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Inverse-moment chiral sum rules

E Golowich
golowich@physics.umass.edu

J Kambor

Inverse-Moment Chiral Sum Rules

Eugene Golowich

Department of Physics and Astronomy
University of Massachusetts
Amherst MA 01003 USA

and

Joachim Kambor

Division de Physique Théorique[1]
Institut de Physique Nucléaire
F-91406 Orsay Cedex, France

Abstract

A general class of inverse-moment sum rules was previously derived by the authors in a chiral perturbation theory (ChPT) study at two-loop order of the isospin and hypercharge vector-current propagators. Here, we address the evaluation of the inverse-moment sum rules in terms of existing data and theoretical constraints. Two kinds of sum rules are seen to occur, those which contain as-yet undetermined $\mathcal{O}(q^6)$ counterterms and those free of such quantities. We use the former to obtain phenomenological evaluations of two $\mathcal{O}(q^6)$ counterterms. Light is shed on the important but difficult issue regarding contributions of higher orders in the ChPT expansion.

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1 Introduction

Recently, we performed a calculation of the isospin and hypercharge vector current propagators ($\Delta_{33}^{\mu\nu}(q^2)$ and $\Delta_{88}^{\mu\nu}(q^2)$) to two-loop order in chiral perturbation theory.[1],[2] For the sake of reference, we display the resulting expression for $\Delta_{88}^{\mu\nu}(q^2)$,

$$
\begin{aligned}
\Pi_{88}(q^2, M_K^2) = {}& -2L_{10}^{(0)}(\mu^2) - 4H_1^{(0)}(\mu^2) \\
& -6\left[i\overline{B}_{21}(q^2, M_K^2) + \frac{1}{192\pi^2}\ln(M_K^2/\mu^2)\right] \\
& +\frac{q^2}{F_0^2}\left[18(i\overline{B}_{21}(q^2, M_K^2) + \frac{1}{192\pi^2}\ln(M_K^2/\mu^2))^2 \right. \\
& \left. -24L_9^{(0)}(\mu^2)(i\overline{B}_{21}(q^2, M_K^2) + \frac{1}{192\pi^2}\ln(M_K^2/\mu^2)) - P(\mu^2)\right] \\
& -\frac{4M_\pi^2}{F_0^2}\left[R(\mu^2) - \frac{1}{3}Q(\mu^2)\right] \\
& -\frac{8M_K^2}{F_0^2}\left[R(\mu^2) + \frac{2}{3}Q(\mu^2) - \frac{3}{32\pi^2}(L_9^{(0)}(\mu^2) + L_{10}^{(0)}(\mu^2))\ln(M_K^2/\mu^2)\right] (1)
\end{aligned}
$$

F_0 is the unrenormalized meson decay constant[2], the constants $L_9^{(0)}(\mu^2)$, $L_{10}^{(0)}(\mu^2)$, $H_1^{(0)}(\mu^2)$ are well-known $\mathcal{O}(q^4)$ counterterms[3] and μ is an arbitrary energy scale. The function $\overline{B}_{21}(q^2, M^2)$, defined in Ref. [1], has a branch point at $q^2 = 4M^2$. Otherwise, it is analytic elsewhere in the q^2 complex plane and in particular, has well-defined derivatives to all orders at the origin $q^2 = 0$. The first three terms on the right hand side of Eq. (1) are generated at one-loop order while all the others occur at two-loop order. Observe the presence of three constants, $P(\mu^2)$, $Q(\mu^2)$ and $R(\mu^2)$. These are new $\mathcal{O}(q^6)$ counterterms which must somehow be determined from experiment. It will be the purpose of this paper to thoroughly analyze 'inverse-moment' sum rules which can serve to phenomenologically constrain the counterterms P and Q. Since the class of such sum rules is of intrinsic theoretical interest, our study will actually have a more general relevance.

The existence of inverse-moment sum rules can be proved from analyticity properties of the vector propagators.[4],[5] Using the asymptotic behavior of the functions $\Pi_{aa}(q^2)$ ($a = 3, 8$ not summed) implied by the operator

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[2]It suffices to take $F_0 = F_\pi \simeq 0.0933$ GeV, with any error occurring at a higher order.

product expansion,[6],[7] it follows that the difference $\Pi_{33} - \Pi_{88}$ satisfies the unsubtracted dispersion relation,[1]

$$(\Pi_{33} - \Pi_{88})(q^2) = \int_{s_0}^{\infty} ds \, \frac{(\rho_{33} - \rho_{88})(s)}{s - q^2 - i\epsilon} \quad , \tag{2}$$

where the $\{\rho_{aa}\}$ are the corresponding spectral functions. The real part of this dispersion relation can be rewritten as a set of sum rules for negative moments of the difference of spectral functions,

$$\int_{s_0}^{\infty} ds \, \frac{(\rho_{33} - \rho_{88})(s)}{s^{n+1}} = \frac{1}{n!} \frac{d^n}{(dq^2)^n} (\Pi_{33} - \Pi_{88})(0) \qquad (n \geq 0) \; . \tag{3}$$

Similar sum rules can be derived for the individual vacuum polarizations. According to the known asymptotic behaviour, however, at least one subtraction is required to obtain convergent sum rules. Thus we have for $a = 3, 8$,

$$\int_{s_0}^{\infty} ds \, \frac{\rho_{aa}(s)}{s^{n+1}} = \frac{1}{n!} \frac{d^n}{(dq^2)^n} \Pi_{aa}(0) \qquad (n \geq 1) \; . \tag{4}$$

For definiteness, we shall focus on the quantities Π_{33} and $\Pi_{33} - \Pi_{88}$ in the analysis to follow.

It is convenient to categorize the inverse-moment sum rules as those which contain $\mathcal{O}(q^6)$ counterterms and those which do not. The former set consists of just two sum rules, obtained respectively by setting $n = 1$ in Eq. (4),

$$\int_{s_0}^{\infty} ds \, \frac{\rho_{33}(s)}{s^2} = -\frac{1}{F_0^2} P(\mu^2) + \frac{1}{480\pi^2} \left(\frac{1}{M_\pi^2} + \frac{1}{2M_K^2} \right)$$
$$+ \frac{1}{8F_0^2} \left(\frac{1}{16\pi^2} \right)^2 \left(1 + \frac{2}{3} \ln \frac{M_\pi^2}{\mu^2} + \frac{1}{3} \ln \frac{M_K^2}{\mu^2} \right)^2$$
$$- \frac{L_9^{(0)}(\mu^2)}{8\pi^2 F_0^2} \left(1 + \frac{2}{3} \ln \frac{M_\pi^2}{\mu^2} + \frac{1}{3} \ln \frac{M_K^2}{\mu^2} \right) \tag{5}$$

and $n = 0$ in Eq. (3),

$$\int_{s_0}^{\infty} ds \, \frac{(\rho_{33} - \rho_{88})(s)}{s} = \frac{16(M_K^2 - M_\pi^2)}{3F_0^2} Q(\mu^2) + \frac{1}{48\pi^2} \ln \frac{M_K^2}{M_\pi^2}$$
$$+ \frac{M_\pi^2}{F_0^2} \cdot \frac{L_9^{(0)}(\mu^2) + L_{10}^{(0)}(\mu^2)}{2\pi^2} \left[M_\pi^2 \ln \frac{M_\pi^2}{\mu^2} - M_K^2 \ln \frac{M_K^2}{\mu^2} \right] \; . \tag{6}$$

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It is from these two relations that we shall obtain determinations respectively of P and Q.

Alternatively, the latter set contains the remaining infinity of inverse-moment sum rules, with $n \geq 1$ in Eq. (3) and $n \geq 2$ in Eq. (4). Each of these sum rules involves only known quantities on the right-hand side. Although lacking direct information on the $\mathcal{O}(q^6)$ counterterms, such relations will nonetheless be seen to play a significant role in helping to properly interpret the inverse-moment sum rules.

2   Data Analysis

Sum rules such as those just discussed will be of practical interest only if one can numerically evaluate the various spectral integrals which appear. Because the s^{-n} moments ($n \geq 1$) strongly emphasize the low energy region, it turns out that the existing database is sufficient to yield reasonably accurate determinations. An extensive treatment of the phenomenological extraction of vector and axialvector spectral functions has been given in Ref. [8]. We refer the reader to that reference for examples of how hadronic production data both in $e^+ e^-$ annihilation and in $\tau(1777)$ decay is used to construct spectral functions.

Because the isospin and hypercharge spectral functions ρ_{33} and ρ_{88} involve isospin-one and isospin-zero channels respectively, we shall arrange the following discussion accordingly.

Isospin-one

(a) *Two-pion*: We adopt the two-pion component of Ref. [8], except for energies less than 400 MeV where we conjoin in a continuous manner the form implied by the two-loop expression for $\Delta_{33}^{\mu\nu}(q^2)$ obtained in Ref. [1]. In this way, we ensure the proper chiral behaviour near threshold up to two-loop order. The agreement between our two-pion spectral function and low-energy data is displayed in Fig. 1.
(b) *Four-pion*: We employ the form appearing in Ref. [8] without modification.
(c) *Isospin-one $K\bar{K}$*: We employ the approach given in Ref. [9] in which one adopts the SU(3) relation between pion and kaon form factors to infer the

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following relation between the corresponding e^+e^- cross sections,

$$\sigma_{K\bar{K}}^{(I=1)}(s) = \frac{\beta_{K^-}^3 + \beta_{K^0}^3}{4\beta_{\pi^-}^3}\sigma_{\pi^+\pi^-}(s) \quad , \tag{7}$$

where $\beta_i \equiv \sqrt{1 - 4M_i^2/s}$. The resulting extraction of the $I = 1$ $K\bar{K}$ spectral function is straightforward, and as a check, is found to yield a branching ratio $B_{\tau\to K^-K^0\nu_\tau}$ in accord with experiment.[9, 10, 11]

(d) *Asymptotic component*: We employ the form appearing in Ref. [8] without modification.

Isospin-zero

(a) $\omega(782)$: The state $\omega(782)$ will contribute to the hypercharge spectral function as a *delta function* rather than as a resonance because we consider only non-anomalous currents and it would require the anomalous vector current to couple $\omega(782)$ to the three-pion continuum. We employ the form

$$\rho_{88}^{(\omega)}(s) = F_\omega^2\delta(s - M_\omega^2) \quad , \tag{8}$$

where the constant F_ω is obtained from the decay rate $\Gamma_{\omega\to\ell^+\ell^-}$ into lepton-antilepton pairs,[5]

$$F_\omega^2 = \frac{9}{4\pi\alpha^2}M_\omega\Gamma_{\omega\to\ell^+\ell^-} \quad , \tag{9}$$

omitting the negligible lepton mass dependence.

(b) $\phi(1020)$: A compilation of cross section data for $e^+e^- \to K\bar{K}$ appears in Ref. [12]. It is possible to infer the $I = 0$ $K\bar{K}$ cross section by first subtracting off the $I = 1$ cross section as expressed by Eq. (7). The $\phi(1020)$ occurs as a resonance just above the $K\bar{K}$ threshold with full width $\Gamma_\phi \simeq 4.43$ MeV. A good fit to the cross section data is obtained from a relativistic Breit-Wigner resonance form,[13]

$$\rho_{88}^{(\phi)}(s) = \frac{1}{\pi} \cdot \frac{F_\phi^2 M_\phi\Gamma_\phi(s)}{(s - M_\phi^2)^2 + (M_\phi\Gamma_\phi(s))^2} \tag{10}$$

where F_ϕ is determined by a relation analogous to Eq. (9) and

$$\Gamma_\phi(s) \equiv \frac{M_\phi}{\sqrt{s}} \cdot \left[\frac{s - 4M_K^2}{M_\phi^2 - 4M_K^2}\right]^{3/2}\Gamma_\phi \quad . \tag{11}$$

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(c) *Nonresonant Isospin-zero $K\bar{K}$*: There is a small nonresonant component to the $I = 0$ $K\bar{K}$ cross section which is present at higher energies. Our fit to the combined resonant and nonresonant components is displayed in Fig. 2.
(d) *Asymptotic component*: We follow the recipe for generating asymptotic form of the $I = 1$ spectral function,[8] except that the $I = 0$ component turns on at a slightly higher energy, as would be expected from a study of the contributing multiparticle thresholds.

3   Analysis

In our quantitative study of the inverse-moment sum rules, we consider separately the two classes defined earlier. First, we analyze the set of sum rules which do not involve the $\mathcal{O}(q^6)$ counterterms and then study those that do. We shall restrict our attention in this section solely to matters of analysis, leaving questions of interpretation to Sect. 4.

<div align="center">

Sum Rules without $\mathcal{O}(q^6)$ counterterms

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For this class of sum rules, the goal is to determine whether evaluations of the left-hand sides of the sum rules agree with those of the right-hand sides. The results are studied as a function of the index n which parameterizes the inverse-moment sum rules (*cf.* Eqs. (3),(4)). As described above, existing data is used as input for numerical evaluation of the spectral integrals which occupy the left-hand sides of the sum rules. The right-hand sides are obtained by performing power series expansions of $\Pi_{aa}(q^2)$ ($a = 3, 8$) to yield analytic expressions for the derivative terms, $(\mathrm{d}/\mathrm{d}q^2)^n \Pi_{aa}(0)/n!$. This was done for the range $n \leq 9$ and in addition, numerical studies were carried out for cases up to $n = 20$. The final step is to focus on the n-dependence of these relations.

A numerical analysis of the sum rules reveals some general patterns:

1. The two-pion component of the data is by far the most numerically important, becoming more so as n is increased.

2. Within the two-pion sector, the $\rho(770)$ resonance plays the major role for $n = 0, 1$, but threshold values become increasingly important thereafter. Some visual insight of this tendency is afforded by Figure 3, which displays the situation for $n = 3$. Observe how pronounced the distortion of the two-pion spectral function produced by the inverse moment has become.

3. For values $n \geq 4$, the $K\bar{K}$ component becomes negligible due to the large energy of the kaon threshold, $s = 4M_K^2$.

4. Correspondingly, the difference in content between sum rules involving Π_{33} on the one hand and $\Pi_{33} - \Pi_{88}$ on the other becomes negligible for $n \geq 4$.

Our main finding regarding these sum rules is that they are generally *not* satisfied. Examples involving Π_{33} are shown in Table 1, where differences between left-hand and right-hand sides of the sum rules are seen to occur quite generally. The pattern of discrepancy is, however, far from arbitrary. Agreement between LHS's and RHS's improves uniformly as n increases. Indeed, for values $n \geq 6$, the sum rules are obeyed to better than a few per cent.

Sum Rules with $\mathcal{O}(q^6)$ counterterms

In the concluding section, we shall provide arguments as to why the RHS's of the two relations in Eqs. (5),(6) are good approximations to the integrals appearing on the LHS's without the need for implicit higher-order contributions. That is, for the two sum rules which contain counterterms, one is justified in neglecting all contributions of higher order than two-loop in the chiral expansion.

We now turn to an evaluation of the counterterms P and Q. Upon using the phenomenological procedure described in Sect. 2 to perform an evaluation of the spectral integral in Eq. (5) we deduce the value

$$P(M_\rho^2) = -(5.6 \pm 0.6) \times 10^{-4} \quad . \tag{12}$$

The choice of scale $\mu = M_\rho$ is a reflection of the important dynamical role played by $\rho(770)$. We shall comment more fully on this point in the concluding section. The error bar accompanying P is an estimate of the uncertainties associated with our phenomenological construction of the spectral function ρ_{33} as well as that arising from the coupling constant $L_9^{(0)}$,

$$
\begin{aligned}
L_9^{(0)}(M_\rho^2) &= L_9^{(0)}(M_\eta^2) + \frac{1}{128\pi^2} \log\left(\frac{M_\eta^2}{M_\rho^2}\right) \quad , \\
&= 0.0071 \pm 0.0003 - 0.0005 = 0.0066 \pm 0.0003 \quad . \tag{13}
\end{aligned}
$$

If we adopt the same procedure for counterterm Q, we obtain from Eq. (6) the estimate

$$Q(M_\rho^2) \sim 6.6 \times 10^{-5} \qquad \text{(Preliminary)} \quad . \tag{14}$$

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We have labeled the above determination 'preliminary' because the existing data sample in the isoscalar channel is incomplete in the four-particle sector. Our concern is that, because the counterterm Q is a measure of $SU(3)$ symmetry breaking, a cancelation between ρ_{33} and ρ_{88} should be evident for each separate n-particle sector. For example, the resonance sector clearly demonstrates this,

$$\int_{s_0}^{\infty} ds \left. \frac{(\rho_{33} - \rho_{88})(s)}{s} \right|_{\rho,\omega,\phi} = 0.0374 - 0.0103 - 0.0204 \simeq 0.0067 \ . \quad (15)$$

For the four-particle sector, however, one finds a large value for the purely isovector four-pion spectral integral ($I_{4\pi} = 0.0107$) which is uncompensated by a corresponding contribution in the isoscalar channel. Unfortunately, there is at present a lack of sufficient data for $e^+ e^- \to K \bar{K} \pi \pi$ to allow an isospin decomposition in that sector. However, in order to obtain some measure of the $K \bar{K} \pi \pi$ contribution, we have treated the $e^+ e^- \to K^+ \bar{K}^- \pi^+ \pi^-$ data of Ref. [14] as if it were purely isoscalar. Although having an obvious uncertainty in the magnitude, this approach should get the threshold and overall energy scale about right. We obtain $I_{K\bar{K}2\pi} \simeq 0.0035$ for the associated spectral integral and finally, the improved determination

$$Q(M_\rho^2) = (3.7 \pm 2.0) \times 10^{-5} \ . \quad (16)$$

It is this value that we shall take for our determination of Q. The large error bars indicate the uncertainty in the four-particle sector.

Often, narrow-width expressions for the $\rho(770)$, $\omega(782)$ and $\phi(1020)$ resonances are used to approximate integrals involving the physical spectral functions.[5],[15] In the narrow-width approximation, the spectral function for a neutral vector meson of mass M_{R} is

$$\rho(s) = F_R^2 \delta(s - M_{\mathrm{R}}^2) \ , \quad (17)$$

where the value of F_R is fixed as in Eqs. (9),(11). Omitting any estimates of error bars, we present here for the sake of comparison the values $P^{(\mathrm{NW})}$ and $Q^{(\mathrm{NW})}$ as obtained in the narrow-width approximation,

$$P^{(\mathrm{NW})}(M_\rho^2) = -4.0 \times 10^{-4} \qquad \text{and} \qquad Q^{(\mathrm{NW})}(M_\rho^2) = 4.2 \times 10^{-5} \ . \quad (18)$$

4 Conclusions

It is useful to restate the approach employed thus far in dispersion-theoretic language. Thus, consider the chiral representation $(\Pi_{33} - \Pi_{88})_\chi^{(2)}$ to two-loop order as obtained in Ref. [1] but expressed now as a dispersion relation.

From known analyticity properties and asymptotic behaviour, we deduce the once-subtracted form,

$$(\Pi_{33} - \Pi_{88})^{(2)}_\chi(q^2) = a^{(2)}_{38} + q^2 \int_{s_0}^\infty ds \, \frac{(\rho_{33} - \rho_{88})^{(2)}_\chi(s)}{s(s - q^2 - i\epsilon)} \quad . \tag{19}$$

In a ChPT framework, the subtraction constant $a^{(2)}_{38}$ corresponds to an $\mathcal{O}(q^6)$ counterterm (essentially the quantity Q). Omitting higher orders and simply equating this representation with that in Eq. (2) implies (for $n \geq 0$)

$$\int_{s_0}^\infty ds \, \frac{(\rho_{33} - \rho_{88})(s)}{s^{n+1}} = a^{(2)}_{38} \delta_{n0} + (1 - \delta_{n0}) \int_{s_0}^\infty ds \, \frac{(\rho_{33} - \rho_{88})^{(2)}_\chi(s)}{s^{n+1}} \quad . \tag{20}$$

This summarizes the content of the sum rules sensitive to $SU(3)$-breaking.

Proceeding in like manner with the $\Pi^{(2)}_{aa\chi}$ ($a = 3, 8$), which would require a twice-subtracted dispersion relation, we have

$$\Pi^{(2)}_{aa\chi}(q^2) = a^{(2)}_{aa} + b^{(2)}_{aa}q^2 + q^4 \int_{s_0}^\infty ds \, \frac{\rho^{(2)}_{aa\chi}(s)}{s^2(s - q^2 - i\epsilon)} \quad , \tag{21}$$

where the $b^{(2)}_{aa}$ are associated with the $\mathcal{O}(q^6)$ counterterm P. The sum rules of relevance to our calculation are

$$\int_{s_0}^\infty ds \, \frac{\rho_{aa}(s)}{s^{n+1}} = b^{(2)}_{aa}\delta_{n1} + (1 - \delta_{n1}) \int_{s_0}^\infty ds \, \frac{\rho^{(2)}_{aa\chi}(s)}{s^{n+1}} \quad (n \geq 1) \quad , \tag{22}$$

where again we have neglected higher orders.

The above analysis contains several points of interest:

1. It demonstrates how the $\mathcal{O}(q^6)$ counterterms of two-loop ChPT, when viewed in a dispersion relation context, appear as subtraction constants.

2. It regains the earlier result that our two-loop ChPT representations lead to an infinity of sum rules, of which just two constrain $\mathcal{O}(q^6)$ counterterms.

3. It suggests the following pattern for higher-loop ChPT treatments — that $\mathcal{O}(q^8)$ counterterms from a three-loop analysis will be constrained by spectral integrals of $\{\rho_{aa}\}/s^3$ ($a = 3, 8$) and of $(\rho_{33} - \rho_{88})/s^2$, and more generally that $\mathcal{O}(q^{2n+2})$ counterterms from an n-loop analysis will appear with spectral integrals involving $\{\rho_{aa}\}/s^n$ ($a = 3, 8$) and $(\rho_{33} - \rho_{88})/s^{n-1}$.

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4. Most importantly, it clarifies why the sum rules without $\mathcal{O}(q^6)$ counterterms must be violated.

The last item deserves comment. Consider for example the isospin polarization function Π_{33}. The point is that the two-loop chiral spectral function $\rho_{33\chi}^{(2)}(s)$ is an adequate approximation to the full physical spectral function $\rho_{33}(s)$ only for energies not too far above the two-pion threshold. A glance at Eq. (22) shows how a discrepancy must arise in the integrations over all energy for the sum rules with $n > 1$. The dispersion expressions for the sum rules also reveal that the discrepancies must decrease as the inverse moment n increases because the threshold region becomes more and more enhanced. As displayed in Fig. 3, the effect is already important for $n = 3$. Table 2 exhibits the relative importance of the low energy region lying above the two-pion threshold ($\sqrt{s} \leq 0.4$ GeV to be precise) to that of the full two-pion component. For $n \geq 6$, the dominance of the threshold region is almost complete and so the sum rules are satisfied to a reasonable degree.

Additional physical insight comes from studying the important role played by the $J = 1$, $I = 0, 1$ resonances, $\rho(770)$, $\omega(782)$ and $\phi(1020)$. In such resonant channels, dynamical effects of the Goldstone bosons are generally subordinate since the discontinuities over the two-pion cuts are too small to compete with a strong resonance. A well-known analog is the electromagnetic charge radius of the pion, where the $\rho(770)$ overwhelms all other effects almost completely.[16] In the ChPT framework, resonances are not treated as explicit degrees of freedom, but instead contribute implicitly in the coupling constants of operator counterterms. Indeed, at $\mathcal{O}(q^4)$ it was shown[17] that low-lying resonances in the vector, axialvector, scalar and pseudoscalar channels saturate the coupling constants $\{L_i\}$ ($i = 1, ..., 10$). Although no such analysis is currently available beyond one-loop order, we anticipate the same behaviour — that a strong resonance will induce large coupling constants for selected counterterms.[18] Known examples of this type occur for the processes $\eta \to \pi\gamma\gamma$[19] and also in $K \to \pi\gamma\gamma$[20].

We are thus led to characterize the terms appearing on the RHS's of the sum rules Eqs. (3),(4) either as 'resonant' or as 'continuum'. A resonant contribution is one which arises from a counterterm which is saturated by resonance exchange, as in Fig. (4a). The continuum contributions are all the others, like those in Fig. (4b), which come purely from rescatterings of the Goldstone bosons or involve a resonance contribution in a subgraph. As is obvious from Eq. (20) and surrounding discussion, the two-loop sum rules without $\mathcal{O}(q^6)$ counterterms have only continuum terms on the RHS. Such

RHS's are recovered entirely from the integrated two-loop $\rho_{aa\chi}^{(2)}$ ($a = 3, 8$) spectral functions.[3] It is thus clear that sum rules of this type are not well suited to determine $\mathcal{O}(q^4)$ counterterms like $L_9^{(0)}$ which appear only in the continuum part of the RHS.

As an example, we can use a simple vector meson exchange picture to model the isospin sum rules. In this vector-dominance (VMD) picture, we expect the contribution at any given order to reflect the effect of a counterterm appearing at that order. Thus, employing the notation of Ref. [17] and working to leading order, we consider the interaction lagrangian

$$\mathcal{L} = \frac{F_V}{2\sqrt{2}} \text{tr}(V_{\mu\nu} f_+^{\mu\nu}), \qquad (23)$$

where $V_{\mu\nu}$ is the octet of vector mesons in the antisymmetric tensor formulation. The VMD form for the isospin polarization function is then easily calculated to be

$$\Pi_{33}^{\text{VMD}}(q^2) = \frac{F_\rho^2}{M_\rho^2 - q^2} = \sum_{n=0}^{\infty} \frac{F_\rho^2}{M_\rho^2} \left(\frac{q^2}{M_\rho^2} \right)^n . \qquad (24)$$

The first derivative of this expression evaluated at $q^2 = 0$ leads to a numerical estimate of the counterterm P. With input parameters $F_\rho = 154$ MeV and $M_\rho = 770$ MeV, we obtain

$$P^{\text{VMD}}(M_\rho^2) = -F_0^2 \frac{F_\rho^2}{M_\rho^2} \simeq -5.8 \times 10^{-4} , \qquad (25)$$

which equals, within errors, the phenomenological determination of Eq. (12). Note also that within the VMD picture, the natural scale for counterterm renormalization is $\mu = M_\rho$. More generally, we list the derivatives of Π_{33}^{VMD} in Table 3, column 2. We see that for $n \leq 3$ the discrepancy between LHS and RHS of the sum rules Eq. (4) is correctly given by the vector meson exchange model within 10% accuracy. For larger n, the discrepancy is small and we expect the corrections to the spectral functions to make up for the small violation still present. Of course there will also be mass corrections to the prediction of Eq. (24). However, we have shown that the large value obtained for the counterterm P as well as the large violation of the sum rules Eq. (4) for small n are well understood in terms of vector meson dominance.

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[3]Although there may be contributions from resonance dominated counterterms (*e.g.* $L_9^{(0)}$), they nonetheless are continuum contributions arising from diagrams like Fig. (4b).

There is no reason for a further large correction of higher order than the order where a resonance dominated counterterm first appears.

The situation is admittedly not as secure for the $SU(3)$ breaking counterterm Q. It is a fact that the nature of $SU(3)$ symmetry breaking is still far from completely understood. This should serve as a warning that the dynamics which underlie the value of Q could present a difficult obstacle. As we have already commented, progress on the phenomenological front will be stalled until improved data becomes available. As a final thought, we suggest that it might be profitable to interpret the isovector and isoscalar four-particle spectral functions in terms of resonances. Substructure in the four-pion sector has been cited as evidence for the isovector states $\rho(1450)$ and $\rho(1700)$.[10] To proceed with this idea, one would need to know more about the resonant couplings than is presently available.

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Table 1 Discrepancy $\Delta(\%)$

n	2	3	4	5	6
$\Delta_n(\%)$	57.	39.	19.	8.	3.

Table 2 Low-energy Contribution to Spectral Integral

n	1	2	3	4	5	6
$I^{2\pi}_{\text{Low}-\text{E}}/I^{2\pi}_{\text{Total}}$	0.07	0.19	0.44	0.64	0.86	0.94

Table 3 Vector meson contribution to Spectral Integral

n	$\frac{1}{n!}\frac{d^n}{(dq^2)^n}\Pi^{\text{VMD}}_{33}(0)$	$\int ds \frac{\rho_{33}-\rho^{(2)}_{33\chi}}{s^{n+1}}$	$\int ds \frac{\rho_{33}}{s^{n+1}}$
1	0.067	0.056	0.083
2	0.114	0.103	0.183
3	0.192	0.228	0.600
4	0.324	0.543	2.944
5	0.564	1.45	19.55
6	0.921	4.40	154.5

Figure Captions

Fig. 1   Low-energy fit to $\rho_{33}(s)$ data.

Fig. 2   Fit to $I = 0$ $K\bar{K}$ cross section.

Fig. 3   Profile of $\rho_{33}(s)/s^4$.

Fig. 4   Contributions of (a) resonant and (b) continuum types.