density ranging between two constants, the source of the harmonic oscillator potential-like function  $V_1(r, \varepsilon)$  which behaves like  $k_1 r^2$  for sufficiently small  $r$ . The static and spherically symmetric metric we started with is distinct from the standard time-dependent Friedmann-Lematre-Robertson-Walker (FLRW) metric. It is not intended to relate to the universe as a whole but rather to the field produced by a single source. There is some superficial similarity between our solution and the so-called dark energy and inflation solutions of modern cosmology since both involve a pressure with an opposite sign of the density. It may be of just academic interest that such potential-like functions not only proportional respectively to  $1/r$  but also to the otherwise ubiquitous  $r^2$  potential correspond to the solution of the Einstein equation under these circumstances. A positive or negative sign of  $k_1$  would give a negative or positive pressure and an attractive or repulsive force that would increase in magnitude with distance. Obviously since there is no evidence for such long range and static increasing forces,  $k_1$  must be virtually infinitesimal if not zero. Note that in nonrelativistic potential theory a constant density would give rise to an attractive Hooke's law force. However, that follows from Gauss' law applied to an inverse square field. The  $r^2$  potential like function discussed here is completely independent from the  $1/r$  potential. Another factor to point out is that the functional form of the density or pressure due to the harmonic oscillator potential is fixed by our solution to the Einstein equations themselves, it is not imposed. The only imposition we made on the density and pressure is that they be the negative of one another. From a mathematical point of view there is no distinction between the solutions discussed here for the inhomogeneous Einstein equation and the one we would have obtained by adding a term  $-pg_{\mu\nu}$  to the left hand side of the Einstein equation and viewing it as an addition to the equation, in analogy to the alternative explanation of dark energy. The difference here is that  $p$  being a constant would be an outcome of the modified Einstein equations and not an imposed functional form.

It was one of Einstein's early goals, although he never succeeded, to incorporate Mach's principle in his general theory of relativity. It has been generally regarded that general relativity does not embody Mach's principle, that is that geometry can exist independent of matter. It was the Schwarzschild solution that seemed to bring this idea its early but reluctant acceptance. That is, a geometry arises from the absence of a source term, from the vacuum. Of course, in the practical applications of the Schwarzschild solution to the precession prob-

lem of Mercury and the bending of light, it was always assumed that looming behind the formal sourceless equation was a real sun. Nevertheless, a possible formal interpretation is that a curved space exists without an identifiable source, thus obviating the need for Mach's principle

Our result has been to replace the Schwarzschild solution to the sourceless spherically symmetric static environment, which then, as now seems to allow the existence of non-trivial spacetime curvature in absence of any matter, with a solution that does not correspond to a sourceless environment but yet leads nevertheless to a metric that can approach the Schwarzschild with arbitrary accuracy in an asymptotic way. In doing so, for this particular case at least, Mach's principle, the idea that geometry emerges as an interaction between an identifiable matter term and geometry is preserved[9].

## A Details of Solution $V_2(r, \varepsilon)$ of $\varepsilon$ -Modified Einstein Equations

Here we consider some details omitted from the text. The expression for  $V_2(r, \varepsilon)$  is obtained below,

$$\begin{aligned}
V_2(r, \varepsilon) &= -k_2(r^2 + \varepsilon^2) \int_r^\infty \frac{dr'}{(r'^2 + \varepsilon^2)^2} = (r' = x\varepsilon) \\
&= -k_2 \frac{(r^2 + \varepsilon^2)}{\varepsilon^3} \int_{r/\varepsilon}^\infty \frac{dx}{(x^2 + 1)^2} = (x = 1/y, dx = -dy/y^2) \\
&= -k_2 \frac{(r^2 + \varepsilon^2)}{\varepsilon^3} \int_0^{\varepsilon/r} \frac{y^2 dy}{(y^2 + 1)^2} \\
&= -k_2 \frac{(r^2 + \varepsilon^2)}{\varepsilon^3} \left[ \int_0^{\varepsilon/r} \frac{dy}{(y^2 + 1)} - \int_0^{\varepsilon/r} \frac{dy}{(y^2 + 1)^2} \right] \\
&= -k_2 \frac{(r^2 + \varepsilon^2)}{\varepsilon^3} \left[ \arctan \varepsilon/r - \left[ \frac{\varepsilon/r}{2(1 + (\varepsilon/r)^2)} + \frac{1}{2} \int_0^{\varepsilon/r} \frac{dy}{(y^2 + 1)} \right] \right] \\
&= -k_2 \frac{(r^2 + \varepsilon^2)}{2\varepsilon^3} \left[ \arctan \varepsilon/r - \frac{\varepsilon/r}{(1 + (\varepsilon/r)^2)} \right] \tag{A.1}
\end{aligned}$$

Next we combine terms to determine the expression for the pressure given in Eq. (??)

$$\begin{aligned}
p &= -\frac{V'}{\bar{r}} - \frac{V}{\bar{r}^2}, \\
\frac{V}{\bar{r}^2} &= k_1 + k_2 \frac{1}{2\varepsilon^3} \left[ \frac{\varepsilon r}{(r^2 + \varepsilon^2)} - \arctan \varepsilon/r \right], \\
V' &= 2k_1 r + k_2 \frac{2r}{2\varepsilon^3} \left[ \frac{\varepsilon r}{(r^2 + \varepsilon^2)} - \arctan \varepsilon/r \right] + k_2 \frac{(r^2 + \varepsilon^2)}{2\varepsilon^3} \left[ \frac{\varepsilon}{(r^2 + \varepsilon^2)} - \frac{2\varepsilon r^2}{(r^2 + \varepsilon^2)^2} + \frac{\varepsilon}{(r^2 + \varepsilon^2)} \right] \\
&= 2k_1 r + k_2 \frac{r}{\varepsilon^3} \left[ \frac{\varepsilon r}{(r^2 + \varepsilon^2)} - \arctan \varepsilon/r \right] + k_2 \frac{(r^2 + \varepsilon^2)}{\varepsilon^2} \left[ \frac{\varepsilon^2}{(r^2 + \varepsilon^2)^2} \right] \\
&= 2k_1 r + k_2 \frac{r}{\varepsilon^3} \left( \frac{\varepsilon r}{(r^2 + \varepsilon^2)} - \arctan \varepsilon/r \right) + \frac{k_2}{(r^2 + \varepsilon^2)} \\
&= 2k_1 r + k_2 \left( \frac{r^2/\varepsilon^2}{(r^2 + \varepsilon^2)} - \frac{r}{\varepsilon^3} \arctan \varepsilon/r + \frac{1}{(r^2 + \varepsilon^2)} \right) \\
&= 2k_1 r + k_2 \left( \frac{1}{\varepsilon^2} - \frac{r}{\varepsilon^3} \arctan \varepsilon/r \right) = 2k_1 r + \frac{rk_2}{\varepsilon^3} \left( \frac{\varepsilon}{r} - \arctan \varepsilon/r \right) \tag{A.2}
\end{aligned}$$

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