

# Decentralized Robust Control via Quadratically Invariant Model Projection

Jong-Han Kim<sup>1</sup>   Sanjay Lall<sup>2</sup>   Walter Merrill<sup>3</sup>   Alireza Behbahani<sup>4</sup>

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## Abstract

We propose a computational approach to systematically find decentralized  $\mathcal{H}_\infty$  suboptimal controllers for general unstructured models. Exploiting the quadratically invariant model projection and the classical robust control synthesis techniques, we show that the original nonconvex problem can be conservatively solved by a series of convex optimization problems.

## 1 Introduction

We consider decentralized control systems, where the controllers are spatially distributed and each controller has access to a different subset of the information. Systematic and efficient synthesis of decentralized control policies in such architectures has been a fundamental and central issue for networked control problems, however many of the crucial questions regarding this issue have not yet been answered clearly.

One of the critical factors limiting these technological developments is that the model-based control synthesis procedures which have been so effective at centralized control do not currently have counterparts for decentralized control. Although good heuristics are known in some cases, and certain special cases have been solved exactly, for the general problem there is currently no method that can in general numerically compute, for example, the optimal mean-square performance achievable by decentralized control, even for the highly specialized scenario of low dimensional linear time-invariant state-space systems. It no longer fits within the existing paradigm (*Riccati* equations, linear matrix inequalities, et cetera) for optimal centralized control problems; this is the key obstacle to the overall problem, and a tractable algorithm for finding the optimal controller, even the optimal linear controller, does not yet exist [2].

Local optimization algorithms have been suggested, [4, 5] for example, of which the convergence largely depends on the choice of initial feasible point. Branch-and-bound techniques were applied to find the globally optimal design [1, 13], however they usually result in extremely large computational load even for small-sized problems. These reflect that finding the globally optimal solution for such decentralized control architectures is not trivial at all.

A recent work introduces the notion of *quadratic invariance* (QI) which characterizes the largest class of convex problems in structured control design problems [9]. It shows that for a large range of practical problems with structured plant models, the decentralized control problem can be formulated as an infinite dimensional convex problem. Furthermore, if one is interested in the  $\mathcal{H}_2$  optimality, the solution can be computed efficiently [8, 9, 12]. However, the following two natural questions still remain unanswered; 1) how we can deal with the unstructured plant models, and 2) how we can solve the quadratically invariant  $\mathcal{H}_\infty$  optimization problems. This paper mainly addresses the first issue. A simple method to conservatively handle the second issue is also introduced.

We resolve the unstructured model issue by introducing the QI projection. We first find the projection of the full model onto the set of sparse models under which the information constraint is quadratically invariant. Then we synthesize a robust controller treating the projection error as the additive model uncertainty. This is a substantially sophisticated extension from the rudimentary approach introduced by the authors [7]. This transforms the original hard problem into iterative convex structured  $\mathcal{H}_\infty$  optimization problems, through which we land on the second issue. This is approximately treated by relaxing the information constraint and solving the multiobjective  $\mathcal{H}_\infty$  optimization problem, which can be cast to semidefinite programming (SDP). Though this convention does not guarantee the global convergence for general problems, it is demonstrated to work quite well on the presented numerical example.

## 2 Decentralized Control Synthesis

For centralized control architectures with widely accepted control objective functions, such as the  $\mathcal{H}_2$  norm or the  $\mathcal{H}_\infty$  norm of the closed-loop system, there are

<sup>1</sup>J.-H. Kim is with the Department of Aeronautics and Astronautics, Stanford University, Stanford, CA. [jonghank@stanford.edu](mailto:jonghank@stanford.edu)

<sup>2</sup>S. Lall is with the Department of Electrical Engineering, and Department of Aeronautics and Astronautics, Stanford University, Stanford, CA. [lall@stanford.edu](mailto:lall@stanford.edu)

<sup>3</sup>W. Merrill is with Scientific Monitoring Inc., Scottsdale, AZ. [walt.merrill@scientificmonitoring.com](mailto:walt.merrill@scientificmonitoring.com)

<sup>4</sup>A. Behbahani is with the Air Force Research Laboratory, Wright-Patterson, AFB, OH. [al.behbahani@wpafb.af.mil](mailto:al.behbahani@wpafb.af.mil)

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several well known methods that solve the problem efficiently. Such problems can be expressed using the following generalized plant description.

$$\underset{K \text{ stabilizing}}{\text{minimize}} \quad \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\|$$

where  $P_{11}$ ,  $P_{12}$ , and  $P_{21}$  describe the input-output interconnection of the models.  $P_{22}$  represents the plant model and  $K$  is the controller to be designed.

The above objective function is not convex in the variable  $K$ , but the problem can be transformed to a convex one by change of variables according to  $Q = K(I - P_{22}K)^{-1}$ . The optimal control is full in general, which requires the centralized control authority.

On the other hand, synthesizing optimal decentralized controls requires additional constraints which makes the problem far harder. The optimal decentralized control problem can be described as follows, by adding an information constraint.

$$\begin{aligned} &\underset{K \text{ stabilizing}}{\text{minimize}} \quad \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\| \\ &\text{subject to} \quad K \text{ satisfies the information constraint} \end{aligned}$$

The last constraint, using a sparsity pattern, defines the decentralized control architectures. The change of variables, which helped the centralized problem, is of no use in this case since it transforms the linear information constraint to a complicated nonconvex one. In general, finding the optimal control for such decentralized setups is very hard, and no algorithm is known to efficiently solve the problem in polynomial time [2].

## 2.1 Quadratic invariance

Quadratic invariance characterizes a simple algebraic condition of the plant and the controller, under which the optimal decentralized control problem reduces to a convex optimization problem [9].

Suppose  $\mathcal{U}$  and  $\mathcal{Y}$  are Banach spaces, and let  $\mathcal{L}(\mathcal{Y}, \mathcal{U})$  be the space of all linear and continuous maps  $K: \mathcal{Y} \rightarrow \mathcal{U}$ . As a general representation of decentralization constraints, we call a subspace  $S \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$  an information constraint.

We consider finding optimal linear controllers, and define the following class of information constraints.

**Definition 1.** Suppose  $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$  and  $S \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$ .  $S$  is called **quadratically invariant** under  $G$  if

$$KGK \in S \quad \text{for all } K \in S$$

For the linear decentralized control problem with a quadratically invariant information constraint, we can show that the optimal controller may be found via convex optimization.

**Theorem 2.** Suppose  $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ , and  $S \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$  is a closed quadratically invariant subspace. Also suppose that the operator  $I - GK$  is invertible for every  $K \in S$ . Then

$$K \in S \iff K(I - GK)^{-1} \in S$$

**Proof.** See [9] for details. ■

The theorem says that the quadratic invariance guarantees the convexity of the information constraint set under the transformation according to  $Q = K(I - GK)^{-1}$ . This gives the equivalent problem.

$$\underset{Q \in \mathcal{RH}_\infty \cap S}{\text{minimize}} \quad \|P_{11} + P_{12}QP_{21}\|$$

This is now an infinite dimensional convex optimization problem, and the  $\mathcal{H}_2$  norm case can be solved by standard methods [8]. This implies that if the system and the controller jointly satisfy some simple algebraic condition, the optimal decentralized control problem may be easily solved.

The notion of quadratic invariance is powerful for some class of sparse plant models. However it is not appropriate for application to general unstructured models, because the only quadratically invariant class of controllers for full plant models are the full (centralized) controllers.

## 2.2 QI projection

In synthesizing the decentralized control laws for general unstructured linear systems, we suggest an intuitive procedure to extensively apply the notion of quadratic invariance to the full unstructured models.

Suppose that a control structure requirement is given by the information constraint  $S$ . We denote by  $S^*$  the set of all linear maps under which  $S$  is quadratically invariant.

**Definition 3.** Suppose  $S \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$  is an information constraint. We define a complementary set  $S^* \subseteq \mathcal{L}(\mathcal{U}, \mathcal{Y})$  as follows.

$$S^* = \left\{ H \mid S \text{ is quadratically invariant under } H \right\}$$

Now the QI projection of  $G$  is defined by the closest point in  $S^*$  from  $G$ .

**Definition 4.** Suppose  $G \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ , and  $S \subseteq \mathcal{L}(\mathcal{Y}, \mathcal{U})$  is an information constraint for which  $S^*$  is nonempty. Then  $\tilde{G}$  is called a **QI projection** of  $G$  if

$$\tilde{G} \in \arg \min_{H \in S^*} \|G - H\|_\infty$$

The projection error is denoted by  $G^\perp = G - \tilde{G}$ .

It is reasonable that we can consider  $\tilde{G}$  as some approximation of  $G$ , and then  $G^\perp$  corresponds to the associated approximation error (or uncertainty). Hence intuition



being the fictitious full perturbation with  $\|\Delta_P\|_\infty \leq 1$ , and let  $\mu_{\hat{\Delta}}(F)$  be the structured singular value of  $F$  associated with the structure  $\hat{\Delta}$ . Then we have the following.

**Lemma 5.** *Consider the standard feedback interconnection of the stable plant  $P$ ,  $\bar{P}$  with the controller  $K$  and the perturbation  $\Delta$  as in Figure 2. Then for the alternative problem in Figure 3, the following statements are equivalent.*

1.  $\|F_l(P(G'), K)\|_\infty \leq 1, \quad \forall G' \in \mathcal{G}$
2.  $\|F_u(F_l(\bar{P}, K), \Delta)\|_\infty \leq 1, \quad \forall \Delta = \delta I, |\delta| \leq 1$
3.  $\mu_{\hat{\Delta}}(F_l(\bar{P}, K)) \leq 1$

**Proof.** This follows from the standard result in the robust control contexts. See [11, 14], for example. ■

The above lemma states that a suboptimal  $K$  for the original problem can be found from the  $\mu$ -suboptimal  $K$  satisfying the last statement. We claim that such suboptimal structured  $K$  may be practically found via a modified  $D$ - $K$  iteration process introduced below. Though the series of coordinatewise convex optimization does not guarantee the global convergence on the general jointly convex function, computational results have shown that it is very reliable in practice [11, 14].

1.  $K$ -step (modified):

$$\begin{aligned} & \underset{K \text{ stabilizing}}{\text{minimize}} && \|DF_l(\bar{P}, K)D^{-1}\|_\infty \\ & \text{subject to} && K \in S \end{aligned}$$

2.  $D$ -step:

$$\underset{D \in \mathcal{D}}{\text{minimize}} \quad \|DF_l(\bar{P}, K)D^{-1}\|_\infty$$

where  $\mathcal{D}$  represents the set of all stable, minimum phase, and structured transfer function matrices satisfying  $D\hat{\Delta} = \hat{\Delta}D$ . A good choice for the initial  $D$  can be  $D = I$ .

Note here that the  $D$ -step problem is identical to that of the classical  $\mu$  synthesis. On the other hand, in the modified  $K$ -step, an information constraint is added to the classical  $K$ -step; this is the hard part. However the problem is convex because  $\bar{P}$  contains  $\tilde{G}$ , and the information constraint  $S$  is quadratically invariant under  $\tilde{G}$ .

More explicitly, note that the solution from the  $D$ -step has the structure of  $D = \text{diag}(d, I)$  with some full, complex matrix  $d$ . The modified  $K$ -step problem can be rewritten as follows.

- $K$  step (modified):

$$\underset{Q \in \mathcal{RH}_\infty \cap S}{\text{minimize}} \quad \|T_{11} + T_{12}QT_{21}\|_\infty$$

where

$$\left[ \begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array} \right] = \left[ \begin{array}{c|c|c} dP_{00}d^{-1} & dP_{01} & dP_{02} \\ P_{10}d^{-1} & P_{11} & P_{12} \\ \hline P_{20}d^{-1} & P_{21} & P_{22} \end{array} \right]$$

and

$$Q = K(I - T_{22}K)^{-1}$$

Obviously,  $S$  is quadratically invariant under  $T_{22}$  (in this example  $T_{22} = P_{22} = -\tilde{G}$ ), therefore the condition  $K \in S$  is equivalent to  $Q \in S$ .

This formulation now implies that the original nonconvex  $\mathcal{H}_\infty$  problem can be conservatively solved by a series of convex  $\mathcal{H}_\infty$  problems.

Recall that the  $\mathcal{H}_2$  problem in this form can be solved efficiently [8], however the  $\mathcal{H}_\infty$  problem is not the case. A series of convergent solutions may be obtained by finite dimensional bases expansion [3, 6]. However such approaches need to handle very large linear matrix inequalities (LMIs), and current off-the-shelf solvers (such as SeDuMi) usually fail even in simple control design problems.

Another approach based on the iterative coordinate-wise optimization can be used to find the optimal  $Q$  elementwisely. Like the  $D$ - $K$  iteration in the  $\mu$ -synthesis, this approach usually finds a sound solution in practice, but without a mathematical guarantee of convergence.

In the next section, we introduce another approach to solve this particular infinite dimensional convex problem conservatively.

## 2.4 Quadratically invariant $\mathcal{H}_\infty$ optimization

We solve the problem by relaxing the information constraint first, and then reformulate the relaxed problem as a multiobjective  $\mathcal{H}_\infty$  optimization problem.

For the illustrated example, let's consider the  $2 \times 2$  nested information pattern as follows.

$$S : \begin{bmatrix} \bullet & \bullet \\ \circ & \bullet \end{bmatrix}$$

Relaxing the information constraint to a norm inequality with some positive  $\epsilon \ll 1$  yields,

$$\begin{aligned} & \underset{Q \in \mathcal{RH}_\infty}{\text{minimize}} && \|T_{11} + T_{12}QT_{21}\|_\infty \\ & \text{subject to} && \|Q_{21}\|_\infty \leq \epsilon \end{aligned}$$

As  $\epsilon \rightarrow 0$ , the optimal value of the above relaxation approaches to the optimal value of the exact problem; more precisely, the difference is bounded by a linear function of  $\epsilon$ . Provided that  $\epsilon$  is sufficiently small, we can reasonably project the relaxed solution onto  $S$ , and can use the projection for the decentralized control purpose. Note that the projection is also stable, thus stabilizes the plant.

The problem can be cast as the following form of multiobjective problem, and the solution can be obtained by an appropriate choice of  $\nu$ .

$$\underset{Q \in \mathcal{RH}_\infty}{\text{minimize}} \quad \|T_{11} + T_{12}QT_{21}\|_\infty + \nu \|Q_{21}\|_\infty$$

For the illustrated problem in Figure 3,  $Q_{21}$  equals to the transfer function from  $r_1$  to  $u_2$ . We can also introduce the weight  $W_2$  to the second term, for computational convenience, and obtain the solution by,

$$\underset{Q \in \mathcal{RH}_\infty}{\text{minimize}} \quad \|T_{11} + T_{12}QT_{21}\|_\infty + \nu' \|T_3QT_4\|_\infty$$

where  $T_3 = \begin{bmatrix} 0 & I \end{bmatrix} W_2$  and  $T_4 = \begin{bmatrix} I & 0 \end{bmatrix}^T$ , with a new constant  $\nu'$  defining the penalty term.

At the cost of introducing conservativeness, the above can be formulated as an SDP with manageable size of LMIs [10], so it can be efficiently solved. However it should be noted that the amount of the conservativeness being entered differs from a specific problem to another. Therefore it is possible that this multiobjective approach with the SDP formulation may result in an unacceptable solution for some problems.

A numerical example on a mixed sensitivity tracking problem, as in Figure 3, is presented in the next section.

### 3 Numerical Example

We use a scaled linear model of the GE F404 turbine engine at a rated thrust condition at 35,000 ft altitude.

$$G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \frac{3.07(s+2.19)(s+1.16)}{(s+0.4)(s+0.90)(s+2.79)} & \frac{-5.87(s+1.64)(s-4.74)}{(s+0.4)(s+0.90)(s+2.79)} \\ \frac{0.37(s+5.23)}{(s+0.90)(s+2.79)} & \frac{27.50(s+2.24)}{(s+0.90)(s+2.79)} \end{bmatrix}$$

As in Figure 1, we are interested in finding a decentralized mixed sensitivity  $\mathcal{H}_\infty$  controller

$$K \in \begin{bmatrix} \bullet & \bullet \\ \circ & \bullet \end{bmatrix}$$

rendering

$$\left\| \begin{bmatrix} W_1(I + GK)^{-1} \\ W_2K(I + GK)^{-1} \end{bmatrix} \right\|_\infty \leq 1$$

	Variable	Description
$u$	$W_f$	Fuel flow rate
	$A_j$	Nozzle area
$y$	$T_{t4.5}$	Turbine total temperature
	$N_2$	Fan speed

Table 1: Input/output variables for the engine model

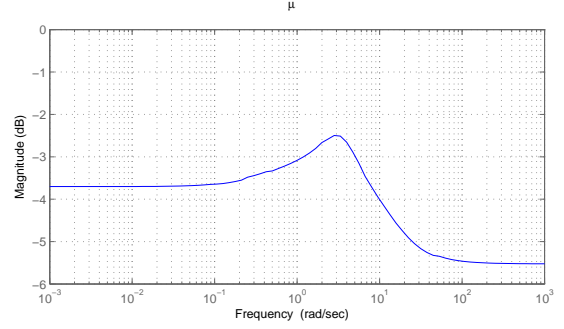


Figure 4: Structured singular value plot

with design parameters given by,

$$W_1 = \frac{0.5(s+5)}{(s+0.0001)}I \quad W_2 = 0.2I$$

Recall from Example 1 that the QI projection for this information pattern is simply,

$$\tilde{G} = \begin{bmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{bmatrix} \quad G^\perp = \begin{bmatrix} 0 & 0 \\ g_{21} & 0 \end{bmatrix}$$

The alternative problem was solved using `cvx` and `SeDuMi`, with  $\nu' = 50$  chosen for the multiobjective  $\mathcal{H}_\infty$  optimization. The  $\mu$ -suboptimality,  $\mu_{\hat{\Delta}}(F_l(\tilde{P}, K)) \leq 1$ , was achieved at the first  $D$ - $K$  iteration (Figure 4), proving the guaranteed performance for all plants in  $\mathcal{G}$ , including the original unstructured plant  $G$  in particular.

The computation results are presented with the comparison of the following three controls.

- $Q_c$  (centralized solution) : obtained from the original problem without the information constraint.  $K_c$  related by  $Q_c = K_c(I + GK_c)^{-1}$ .
- $Q_r$  (close-to-decentralized solution) : obtained from the relaxed multiobjective problem.  $K_r$  related by  $Q_r = K_r(I + \tilde{G}K_r)^{-1}$ .
- $Q_d$  (decentralized solution) : obtained by projecting  $Q_r$  onto  $S$ .  $K_d$  related by  $Q_d = K_d(I + \tilde{G}K_d)^{-1}$ .

The sensitivity function and the step response on the full unstructured plant  $G$  are plotted in Figure 5 and Figure 6. In these plots,  $K_r$  and  $K_d$  produce the almost overlapping plots, by which we indirectly recognize that  $Q_r$  is very close to  $S$ . More quantitatively, the elementwise size of  $Q_r$  in the  $\mathcal{H}_\infty$  norm, is shown below, which indicates that the relaxed multiobjective problem successfully finds a close-to-sparse solution.

$$\begin{bmatrix} \|Q_{r,11}\|_\infty & \|Q_{r,12}\|_\infty \\ \|Q_{r,21}\|_\infty & \|Q_{r,22}\|_\infty \end{bmatrix} = \begin{bmatrix} 1.2291 & 0.3123 \\ 0.0008 & 0.1308 \end{bmatrix}$$

Achieved closed loop performance on the original unstructured plant  $G$  is displayed in Table 2. Again, note that  $K_r$  and  $K_d$  render the practically equal performance, and all three controllers successfully achieve the required suboptimality.



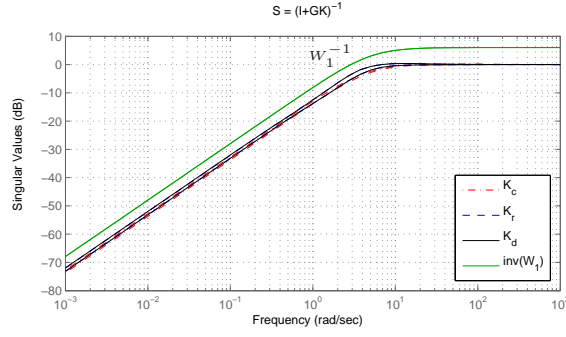


Figure 5: Sensitivity function

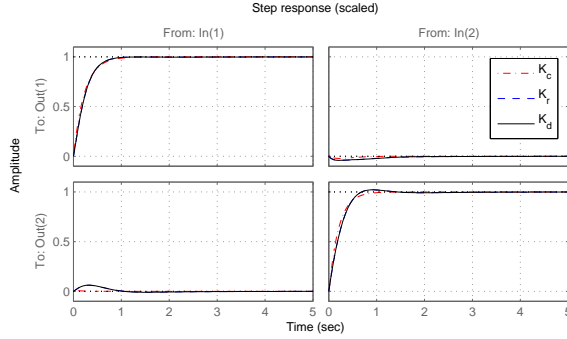


Figure 6: Step response

## 4 Concluding Remarks

We proposed a decentralized  $\mathcal{H}_\infty$  synthesis algorithm for general full plants. We define the QI projection of the original unstructured plant model, and develop a slightly modified  $\mu$ -synthesis technique on the projected model. The approach conservatively guarantees the suboptimal performance by the structured singular value condition with a structured  $K$  constraint. The modified  $\mu$ -synthesis results in a series of the quadratically invariant  $\mathcal{H}_\infty$  optimization problems, which we approximately solve by the  $\epsilon$ -norm relaxed multiobjective  $\mathcal{H}_\infty$  optimization.

An issue in the presented approach was that the multi-objective formulation for the quadratically invariant  $\mathcal{H}_\infty$  problem does not provide the globally optimal solution, although it found a sound solution in the particular numerical example. The authors believe that one may be able to find an efficient realization of the globally optimal solution, because the problem is in convex form. This can be a natural extension of the presented work.

Another extension may be implementation of this re-

Controller	$\ F_l(P, K)\ _\infty$
$K_c$	0.5770
$K_r$	0.6784
$K_d$	0.6794

Table 2: Achieved closed loop performance on  $G$

sult to practical problems with more realistic plant models.

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