Rarefaction Shock Waves in Collisionless Plasma with Electronic Beam

Victor Ts. Gurovich† and Leonid G. Fel‡

†Department of Physics, Technion, Haifa 32000, Israel
‡Department of Civil Engineering, Technion, Haifa 32000, Israel

Abstract

We show that an electronic beam passing through the collisionless plasma of the ”cold” ions and the ”hot” Boltzmann electrons can give rise to the propagation of the supersonic ion-acoustic rarefaction shock waves. These waves are analogous to those predicted by Zeldovich [5] in gasodynamics and complementary to the ion-acoustic compression shock waves in collisionless plasma described by Sagdeev [3].

Keywords: Collisionless plasma, Supersonic ion-acoustic rarefaction shock waves

PACS 2006: Primary – 47.40.-x, Secondary – 52.35.Tc.

1 Introduction

It is well known that the supersonic compression shock (CS) waves can propagate in the ideal gas, but their alternative - the supersonic rarefaction shock (RS) waves - cannot. In 1937 Zeldovich [5] has shown that the interaction between the gas molecules possessing the repulsive and attractive parts (e.g., the Van der Waals' gas) results in substantial changes: the only RS waves could been stably propagated, but the CS waves are unstable.

In the 1960th Sagdeev [3] has predicted the CS wave which may propagate in the collisionless (CL) plasma. This appears when the CL plasma comprises the ”cold” ions and the ”hot” Boltzmann electrons and the mean free path is determined by the Debye radius of electrons. The shock wave is arisen by the supersonic compression soliton disturbed by dissipative effects. Such CS waves are analogous to those observed in gasodynamics.

Regarding the RS waves in the CL plasma, a study of such waves dates back to the papers [1] and [4] for the media comprised the electrons with bi-Maxwellian distribution function and ”cold” ions.

In the present article we suggest another kind of the RS waves generation in the CL plasma based on the supersonic rarefaction solitons. We consider a problem in two steps. First, we show that the supersonic rarefaction soliton (of density and electric potential) can propagate in the CL plasma supplemented by the electronic beam. Next, being partially reflected by the soliton's electric potential, the beam disturbs its symmetric shape in such a way, that behind the maximal value of potential there appear nonlinear oscillations.

2 Setup of the Problem

Consider the CL plasma with the Boltzmann distribution $N_e = N_{eo}\exp\left(e\phi(x)/\kappa T_e\right)$ of the "hot" electrons, where N_{eo} is a concentration of hot electrons in homogeneous plasma, T_e is a temperature of electrons, e and κ stand for the charge of electron and the Boltzmann constant, respectively. The electric potential $\phi(x)$ is varying along the x direction together with the charge density. Such plasma with electronic beam can be realized in experiments with a hollow anode plasma source [2]. In what follows, we use the renormalized potential $\psi(x) = -|e|\phi(x)/\kappa T_e$ and choose the reference frame related to the electric potential and normalized coordinate $\xi = x/D$ where $D = \sqrt{\varepsilon_0 T_e/N_{io}e^2}$ denotes the Debay radius and N_{io} and ε_0 stand for homogeneous ions density and dielectric vacuum permittivity, respectively. In this reference frame the ions are running with velocity $V(\xi)$ toward the wave front which is defined by requirement $V \to U$ when $\psi \to 0$. Here U denotes the velocity of the homogeneous flow of plasma.

Equations of the energy and mass conservations for cold ions describe their 1-dim stationary motion and lead [3] to the non homogeneous ions density $N_i(\xi)$,

$$N_i(\xi) = \frac{N_{io}}{\sqrt{1+2\psi(\xi)/\mathsf{M}^2}}\ , \quad \mathsf{M} = \frac{U}{C}\ , \quad C = \sqrt{\frac{\kappa T_e}{m_i}}\ , \tag{1}$$

where m_i and C denote the mass of the ion and the velocity of the ion sound. The Max number for the ions motion is denoted by M.

Let $\mathcal{E} = mv^2/(2\kappa T_e) + \psi(\xi)$ be an energy of the single electron of the beam in the κT_e units. Then the density $N_b(\psi)$ of the beam electrons is given by the distribution function $f(\xi, v)$,

$$N_b(\xi) = \int_0^\infty f(\xi, v)dv = \sqrt{\frac{\kappa T_e}{2m}} \int_0^\infty \frac{f(\mathcal{E})d\mathcal{E}}{\sqrt{\mathcal{E} - \psi(\xi)}}\ , \quad N_{bo} = \sqrt{\frac{\kappa T_e}{2m}} \int_0^\infty \frac{f(\mathcal{E})d\mathcal{E}}{\sqrt{\mathcal{E}}}\ , \tag{2}$$

where N_{bo} denotes a density of the electronic beam in homogeneous plasma, and m and v stand for mass and velocity of electron.

The dimensionless potential $\psi(\xi)$ is a smooth function and obeys the Poisson equation,

$$\frac{d^2\psi}{d\xi^2} = \frac{1}{\sqrt{1+2\psi(\xi)/\mathsf{M}^2}} - Ae^{-\psi(\xi)} - B(\xi)\ , \tag{3}$$

where

$$B(\xi) = \frac{N_b(\xi)}{N_{io}} \; , \quad B(0) = \frac{N_{bo}}{N_{io}} \; , \quad A = \frac{N_{eo}}{N_{io}} \; , \quad A + B(0) = 1 \; , \quad A < 1 \; . \tag{4}$$

The last equality holds due to the quasineutrality of homogeneous plasma.

Regarding $N_b(\psi)$, in this paper we deal with two different types of the density distribution of electronic beam in accordance with two posed problems: the existence of the supersonic rarefaction soliton (see section 3) and appearance of the supersonic RS waves (see section 4).

3 The Rarefaction Soliton

Consider the electronic beam with a distribution function $f(\mathcal{E})$ given as follows,

$$f(\mathcal{E}) > 0 \quad \text{if} \quad \psi \ll \mathcal{E}_1 \leq \mathcal{E} \leq \mathcal{E}_2 \quad \text{and} \quad f(\mathcal{E}) = 0 \quad \text{if} \quad \mathcal{E} < \mathcal{E}_1 \text{ or } \mathcal{E} > \mathcal{E}_2 \; . \tag{5}$$

Then, by (2), (4) and (5) we have $B(\xi) \simeq B(0)$, and equation (3) can been integrated as follows,

$$\frac{1}{2}\left(\frac{d\psi}{d\xi}\right)^2 = \mathsf{M}^2\left(\sqrt{1+\frac{2\psi(\xi)}{\mathsf{M}^2}}-1\right) + A\left(e^{-\psi}-1\right) - B(0)\psi + K \; , \quad K = const \; . \tag{6}$$

Equation (6) gives rise to soliton solution if $d\psi/d\xi \to 0$ when $\psi \to 0$, that immediately requires $K = 0$. Rewrite equation (6) for $\psi \ll 1$ and preserving the ψ- and ψ^2 terms,

$$\frac{1}{2}\left(\frac{d\psi}{d\xi}\right)^2 = \psi(1 - A + B(0)) + \frac{\psi^2}{2}\left(A - \frac{1}{\mathsf{M}^2}\right) \; . \tag{7}$$

The linear in ψ term in (7) is vanishing due to (4), and what is left results in $\mathsf{M} \geq 1/\sqrt{A}$. Keeping in mind $A < 1$ we get the supersonic rarefaction soliton, $\mathsf{M} > 1$.

Since $\psi(\xi)$ is a smooth function everywhere (also when $\psi \geq 1$) and $\psi(\xi)$ arrives its maximal value ψ_{max} then by (6) we get

$$\mathsf{M}^2\left(\sqrt{1+\frac{2\psi_{max}}{\mathsf{M}^2}}-1\right) + A\left(e^{-\psi_{max}}-1\right) - B(0)\psi_{max} = 0 \; . \tag{8}$$

The last equation can be resolved analytically as $\mathsf{M} = \mathsf{M}(A, B(0), \psi_{max})$. E.g., for the experimental date [2] $B(0) \simeq 0.3$, $A \simeq 0.7$ and $\psi_{max} \simeq 20$ we obtain $\mathsf{M} \simeq 1.035$. In Figure 1 we present the phase portrait of equation (7) for the parameters given above. The center (*the singular point*) is corresponded to $K \simeq 0.579$ while the separatrix (*soliton*) is corresponded to $K = 0$.

Up to the sign of the potential $\psi(\xi)$ the phase portrait at Figure 1 is similar to that of the nonlinear ion-acoustic travelling waves of compression [3]. The main difference appears: in the latter case the maximal value of the ion density $N_i(\xi)$ and the maximal value of the potential $\psi(\xi)$ occur at the same coordinate ξ, while in the former case the minimal value of the ion density occur when the potential arrives it maximal value. This is why the soliton solution $\psi(\xi)$ is accompanied by the ionic rarefaction.

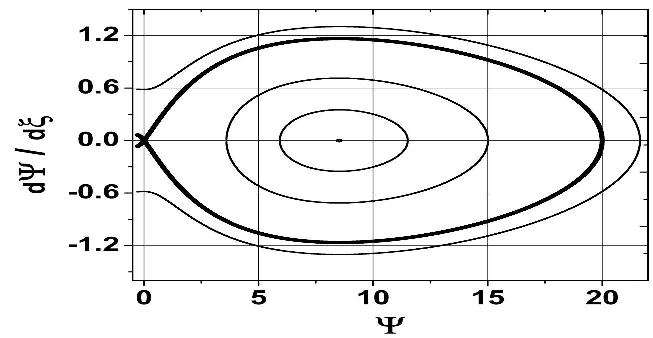

Figure 1: Phase portrait of Eqn (7) for travelling waves of rarefaction; a soliton is drawn in bold.

4 The Rarefaction Shock Waves

In the model of the CS waves [3] a symmetric shape of soliton is disturbed when a small portion of ions (with a lower energy) is reflected by potential barrier, but a large portion of ions (with a higher energy) is passed throughout it. In our case the rarefaction soliton will be disturbed by electrons of the beam which are reflected by the barrier.

Choose the step-like distribution function $f(\mathcal{E})$,

$$f(\mathcal{E}) = \sqrt{\frac{8m}{\kappa T_e}} \frac{N_{bo}}{\sqrt{\mathcal{E}_2} - \sqrt{\mathcal{E}_1}}\ \Theta(\mathcal{E}_2 - \mathcal{E})\Theta(\mathcal{E} - \mathcal{E}_1)\ , \tag{9}$$

where $\Theta(a)$ denotes the Heviside step function. Substituting (9) into (2) we get a relative density of the transmitted electronic beam,

$$\vartheta_1(\xi) = \frac{B(0)}{2(\sqrt{\mathcal{E}_2} - \sqrt{\mathcal{E}_1})} \int_{\mathcal{E}_1}^{\mathcal{E}_2} \frac{d\,\mathcal{E}}{\sqrt{\mathcal{E} - \psi(\xi)}} = B(0) \frac{\sqrt{\mathcal{E}_2 - \psi(\xi)} - \sqrt{\mathcal{E}_1 - \psi(\xi)}}{\sqrt{\mathcal{E}_2} - \sqrt{\mathcal{E}_1}} \qquad \mathcal{E}_2 > \mathcal{E}_1\ . \tag{10}$$

Formula (10) holds for the monotone growing potential $\psi(\xi)$ when $\psi(\xi) \leq \mathcal{E}_1 \leq \mathcal{E} \leq \mathcal{E}_2$, or $\xi \leq \xi_1$, where $\psi(\xi_1) = \mathcal{E}_1$ (see Figure 2). Here the incident electrons do not reflected by barrier but are transmitted if $\mathcal{E} > \mathcal{E}_1$.

Next, let us focus on the other case, $\mathcal{E}_1 < \psi(\xi) < \mathcal{E} < \mathcal{E}_2$, where $\psi_{max} = \psi(\xi_{max})$ and ξ_{max} stands for location of maximal soliton potential. Here the beam is partly penetrated into the soliton potential when $\xi_1 \leq \xi \leq \xi_{max}$, but the rest of electrons are reflected from the barrier when $\psi(\xi) \geq \mathcal{E}_1$, or $\xi \geq \xi_1$. It leads to the changes in formula (10)

$$\vartheta_2(\xi) = \overline{B}\sqrt{\mathcal{E}_2 - \psi(\xi)}\ , \quad \xi_1 \leq \xi \leq \xi_{max}\ , \quad \overline{B} = \frac{B(0)}{\sqrt{\mathcal{E}_2} - \sqrt{\mathcal{E}_1}}\ , \tag{11}$$

Finally, consider the last case $\psi_{max} < \mathcal{E} < \mathcal{E}_2$ (electrons do not reflect by the barrier)

$$\vartheta_3(\xi) = \overline{B}\left(\sqrt{\mathcal{E}_2 - \psi(\xi)} - \sqrt{\psi_{max} - \psi(\xi)}\right)\ . \tag{12}$$

For convenience, unify three functions $\vartheta_i(\xi)$, $i = 1, 2, 3$, given in different ranges of ξ by one $I(\xi)$ given in whole range of ξ,

$$\begin{aligned} I(\xi) \quad &= \quad \overline{B} \left(\sqrt{\mathcal{E}_2 - \psi(\xi)} - \Theta\left(\mathcal{E}_1 - \psi(\xi)\right) \Theta\left(\xi_{max} - \xi\right) \sqrt{\mathcal{E}_1 - \psi(\xi)} - \right. \\ & \left. \Theta\left(\xi - \xi_{max}\right) \sqrt{\psi_{max} - \psi(\xi)} \right) . \end{aligned} \tag{13}$$

In the similar way we can construct the relative density $R(\xi)$ of the electronic beam reflected from the barrier,

$$R(\xi) = \overline{B} \left(\sqrt{\psi_{max} - \psi(\xi)} - \Theta\left(\mathcal{E}_1 - \psi(\xi)\right) \sqrt{\mathcal{E}_1 - \psi(\xi)} \right) \ , \quad \xi \leq \xi_{max} \ . \tag{14}$$

A choice of arguments in the Θ-functions in (13) and (14) is motivated by need to represent the Poisson equation (3) as an autonomous differential equation.

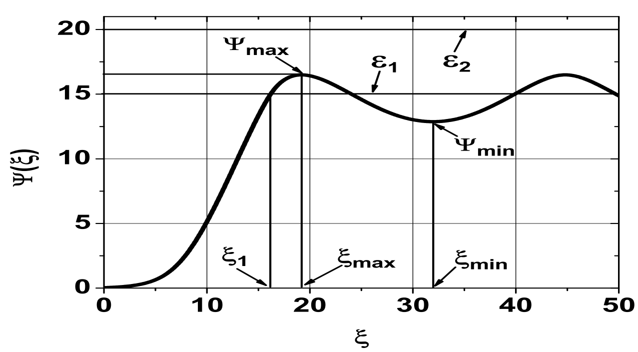

Figure 2: The electric potential $\psi(\xi)$ for the nonlinear RS wave. Its 1st maximum ψ_{max} satisfies inequality $\mathcal{E}_1 < \psi_{max} < \mathcal{E}_2$.

Substitute the entire density $B(\xi) = I(\xi) + R(\xi)$ of the beam into equation (3) and obtain its 1st integral,

$$\frac{1}{2} \left(\frac{d\psi}{d\xi} \right)^2 - V_- \left(\psi \right) = 0 \ , \quad \xi \leq \xi_{max} \ , \tag{15}$$

where the quasipotential $V_-(\psi)$ is given by

$$\begin{aligned} V_-(\psi) \quad &= \quad \mathsf{M}^2 \left(\sqrt{1 + \frac{2\psi}{\mathsf{M}^2}} - 1 \right) + A \left(e^{-\psi} - 1 \right) + \\ & \frac{2\overline{B}}{3} \left[(\mathcal{E}_2 - \psi)^{3/2} + (\psi_{max} - \psi)^{3/2} - 2(\mathcal{E}_1 - \psi)^{3/2} \Theta(\mathcal{E}_1 - \psi) + 2\mathcal{E}_1^{3/2} - \mathcal{E}_2^{3/2} - \psi_{max}^{3/2} \right] . \end{aligned} \tag{16}$$

In (15) we have taken a zero's value for the integration constant (see (6) with $K = 0$) to provide the requirement $d\psi/d\xi \to 0$ when $\psi \to 0$.

Rewrite the quasipotential (16) for $\psi \ll 1$ and preserve the non-linear terms up to ψ^2,

$$\begin{aligned} V_-(\psi) &= \psi\left[1 - A - \overline{B}\left(\sqrt{\mathcal{E}_2} + \sqrt{\psi_{max}} - 2\sqrt{\mathcal{E}_1}\right)\right] \\ &+ \frac{\psi^2}{2}\left[-\frac{1}{\mathsf{M}^2} + A + \frac{\overline{B}}{2}\left(\frac{1}{\sqrt{\mathcal{E}_2}} + \frac{1}{\sqrt{\psi_{max}}} - \frac{2}{\sqrt{\mathcal{E}_1}}\right)\right] . \end{aligned} \tag{17}$$

Note that by (10,14) the term $\overline{B}\left(\sqrt{\mathcal{E}_2} + \sqrt{\psi_{max}} - 2\sqrt{\mathcal{E}_1}\right) = \overline{B}\left(\sqrt{\mathcal{E}_2} - \sqrt{\mathcal{E}_1}\right) + \overline{B}\left(\sqrt{\psi_{max}} - \sqrt{\mathcal{E}_1}\right)$ which enters into the linear in ψ term in (17), gives a total density of the electronic beam including incidence and reflection as well. Then the whole linear in ψ term in (17) disappears due to the quasineutrality of homogeneous plasma in the general case (when beam's reflection exists),

$$A + \overline{B}\left(\sqrt{\mathcal{E}_2} + \sqrt{\psi_{max}} - 2\sqrt{\mathcal{E}_1}\right) = 1 \quad \rightarrow \quad A < 1 . \tag{18}$$

Substituting the quadratic in ψ term of (17) into (15) we find the lower bound for M,

$$\frac{1}{\mathsf{M}^2} \leq A + \frac{\overline{B}}{2}\left(\frac{1}{\sqrt{\mathcal{E}_2}} + \frac{1}{\sqrt{\psi_{max}}} - \frac{2}{\sqrt{\mathcal{E}_1}}\right) \quad \rightarrow \quad \frac{1}{\mathsf{M}^2} < A . \tag{19}$$

Combining (18) and (19) we arrive at $\mathsf{M} > 1/A > 1$, i.e., the supersonic RS wave.

An exact value of the Mach number can be found if we consider equation (15) for $\xi = \xi_{max}$ and $\psi = \psi_{max} > \mathcal{E}_1$, i.e., when $d\psi/d\xi = 0$,

$$\sqrt{1 + \frac{2\psi_{max}}{\mathsf{M}^2}} - 1 = \frac{A}{\mathsf{M}^2}\left(1 - e^{-\psi_{max}}\right) + \frac{2\overline{B}}{3\mathsf{M}^2}\left[\mathcal{E}_2^{3/2} + \psi_{max}^{3/2} - (\mathcal{E}_2 - \psi_{max})^{3/2} - 2\mathcal{E}_1^{3/2}\right] . \tag{20}$$

The last equality together with a quasineutrality condition (18) allow to obtain M and A if the other four parameters $\overline{B}$, $\mathcal{E}_1$, $\mathcal{E}_2$ and ψ_{max} are given.

Now, consider the Poisson equation (3) in the range $\xi \geq \xi_{max}$

$$\frac{1}{2}\left(\frac{d\psi}{d\xi}\right)^2 - V_+(\psi) = 0 , \quad \xi \geq \xi_{max} , \tag{21}$$

where the quasipotential $V_+(\psi)$ is given by

$$\begin{aligned} V_+(\psi) &= \mathsf{M}^2\left(\sqrt{1 + \frac{2\psi}{\mathsf{M}^2}} - \sqrt{1 + \frac{2\psi_{max}}{\mathsf{M}^2}}\right) + A\left(e^{-\psi} - e^{-\psi_{max}}\right) + \\ & \frac{2\overline{B}}{3}\left[(\mathcal{E}_2 - \psi)^{3/2} - (\psi_{max} - \psi)^{3/2} - (\mathcal{E}_2 - \psi_{max})^{3/2}\right] . \end{aligned} \tag{22}$$

In (22) we have taken non-zero's value for the integration constant (see (6) with $K \neq 0$) to provide the requirement $V_+(\psi_{max}) = 0$. Note that if the potential $\psi(\xi)$ arrives also its minimal value ψ_{min} at $\xi_{min} > \xi_{max}$ (see Figure 2), then $V_+(\psi_{min}) = 0$ as it follows from (21). By equalities (18), (20) and (22) it follows also that the value ψ_{min} is completely determined by four parameters $\overline{B}$, $\mathcal{E}_1$, $\mathcal{E}_2$ and ψ_{max}.

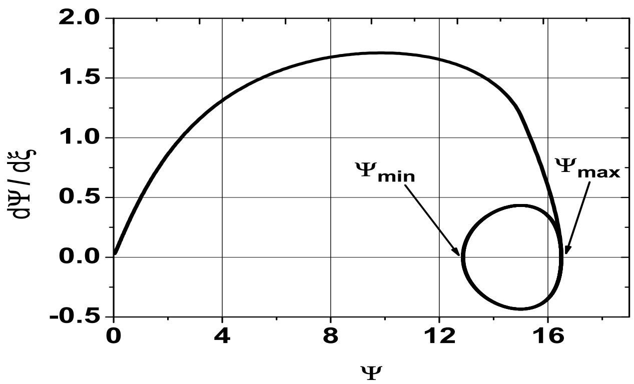

Figure 3: Phase portrait of equations (15) and (21) for the RS waves.

In Figure 3 we present the phase portraits of two first integrals (15) and (21) of the Poisson equation (3) for the date $\mathcal{E}_1 = 15$, $\mathcal{E}_2 = 20$, $\psi_{max} \simeq 16.5$ and $B(0) \simeq 0.15$ taken from experiments [2]. The values $A \simeq 0.803$, M $\simeq 1.433$ and $\psi_{min} \simeq 12.86$ were found by (18), (20) and (22), respectively.

Both portraits are unified in one smooth curve. Its 1st part (an arch in the upper half-plane, $d\psi/d\xi \geq 0$) is related to the soliton-like behavior of the electric potential ψ at the front of the RS waves, while its 2nd part (a closed loop) is related to the nonlinear oscillations of ψ.

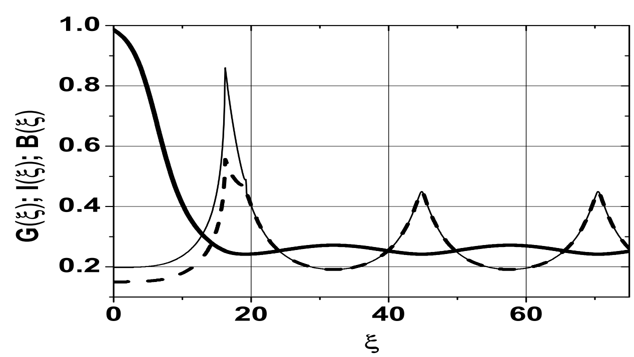

Figure 4: Densities' distribution of the total electronic beam $B(\xi)$, including reflections (*plain*); of the transmitted electronic beam $I(\xi)$ (*dashed*), and of the ions $G(\xi)$ (*bold*).

In Figure 4 we present three densities' distributions: the total density of the electronic beam $B(\xi)$ when the reflected electrons are accounted for; the density of the transmitted electronic beam $I(\xi)$ and also the density of the ions in plasma, $G(\xi)$. The maximal value of $G(\xi)$ is arrived before the wave front ($\xi = 0$) of electric potential $\psi(\xi)$, while the density of the ions in the region $\xi > 0$ is much less. This manifests that we get the rarefaction shock wave.

We appreciate useful discussions with L.P.Pitaevsky, Y.E.Krasik and I.D.Kaganovich.

References

[1] B.Bezzerides, D.W.Forslund and E.L.Lindman, *Physics Fluids*, **21**, 2179-2185 (1978)

[2] V.T.Gurovich, J.Z.Gleizer, Y.Bliokh and Y.E.Krasik, *Physics of Plasmas*, **13**, 073506, (2006)

[3] R.Z.Sagdeev, *Rev. Plasma Phys.*, **4**, 23 (1966)

[4] I.M.Wickens, J.E.Allen and P.T.Rumsby, *Phys. Rev. Lett.*, **41**, 243-246 (1978)

[5] Ya.B.Zeldovich, *Zh. Eksp. Teor. Fiz.*, **16**, 363 (1946)