Classification of poset-block spaces admitting MacWilliams-type identity

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***Abstract*—In this work we prove that a poset-block space admits a MacWilliams-type identity if and only if the poset is hierarchical and at any level of the poset, all the blocks have the same dimension. When the poset-block admits the MacWilliams-type identity we explicit the relation between the weight enumerators of a code and its dual.**

***Index Terms*—Poset-block codes, MacWilliams identity, weight distribution, MacWilliams-type identity.**

I. Introduction

Due to both the interest in generalizing classic problems in coding theory and to applications in cryptography, experimental designs and high-dimensional numerical integration (see for example [1] and [2]), by the mid 1990s researches began to study codes considering metrics others than the usual Hamming metric over $\mathbb{F}_q^n$. Among those families of metrics are the poset metrics [3] and the block metrics [2]. Much of the classical theory has been generalized to codes in spaces endowed with a poset metric, as can be seen, for example, in [4], [5], [6] and [7].

In 2008 Firer *et al* [8] presented the family of metrics called poset-block that generalizes all the previous ones. In this work we generalize to poset-block spaces the characterization given in [5] for poset-metric spaces of poset-block metrics admitting MacWilliams-type identity.

Let $[m] := \{1, 2, \cdots, m\}$ be a finite set. If $\preccurlyeq$ is a partial order relation in $[m]$, we say $P := ([m], \preccurlyeq)$ is a *poset* and denote by $\preccurlyeq_P$ the order in P. An *ideal* in a poset is a nonempty subset $I \subset [m]$ such that, for $i \in I$ and $j \in [m]$, if $j \preccurlyeq_P i$ then $j \in I$. Given $A \subset [m]$, we denote by $\langle A \rangle_P$ the smaller ideal of P containing A. If $A = \{i\}$, we will denote by $\langle i \rangle_P$ the ideal $\langle \{i\} \rangle_P$. A *chain* in a poset P is a subset of $[m]$ such that every two elements are comparable.

Let $\mathbb{F}_q$ be a finite field and $\mathbb{F}_q^n$ the vector space of n-tuples over $\mathbb{F}_q$. Given $m \in [n]$, P a poset over $[m]$ and $\pi : [m] \to \mathbb{N}$ a map such that $n = \sum_{i=1}^{m} \pi(i)$, we say that π is a *labeling* of the poset P and that the pair (P, π) is a *poset-block* structure over $[m]$.

We denote $k_i = \pi(i)$, and consider the vector space over $\mathbb{F}_q$

$$V := \mathbb{F}_q^{k_1} \times \mathbb{F}_q^{k_2} \times \cdots \times \mathbb{F}_q^{k_m},$$

isomorphic to $\mathbb{F}_q^n$. Given $u \in \mathbb{F}_q^n$, there is a unique decomposition $u = (u_1, \cdots, u_m)$ with $u_i \in \mathbb{F}_q^{k_i}$, $i \in [m]$. The π-*support* and the (P, π)-*weight* of u are defined respectively as

$$supp_\pi(u) := \{i \in [m] : u_i \neq 0 \in \mathbb{F}_q^{k_i}\}$$

and

$$w_{(P,\pi)}(u) := |\langle supp_\pi(u) \rangle_P|,$$

where $|.|$ denotes the cardinality of the given set. For $u, v \in \mathbb{F}_q^n$,

$$d_{(P,\pi)}(u, v) := w_{(P,\pi)}(u - v)$$

defines a metric over $\mathbb{F}_q^n$ called *poset-block metric*, or just (P, π)-*distance* between u and v.

We note that when $\pi(i) = 1$ for every $i \in [m]$ the (P, π)-distance is usual poset distance introduced in [3], while imposing P to be a trivial poset ($i \preccurlyeq j \iff i = j$) turns the (P, π)-distance into the block distance defined in [2]. Interweaving the poset and the block structures opens a wide range of possibilities for searching for codes with interesting metric characteristics, such as perfect codes, since poset and block metrics have opposite effects on distances: while enlarging the relations on a poset enlarges the distances (hence "shrinks" metric balls), enlarging the blocks diminishes distances (hence "blows" metric balls).

Concerned with MacWilliams-type identities, dual posets play a crucial role:

Definition 1: Given a poset P over $[m]$, the *dual poset* is the poset $\overline{P}$ defined by the relations

$$i \preccurlyeq_P j \iff j \preccurlyeq_{\overline{P}} i$$

for every $i, j \in [m]$. The pair $(\overline{P}, \pi)$ is called the *dual poset-block*.

Given $j \in [m]$, the *rank* of j, denoted by $h_P(j)$, is

$$h_P(j) := max\{|C| : C \subset \langle j \rangle_P \text{ and } C \text{ is a chain}\}.$$

The *height* $h(P)$ *of* P is the maximal rank of the elements of $[m]$. The i-*level* of P is $\Gamma_P^i := \{j \in [m] : h_P(j) = i\}$. We define $b_i = \sum_{j \in \Gamma_P^i} k_j$ as the sum of the dimensions of the blocks associated by π to the i-level of P, and we call it the *dimension of* Γ_P^i.

A poset-block (P, π) is said to be *hierarchical* if given $j_1 \in \Gamma_P^i$ we have that $j_1 \preccurlyeq_P j$ for all $j \in \Gamma_P^{i+1}$. Defining a hierarchical poset on $[m]$ is equivalent to choosing an ordered partition of $[m]$ (the partition defined by the different levels), thus it is a quite large set of posets (or poset metrics) including, as a particular case, the block structures presented in [2] when the poset structure is trivial ($h(P) = 1$), the Niederreiter-Rosenbloom-Tsfasman metric (see [9]) with a unique chain when $h(P) = m$ and the block structure is trivial ($k_i = 1$ for

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every $i \in [m]$) and the usual Hamming structure when both the poset and the block structures are trivial.

Given a poset-block (P,π) over $[m]$ such that $|\Gamma_P^i| = m_i$, let σ be a permutation of $[m]$ such that $\{\sigma^{-1}(r_i+1),\cdots,\sigma^{-1}(r_i+m_i)\} = \Gamma_P^i$ where $r_i = m_1+\cdots+m_{i-1}$ and $m_0 = 0$. We let P_1 be the poset induced by σ, ie, the poset in which $\sigma(j_1) \preccurlyeq_{P_1} \sigma(j_2)$ if $j_1 \preccurlyeq_P j_2$. Obviously, P_1 and P are isomorphic posets. If we put $\pi_1(i) = \pi(\sigma^{-1}(i)) = k_i'$, then the map

$$g : (\mathbb{F}_q^{k_1} \times \cdots \times \mathbb{F}_q^{k_m}, d_{(P,\pi)}) \to (\mathbb{F}_q^{k_1'} \times \cdots \times \mathbb{F}_q^{k_m'}, d_{(P_1,\pi_1)})$$

$$(v_1,\cdots,v_m) \mapsto (v_{\sigma(1)},\cdots,v_{\sigma(m)})$$

is, by construction, a linear isometry. Hence, up to a linear isometry, we can and will assume that $\Gamma_P^i = \{r_i+1,\cdots,r_i+m_i\}$, and in this case we say (P,π) has a *natural labeling*. Hence, given $u \in \mathbb{F}_q^n$ we may decompose it as

$$u = \sum_{i=1}^{h(P)} \sum_{j=1}^{m_i} \sum_{l=1}^{k_{(r_i+j)}} u_{r_i+j}^l e_{s(i,j,l)}$$

where $u_{r_i+j}^l \in \mathbb{F}_q$ are scalars and

$$\left\{e_{s(i,j,l)} : 1 \leqslant l \leqslant k_{(r_i+j)}, 1 \leqslant j \leqslant m_i, 1 \leqslant i \leqslant h(P)\right\}$$

is the usual basis of $\mathbb{F}_q^n$, with $s(i,j,l) = l + \sum_{t=0}^{r_i+j-1} k_t$ and $k_0 = 0$.

A $[n,k,\delta]_q$ linear (P,π)-*code* is a k-dimensional subspace $\mathcal{C} \subset \mathbb{F}_q^n$ where $\mathbb{F}_q^n$ is equipped with the poset-block metric $d_{(P,\pi)}$ and

$$\delta = min\{w_{(P,\pi)}(v) : 0 \neq v \in \mathcal{C}\}$$

is the (P,π)-*minimum distance* of $\mathcal{C}$.

Definition 2: Let $\mathcal{C}$ be a linear (P,π)-code. Its *dual code* is defined as

$$\mathcal{C}^\perp = \{x \in \mathbb{F}_q^n : x \cdot u = 0 \ \forall \ u \in \mathcal{C}\}$$

where $x \cdot u$ is the usual formal inner product. We remark that $\mathcal{C}^\perp$ is an $(n-k)$-dimensional linear code. Along this work, $\mathcal{C}^\perp$ is considered to be a linear $(\overline{P},\pi)$-code with parameters $[n,n-k]_q$ and we denote by $\delta^\perp$ its minimal distance (according to the $(\overline{P},\pi)$-metric).

Given a linear (P,π)-code $\mathcal{C}$, the (P,π)-*weight enumerator of* $\mathcal{C}$ is the polynomial

$$W_{\mathcal{C},(P,\pi)}(x) = \sum_{u\in\mathcal{C}} x^{w_{(P,\pi)}(u)} = \sum_{i=0}^{m} A_{i,(P,\pi)}(\mathcal{C})x^i,$$

where $A_{i,(P,\pi)}(\mathcal{C}) = |\{u \in \mathcal{C} : w_{(P,\pi)}(u) = i\}|$. When no confusion may arise, we will use a simplified notation for those coefficients: $A_i = A_{i,(P,\pi)}(\mathcal{C})$ and $\overline{A}_i = A_{i,(\overline{P},\pi)}(\mathcal{C}^\perp)$.

Note that

$$V := \mathbb{F}_q^{b_1} \times \cdots \times \mathbb{F}_q^{b_t}$$

is a vector space over $\mathbb{F}_q$ isomorphic to $\mathbb{F}_q^n$, so that given $u \in \mathbb{F}_q^n$ we can write $u = (u^1,\cdots,u^t)$ where $u^i \in \mathbb{F}_q^{b_i}$ and $u^i = (u_{r_i+1},\cdots,u_{r_i+m_i})$ is such that $u_{r_i+j} \in \mathbb{F}_q^{k_{(r_i+j)}}$.

If P is a poset with t levels, the *leveled* (P,π)-*weight enumerator of* $\mathcal{C}$ is the formal expression

$$W_{\mathcal{C},(P,\pi)}(x;y_0,\cdots,y_t) := \sum_{u\in\mathcal{C}} x^{w_{(P,\pi)}(u)} y_{s_P(u)},$$

where $s_P(u) = max\{i : u^i \in \mathbb{F}_q^{b_i}\backslash\{0\}\}$ and $s_P(0) = 0$. This definition is similar to the one used in [5] in the classification of poset metrics that admits MacWilliams-type identity, ie, the case where the block structure is trivial. It is clear that $W_{\mathcal{C},(P,\pi)}(x) = W_{\mathcal{C},(P,\pi)}(x;1,\cdots,1)$.

Definition 3: We say that a poset-block (P,π) admits a MacWilliams-type identity (*MW-I*) if the $(\overline{P},\pi)$-weight enumerator of $\mathcal{C}^\perp$ is uniquely determined by the (P,π)-weight enumerator of $\mathcal{C}$ for every linear (P,π)-code $\mathcal{C}$.

MacWilliams-type identities in the context of poset codes have interested researchers (see [4], [10] and [11]) since they establish a relation between important invariants of a high information rate code with those of a low dimension code, that are much easier to compute. In 2005, Kim and Oh [5] proved that a poset space admits a MW-I if and only if the poset is hierarchical. In this work we extend this result to the instances that remained open: the instance of poset-block (and block metrics as a particular case).

II. MacWilliams-type identity in (P,π) spaces

The example below shows that the condition established in [5] is not sufficient to ensure MacWilliams-type identity in (P,π) spaces.

Example 1: Let $P = \{1,2,3\}$ be the hierarchical poset with partial order defined by the relations $1 \preccurlyeq_P 2$ and $1 \preccurlyeq_P 3$ so that the dual poset $\overline{P}$ is defined by the relations $2 \preccurlyeq_{\overline{P}} 1$ and $3 \preccurlyeq_{\overline{P}} 1$. Define $\pi : [3] \to \mathbb{N}$ by $\pi(1) = 1$, $\pi(2) = 1$ and $\pi(3) = 2$. Then, direct computations shows that the linear codes

$$\mathcal{C}_1 = \{(0,0,0,0),(0,0,1,0)\}$$

and

$$\mathcal{C}_2 = \{(0,0,0,0),(0,1,0,0)\}$$

over $\mathbb{F}_2^4$ has the same (P,π)-weight enumerator:

$$W_{\mathcal{C}_1,(P,\pi)}(x) = 1 + x^2 = W_{\mathcal{C}_2,(P,\pi)}(x).$$

However,

$$W_{\mathcal{C}_1^\perp,(\overline{P},\pi)}(x) = 1 + 2x + x^2 + 4x^3$$

and

$$W_{\mathcal{C}_2^\perp,(\overline{P},\pi)}(x) = 1 + 3x + 4x^3,$$

so that MW-I does not hold.

A. Necessary condition for MacWilliams-type identity

Let (P,π) be a poset-block in $[m]$ with t levels such that $|\Gamma_P^i| = m_i$ for $i \in [t]$. The three lemmas below are the equivalent, for the poset-block case, of Lemmas (2.1)–(2.4) in [5]. Despite the fact their proofs for poset-block being more delicate than in the case of posets (where the blocks are trivial), they are quite similar.

Lemma 1: Given $u \in \mathbb{F}_q^n$ then $w_{(\overline{P},\pi)}(u) = m \Leftrightarrow supp_\pi(u) \supset \Gamma_P^1$. Furthermore, if u satisfies $supp_\pi(u) \subset \Gamma_P^1$, we have that

$$q^{n-b_1} \bigm| |\{v \in \mathbb{F}_q^n : u \cdot v = 0 \text{ and } w_{(\overline{P},\pi)}(v) = m\}|.$$

where $a\bigm|b$ means a divides b and b_1 is the dimension of Γ_P^1.

Proof: The first affirmation is evident. Let $u \in \mathbb{F}_q^n$ such that $supp_\pi(u) \subset \Gamma_P^1$. Without loss of generality we can assume that $\Gamma_P^1 = [m_1]$ and $u = (u_1, \cdots, u_i, 0, \cdots, 0)$ where $i \leqslant m_1$ and $u_j \in \mathbb{F}_q^{k_j} \backslash \{0\}$ for all $j \in [i]$. Set

$$A := \{(v_1, \cdots, v_i) : v_j \in \mathbb{F}_q^{k_j} \backslash \{0\} \ \forall \ j \in [i] \text{ and } u_1 \cdot v_1 + \cdots + u_i \cdot v_i = 0\}.$$

In each $\mathbb{F}_q^{k_j}$ space we have $q^{k_j} - 1$ non null vectors, then we have $\prod_{j=i+1}^{m_1} (q^{k_j} - 1)$ possibilities of vectors in the blocks associated to elements of the subset $\{i+1, \cdots, m_1\}$ of $[m]$, since we do not impose restrictions in the $m - m_1$ remaining blocks, by first claim it follows that

$$|\{v \in \mathbb{F}_q^n : u \cdot v = 0 \text{ and } w_{(\overline{P},\pi)}(v) = m\}| = q^{n-b_1} |A| \prod_{j=i+1}^{m_1} (q^{k_j} - 1).$$

■

Lemma 2: If a poset-block (P, π) admits a MW-I, then $j \preceq_P i$ for every $i \in \Gamma_P^2$ and $j \in \Gamma_P^1$.

Proof: Assuming $\Gamma_P^2 \neq \emptyset$, it follows that $m > m_1$. Suppose there is $i \in \Gamma_P^2$ that is not comparable to some $j \in \Gamma_P^1$, that is, such that $|\langle i \rangle_P| < 1 + |\Gamma_P^1|$. In this instance there are $u, v \in \mathbb{F}_q^n$ such that $supp_\pi(u) = \{i\}$, $supp_\pi(v) \subset \Gamma_P^1$ and $|\langle supp_\pi(u) \rangle_P| = |\langle supp_\pi(v) \rangle_P|$. Without loss of generality we can admit that $u = e_{s(2,1,1)}$. If $\mathcal{C}_u$ and $\mathcal{C}_v$ are two one-dimensional linear (P, π)-codes generated by u and v respectively, then $\mathcal{C}_u$ and $\mathcal{C}_v$ have same (P, π)-weight enumerator. Assuming the MW-I in (P, π), $\mathcal{C}_u^\perp$ and $\mathcal{C}_v^\perp$ must have the same $(\overline{P}, \pi)$-weight enumerator. If $x \in \mathcal{C}_u^\perp$ then $x_{r_2+1}^1 = 0$. Furthermore, by Lemma 1 $w_{(\overline{P},\pi)}(x) = m$ if and only if $\Gamma_P^1 \subset supp_\pi(x)$, so that

$$|\{x \in \mathcal{C}_u^\perp : w_{(\overline{P},\pi)}(x) = m\}| = |\{x \in \mathbb{F}_q^n : x_{r_2+1}^1 = 0 \text{ and } \Gamma_P^1 \subset supp_\pi(x)\}|.$$

Set

$$A := \{x_i \in \mathbb{F}_q^{k_i} : x_{r_2+1}^1 = 0\}$$

and

$$B := \{(x_1, \cdots, x_m) : x_j \neq 0 \ \forall \ j \in [m_1] \text{ and } x_i = 0\}.$$

Since $i \notin \Gamma_P^1$, $|A| = q^{k_i - 1}$ and $|B| = q^{n-k_i-b_1} \prod_{j=1}^{m_1} (q^{k_j} - 1)$, it follows that

$$|\{x \in \mathcal{C}_u^\perp : w_{(\overline{P},\pi)}(x) = m\}| = |B||A| = q^{n-b_1-1} \prod_{j=1}^{m_1} (q^{k_j} - 1). \quad (1)$$

On the other hand

$$\{x \in \mathcal{C}_v^\perp : w_{(\overline{P},\pi)}(x) = m\} = \{x \in \mathbb{F}_q^n : x \cdot v = 0 \text{ and } w_{(\overline{P},\pi)}(x) = m\}, \quad (2)$$

hence, by Lemma 1 and by Equations (1) and (2) it follows that

$$q \bigm| \prod_{j=1}^{m_1} (q^{k_j} - 1),$$

a contradiction because q is power of a prime. Therefore $|\langle i \rangle_P| = 1 + |\Gamma_P^1|$, ie, $j \preceq_P i$ for all $j \in \Gamma_P^1$. ■

Let $P^j = P \backslash \cup_{i=1}^j \Gamma_P^i$. Consider on P^j the order induced by P and let $\pi^j = \pi|_{[m] \backslash \cup_{i=1}^j \Gamma_P^i}$ be the restriction of π to $[m] \backslash \cup_{i=1}^j \Gamma_P^i$.

Lemma 3: If a poset-block (P, π) admits the MW-I, then the poset-block (P^1, π^1) also admits.

Proof: If $m = m_1$ we have that $[m] \backslash \Gamma_P^1 = \emptyset$ and there is nothing to be proved. Let us assume that $m > m_1$ and let $\mathcal{C}_1'$ and $\mathcal{C}_2'$ be linear (P^1, π^1)-codes with length $n - b_1$ and same (P^1, π^1)-weight enumerator. For $i = 1, 2$, let

$$\mathcal{C}_i := \mathbb{F}_q^{b_1} \oplus \mathcal{C}_i' = \{(u, v) : u \in \mathbb{F}_q^{b_1} \text{ and } v \in \mathcal{C}_i'\}$$

be linear (P, π)-codes with length n and same (P, π)-weight enumerator. Since (P, π) admits MW-I, $\mathcal{C}_1^\perp$ and $\mathcal{C}_2^\perp$ have the same $(\overline{P}, \pi)$-weight enumerator. Furthermore, the dual codes $\mathcal{C}_1^\perp$ and $\mathcal{C}_2^\perp$ can be described as

$$\mathcal{C}_i^\perp = \{(u, v) \in \mathbb{F}_q^{b_1} \times \mathbb{F}_q^{n-b_1} : (u, v) \cdot (a, b) = 0 \ \forall \ a \in \mathbb{F}_q^{b_1} \text{ and } b \in \mathcal{C}_i'\}.$$

Being $b \in \mathcal{C}_i'$ the null code-word of $\mathcal{C}_i'$, by definition of $\mathcal{C}_i^\perp$ it follows that u is the null element of $\mathbb{F}_q^{b_1}$, hence

$$\mathcal{C}_i^\perp = \{(u, v) : u = 0 \in \mathbb{F}_q^{b_1} \text{ and } v \in \mathcal{C}_i'^\perp\}.$$

Therefore, by puncturing the codes $\mathcal{C}_1^\perp$ and $\mathcal{C}_2^\perp$ in the first b_1 coordinates, it follows that $\mathcal{C}_1'^\perp$ and $\mathcal{C}_2'^\perp$ have the same (P^1, π^1)-weight enumerator. ■

By induction, using Lemmas 2 and 3 we have the following necessary condition for a poset-block (P, π) to admit a MW-I.

Proposition 1: If (P, π) admits the MW-I, then P is a hierarchical poset.

By Example 1 we can conclude that the previous condition is not sufficient to assure an MW-I and the following is also necessary:

Proposition 2: Suppose that (P, π) admits a MW-I. Then, $\pi(j_1) = \pi(j_2)$ for all $j_1, j_2 \in \Gamma_P^i$ and every $1 \leqslant i \leqslant h(P)$, ie, blocks at the same level have the same dimension.

Proof: Given $i \in [h(P)]$ consider $j_1, j_2 \in \Gamma_P^i$ and assume $\pi(j_1) \leqslant \pi(j_2)$. Let $\mathcal{C}_u$ and $\mathcal{C}_v$ be the one-dimensional linear (P, π)-codes with length n generated by $u = e_{s(i,j_1-r_i,1)}$ and $v = e_{s(i,j_2-r_i,1)}$ respectively, where $r_i = m_1 + \cdots + m_{i-1}$. By Proposition 1 the poset P is hierarchical, and since there are

$$(q^{k_{j_1}-1} - 1) + \sum_{\substack{j \in \Gamma_P^i \\ j \neq j_1}} (q^{k_j} - 1)$$

elements in $\mathcal{C}_u^\perp$ with support contained in a unique block at the i-level of P, then

$$A_{m_{i+1}+\cdots+m_t+1,(\overline{P},\pi)}(\mathcal{C}_u^\perp) = (q^{k_{j_1}-1}-1)\prod_{\substack{j\in\Gamma_P^l\\ i<l\leqslant t}} q^{k_j} + \sum_{\substack{j\in\Gamma_P^i\\ j\neq j_1}}(q^{k_j}-1)\prod_{\substack{j\in\Gamma_P^l\\ i<l\leqslant t}} q^{k_j}$$

since, when considering the dual poset $\overline{P}$, there are no restrictions on the coordinates in the blocks belonging to levels higher (in P) than i. In a similar way we find that

$$A_{m_{i+1}+\cdots+m_t+1,(\overline{P},\pi)}(\mathcal{C}_v^\perp) = (q^{k_{j_2}-1}-1)\prod_{\substack{j\in\Gamma_P^l\\ i<l\leqslant t}} q^{k_j} + \sum_{\substack{j\in\Gamma_P^i\\ j\neq j_2}}(q^{k_j}-1)\prod_{\substack{j\in\Gamma_P^l\\ i<l\leqslant t}} q^{k_j}.$$

Assuming that (P,π) admits a MW-I it follows that

$$A_{m_{i+1}+\cdots+m_t+1,(\overline{P},\pi)}(\mathcal{C}_u^\perp) = A_{m_{i+1}+\cdots+m_t+1,(\overline{P},\pi)}(\mathcal{C}_v^\perp),$$

ie, $\pi(j_1)=\pi(j_2)$. ■

From the two previous propositions it follows that:

Theorem 1: If (P,π) admits a MacWilliams-type identity then P is a hierarchical poset and blocks at the same level have the same dimension.

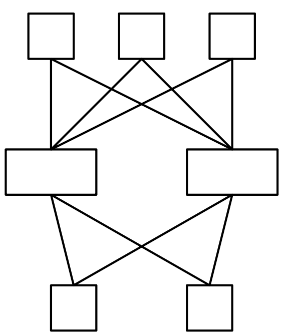

Fig. 1. Diagram of a typical hierarchical poset-block with blocks of equal dimension at each level.

B. Sufficient condition for MacWilliams-type identity

In this section we will prove that the conditions found to be necessary will also be sufficient. Let (P,π) be a hierarchical poset-block over $[m]$ with t levels such that $|\Gamma_P^i| = m_i$, with $i\in[t]$. As before, we let $m_0=0$ and $r_i = m_1+\cdots+m_{i-1}$. We can assume without loss of generality that $\Gamma_P^i = \{r_i+1,\cdots,r_i+m_i\}$. Let $d_i=\pi(r_i+j)$ for every $j\in[m_i]$, ie, blocks at the same level have the same dimension. Under this condition the dimension of the i-level is given by

$$b_i = \sum_{j=1}^{m_i}\pi(r_i+j) = m_i d_i.$$

We note that $n = b_1+\cdots+b_t$ and $m = m_1+\cdots+m_t$. Given $i\in\{0,1,\cdots,t\}$, set

- $\widehat{b_i} = n-(b_1+\cdots+b_i)$;
- $\widehat{m_i} = m-(m_1+\cdots+m_i)$ and
- $\widetilde{u^{i+1}} = (u^{i+1},\cdots,u^t)\in\mathbb{F}_q^{\widehat{b_i}}$.

With this definitions we have that $w_{(\overline{P},\pi)}(u) = \widehat{m_i} + w_{\pi_i}(u^i)$ where $w_{\pi_i}(u^i)$ is the $(\Gamma_P^i, \pi|_{\Gamma_P^i})$-weight of u^i, the block weight as introduced in [2]. Given a linear (P,π)-code $\mathcal{C}$, the set

$$\mathcal{C}_i = \{u\in\mathcal{C} : \widetilde{u^{i+1}} = 0\}$$

is a subcode of $\mathcal{C}$ that can be decomposed as $\mathcal{C}_i = \mathcal{C}_i^0 \sqcup \mathcal{C}_i^1$ where

$$\mathcal{C}_i^0 = \{u\in\mathcal{C}_i : u^i = 0\} \text{ and } \mathcal{C}_i^1 = \{u\in\mathcal{C}_i : u^i \neq 0\}.$$

Given $i\in[t]$, *the weight enumerator of the i-level* of P is defined as

$$LW^{(i)}_{\mathcal{C},(P,\pi)}(x) := \sum_{j=1}^{m_i} A_{r_i+j}x^{r_i+j}. \tag{3}$$

The coefficients of this polynomial represent the weight distribution of code-words such that its support contains elements in the i-level and do not contain elements that are above the i-level. If we define $LW^{(0)}_{\mathcal{C},(P,\pi)}(x) = A_0$, it is clear that

$$W_{\mathcal{C},(P,\pi)}(x;y_0,y_1,\cdots,y_t) = \sum_{i=0}^{t} LW^{(i)}_{\mathcal{C},(P,\pi)}(x)y_i. \tag{4}$$

If for each $i\in[t]$ we have that $y_j=1$ for $j\leqslant i$ and $y_j=0$ for all $j>i$, then the leveled (P,π)-weight enumerator of $\mathcal{C}$ coincides with the (P,π)-weight enumerator of $\mathcal{C}_i$, hence,

$$\begin{aligned} W_{\mathcal{C}_i,(P,\pi)}(x) - W_{\mathcal{C}_{i-1},(P,\pi)}(x) &= LW^{(i)}_{\mathcal{C},(P,\pi)}(x) \\ &= \sum_{u\in\mathcal{C}_i^1} x^{w_{(P,\pi)}(u)}. \end{aligned} \tag{5}$$

We introduce now some concepts related to additive characters, that will be used in the proof in a way similar to what was done first by MacWilliams [12] in the classical case and later in the poset case (see [4], [5] and [11]).

Definition 4: An additive character χ in $\mathbb{F}_q$ is an homomorphism of the additive group $\mathbb{F}_q$ into the multiplicative group of complex numbers with norm 1. If $\chi\equiv 1$, we say that χ is the trivial additive character.

Lemma 4: Let χ be a non trivial additive character of $\mathbb{F}_q$ and α a fix element of $\mathbb{F}_q^j$. Then

$$\sum_{\beta\in\mathbb{F}_q^j}\chi(\alpha\cdot\beta) = \begin{cases} q^j, & \text{if } \alpha \text{ is null} \\ 0, & \text{otherwise}\end{cases}$$

Lemma 5: Let χ be a non trivial additive character of $\mathbb{F}_q$. For any linear code $\mathcal{C}\subset\mathbb{F}_q^n$

$$\sum_{v\in\mathcal{C}}\chi(u\cdot v) = \begin{cases} 0, & \text{if } \quad u\in\mathbb{F}_q^n\backslash\mathcal{C}^\perp \\ |\mathcal{C}|, & \text{if } \quad u\in\mathcal{C}^\perp\end{cases}$$

Definition 5: (Hadamard Transform) Let f be a complex function defined in $\mathbb{F}_q^n$. The Hadamard transform of f is

$$\widehat{f}(u) = \sum_{v\in\mathbb{F}_q^n}\chi(u\cdot v)f(v).$$

The proof of the following lemma may be found in [13].

Lemma 6: (Discrete Poisson Summation Formula) Let $\mathcal{C} \subset \mathbb{F}_q^n$ be a linear code and f a complex function defined on $\mathbb{F}_q^n$. Then

$$\sum_{v \in \mathcal{C}^\perp} f(v) = \frac{1}{|\mathcal{C}|} \sum_{u \in \mathcal{C}} \widehat{f}(u). \tag{6}$$

In case both the block and the poset structures are trivial (the Hamming case), the use of the discrete Poisson summation formula to establish the MacWilliams identity is simple: just consider $f(u) = x^{w_H(u)}$ and apply the discrete Poisson summation formula to the Hadamard transform $\widehat{f}(u) = (1 + (q-1)x)^{n - w_H(u)}(1-x)^{w_H(u)}$ (as in [12]). If $f(u) = x^{w_{(\overline{P},\pi)}(u)} z_{s_{\overline{P}}(u)}$, where $s_{\overline{P}}(u) = min\{i : u^i \in \mathbb{F}_q^{b_i} \backslash \{0\}\}$ and $s_{\overline{P}}(0) = t+1$, then

$$\sum_{u \in \mathcal{C}^\perp} f(u) = W_{\mathcal{C}^\perp, (\overline{P},\pi)}(x; z_{t+1}, \cdots, z_1). \tag{7}$$

Therefore we will extend this result determining the Hadamard transform of the function $f(u) = x^{w_{(\overline{P},\pi)}(u)} z_{s_{\overline{P}}(u)}$.

Given $i \in \{0, \cdots, t\}$, we set

$$B_i = \{u \in \mathbb{F}_q^n : u^j = 0 \ \forall \ 1 \leqslant j \leqslant i \text{ and } u^{i+1} \neq 0\}$$

and then

$$\begin{aligned}
\widehat{f}(u) &= \sum_{v \in \mathbb{F}_q^n} \chi(u \cdot v) f(v) \\
&= \sum_{i=0}^{t} \sum_{v \in B_i} \chi(u \cdot v) f(v) \\
&= \sum_{i=0}^{t} \sum_{v \in B_i} \chi(u \cdot v) x^{w_{(\overline{P},\pi)}(v)} z_{s_{\overline{P}}(v)}.
\end{aligned}$$

Defining $S_i(u) = \sum_{v \in B_i} \chi(u \cdot v) x^{w_{(\overline{P},\pi)}(v)} z_{s_{\overline{P}}(v)}$, since $B_t = \{0\}$ it follows that

$$\widehat{f}(u) = z_{t+1} + \sum_{i=1}^{t} S_{i-1}(u). \tag{8}$$

The proof of the sufficiency condition will be done with the aid of four lemmas that allow us to determine $\sum_{u \in \mathcal{C}} \widehat{f}(u)$ as a function of the leveled weight enumerator of $\mathcal{C}$. From Equation (8) and assuming that the poset is hierarchical and that blocks at the same level has the same dimension, we get the following four lemmas.

Lemma 7: To $i \in [t]$, denote $\gamma_i = (q^{d_i} - 1)$, then for all $u \in \mathbb{F}_q^n$ we have that

$$S_{i-1}(u) = z_i x^{\widehat{m_i}} q^{\widehat{b_i}} [(1-x)^{w_{\pi_i}(u^i)} (1 + \gamma_i x)^{m_i - w_{\pi_i}(u^i)} - 1]$$

if $\widetilde{u^{i+1}}$ is a null vector and $S_{i-1}(u) = 0$ if $\widetilde{u^{i+1}}$ is not a null vector.

Proof: Since P is a hierarchical poset, if $v \in B_{i-1}$, then $w_{(\overline{P},\pi)}(v) = \widehat{m_i} + w_{\pi_i}(v^i)$, and we denote $v = (v^1, \cdots, v^i, \widetilde{v^{i+1}})$. By definition of $S_{i-1}(u)$ and since a character is an additive homomorphism, we have that

$$S_{i-1}(u) = z_i x^{\widehat{m_i}} \sum_{\widetilde{v^{i+1}} \in \mathbb{F}_q^{\widehat{b_i}}} \chi(\widetilde{u^{i+1}} \cdot \widetilde{v^{i+1}}) \times \sum_{v^i \in \mathbb{F}_q^{b_i} \backslash \{0\}} \chi(u^i \cdot v^i) x^{w_{\pi_i}(v^i)}. \tag{9}$$

By Lemma 4

$$\sum_{\widetilde{v^{i+1}} \in \mathbb{F}_q^{\widehat{b_i}}} \chi(\widetilde{u^{i+1}} \cdot \widetilde{v^{i+1}}) = \begin{cases} q^{\widehat{b_i}}, & \text{if } \widetilde{u^{i+1}} \text{ is null} \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

Being $r_i = m_1 + \cdots + m_{i-1}$ and χ a non trivial additive character, since $v_{r_i+j} \in \mathbb{F}_q^{d_i}$ for every $j \in \{1, \cdots, m_i\}$ and $w_{\pi_i}(v^i) = \sum_{j=1}^{m_i} \delta(v_{r_i+j})$ where $\delta(u)$ is the Kronecker function (it returns 1 if u is not null and 0 otherwise), it follows that

$$\sum_{v^i \in \mathbb{F}_q^{b_i}} \chi(u^i \cdot v^i) x^{w_{\pi_i}(v^i)} = \prod_{j=1}^{m_i} \sum_{v_{r_i+j} \in \mathbb{F}_q^{d_i}} \chi(u_{r_i+j} \cdot v_{r_i+j}) x^{\delta(v_{r_i+j})}.$$

Therefore, if u_{r_i+j} is a null vector, then

$$\sum_{v_{r_i+j} \in \mathbb{F}_q^{d_i}} \chi(u_{r_i+j} \cdot v_{r_i+j}) x^{\delta(v_{r_i+j})} = 1 + \gamma_i x.$$

If u_{r_i+j} is not a null vector, since $u_{r_i+j} \notin (\mathbb{F}_q^{d_i})^\perp$, then by Lemma 5

$$\sum_{v_{r_i+j} \in \mathbb{F}_q^{d_i}} \chi(u_{r_i+j} \cdot v_{r_i+j}) x^{\delta(v_{r_i+j})} = 1 + x \sum_{v_{r_i+j} \in \mathbb{F}_q^{d_i} \backslash \{0\}} \chi(u_{r_i+j} \cdot v_{r_i+j}) = 1 - x,$$

hence

$$\sum_{v^i \in \mathbb{F}_q^{b_i} \backslash \{0\}} \chi(u^i \cdot v^i) x^{w_{\pi_i}(v^i)} = (1-x)^{w_{\pi_i}(u^i)} (1 + \gamma_i x)^{m_i - w_{\pi_i}(u^i)} - 1. \tag{11}$$

The result follows from Equations (9), (10) and (11). ∎

Lemma 8: Given $i \in [t]$, define

$$Q_i(x) := \frac{1-x}{1+\gamma_i x},$$

$$a_i(x) := q^{\widehat{b_i}} \left(\frac{1+\gamma_i x}{x}\right)^{m - \widehat{m_i}} (1-x)^{\widehat{m_{i-1}}}$$

and

$$c_i(x) := x^{\widehat{m_i}} q^{\widehat{b_i}} \left(\frac{1-x}{Q_i(x)}\right)^{m_i},$$

where $\gamma_i = q^{d_i} - 1$. Then,

$$\begin{aligned}
\sum_{u \in \mathcal{C}} \widehat{f}(u) =& |\mathcal{C}| z_{t+1} \\
&+ \left(\frac{x}{1-x}\right)^m \sum_{i=1}^{t} a_i(x) z_i LW^{(i)}_{\mathcal{C},(P,\pi)}(Q_i(x)) \\
&+ \sum_{i=1}^{t} z_i c_i(x) |\mathcal{C}_{i-1}| - \sum_{i=1}^{t} z_i x^{\widehat{m_i}} q^{\widehat{b_i}} |\mathcal{C}_i|.
\end{aligned} \tag{12}$$

Proof: If $u \notin \mathcal{C}_i$ then $\widetilde{u^{i+1}}$ is not a null vector and by Lemma 7 we find that

$$\sum_{u\in\mathcal{C}} S_{i-1}(u) = \sum_{u\in\mathcal{C}_i} S_{i-1}(u) + \sum_{u\in\mathcal{C}\setminus\mathcal{C}_i} S_{i-1}(u) = z_i x^{\widehat{m_i}} q^{\widehat{b_i}} \left[\left(\frac{1-x}{Q_i(x)}\right)^{m_i} \sum_{u\in\mathcal{C}_i} Q_i(x)^{w_{\pi_i}(u^i)} - |\mathcal{C}_i|\right]. \quad (13)$$

If $u \in \mathcal{C}_i^1$, then $w_{(P,\pi)}(u) = w_{\pi_i}(u^i) + (m - \widehat{m_{i-1}})$, and if $u \in \mathcal{C}_i^0$ we have $w_{\pi_i}(u^i) = 0$. Since $\mathcal{C}_{i-1} = \mathcal{C}_i^0$, then $|\mathcal{C}_{i-1}| = |\mathcal{C}_i^0|$ and hence

$$\sum_{u\in\mathcal{C}_i} Q_i(x)^{w_{\pi_i}(u^i)} = \sum_{u\in\mathcal{C}_i^1} Q_i(x)^{w_{\pi_i}(u^i)} + \sum_{u\in\mathcal{C}_i^0} Q_i(x)^{w_{\pi_i}(u^i)} = \frac{1}{Q_i(x)^{m-\widehat{m_{i-1}}}} \sum_{u\in\mathcal{C}_i^1} Q_i(x)^{w_{(P,\pi)}(u)} + |\mathcal{C}_{i-1}|. \quad (14)$$

Since $m - \widehat{m_{i+1}} + m_i = m - \widehat{m_i}$ and by Equation (3) we have that $\sum_{u\in\mathcal{C}_i^1} Q_i(x)^{w_{(P,\pi)}(u)} = LW^{(i)}_{\mathcal{C},(P,\pi)}(Q_i(x))$, by replacing Equation (14) into (13) it follows that

$$\sum_{u\in\mathcal{C}} S_{i-1}(u) = \left(\frac{x}{1-x}\right)^m q^{\widehat{b_i}} \left(\frac{1+\gamma_i x}{x}\right)^{m-\widehat{m_i}} \times (1-x)^{\widehat{m_{i-1}}} z_i LW^{(i)}_{\mathcal{C},(P,\pi)}(Q_i(x)) + z_i x^{\widehat{m_i}} q^{\widehat{b_i}} \left[\left(\frac{1-x}{Q_i(x)}\right)^{m_i} |\mathcal{C}_{i-1}| - |\mathcal{C}_i|\right]. \quad (15)$$

By Identity (8), $\widehat{f}(u) = z_{t+1} + \sum_{i=1}^t S_{i-1}(u)$, then by Equation (15)

$$\begin{aligned}\sum_{u\in\mathcal{C}} \widehat{f}(u) &= |\mathcal{C}| z_{t+1} + \sum_{i=1}^t \sum_{u\in\mathcal{C}} S_{i-1}(u) \\ &= |\mathcal{C}| z_{t+1} + \left(\frac{x}{1-x}\right)^m \sum_{i=1}^t a_i(x) z_i LW^{(i)}_{\mathcal{C},(P,\pi)}(Q_i(x)) \\ &+ \sum_{i=1}^t z_i c_i(x)|\mathcal{C}_{i-1}| - \sum_{i=1}^t z_i x^{\widehat{m_i}} q^{\widehat{b_i}} |\mathcal{C}_i|.\end{aligned}$$

■

In the definition of $W_{\mathcal{C},(P,\pi)}(x; y_0, \cdots, y_t)$, the $y_i's$ were considered as formal symbols. In two next lemmas we consider specific situations that will determine the weight enumerator in the stated conditions.

Lemma 9: Let

$$g_j = \begin{cases} \sum_{i=j+1}^t c_i(x) z_i, & \text{if } 0 \leqslant j \leqslant t-1 \\ 0, & \text{if } j = t \end{cases}.$$

Then

$$\sum_{i=1}^t z_i c_i(x) |\mathcal{C}_{i-1}| = W_{\mathcal{C},(P,\pi)}(1; g_0, \cdots, g_t).$$

Proof: Since $r_i = m_1 + \cdots + m_{i-1}$ and

$$|\mathcal{C}_i| = A_0 + A_1 + \cdots + A_{r_i+m_i} = \sum_{j=0}^i LW^{(j)}_{\mathcal{C},(P,\pi)}(1), \quad (16)$$

then

$$\begin{aligned}\sum_{i=1}^t z_i c_i(x)|\mathcal{C}_{i-1}| = \\ & A_0(c_1(x)z_1 + c_1(x)z_2 + \cdots + c_t(x)z_t) \\ & + (A_1 + \cdots + A_{m_1})(c_2(x)z_2 + \cdots + c_t(x)z_t) \\ & + \cdots + \\ & + (A_{m_1+\cdots+m_{t-2}+1} + \cdots + A_{m_1+\cdots+m_{t-1}})c_t(x)z_t \\ & = \sum_{i=0}^t LW^{(i)}_{\mathcal{C},(P,\pi)}(1) g_i\end{aligned}$$

hence the result follows from Identity (4). ■

The proof of the next lemma is omitted since it follows the same steps as in the proof of Lemma 9.

Lemma 10: Let

$$h_j = \begin{cases} \sum_{i=j}^t z_i x^{\widehat{m_i}} q^{\widehat{b_i}}, & \text{if } 1 \leqslant j \leqslant t \\ \sum_{i=1}^t z_i x^{\widehat{m_i}} q^{\widehat{b_i}}, & \text{if } j = 0 \end{cases}.$$

Then

$$\sum_{i=1}^t z_i x^{\widehat{m_i}} q^{\widehat{b_i}} |\mathcal{C}_i| = W_{\mathcal{C},(P,\pi)}(1; h_0, \cdots, h_t).$$

Before we proceed to prove the next theorem we recall we are assuming the following collection of conditions and notations:

- (P, π) a poset-block over $[m]$ with t levels;
- P is hierarchical;
- $r_i = m_1 + \cdots + m_{i-1}$;
- $\Gamma_P^i = \{r_i + 1, \cdots, r_i + m_i\}$;
- $d_i = \pi(r_i + j)$ for every $j \in \{1, \cdots, m_i\}$;
- $b_i = m_i d_i$ is such that $\sum_{i=1}^t b_i = n$.

Now we can prove that necessary conditions stated in Theorem 1 are also sufficient to have a MW-I.

Theorem 2: Under the conditions above stated, the poset-block (P, π) admits a MacWilliams-type identity.

Proof: By (6) and (7) we have that

$$W_{\mathcal{C}^\perp,(\overline{P},\pi)}(x; z_{t+1}, \cdots, z_1) = \frac{1}{|\mathcal{C}|} \sum_{u\in\mathcal{C}} \widehat{f}(u). \quad (17)$$

Considering Equation (4) we have that

$$a_i(x) z_i LW^{(i)}_{\mathcal{C},(P,\pi)}(Q_i(x)) = W_{\mathcal{C},(P,\pi)}(Q_i(x); y_0, \cdots, y_t),$$

for every $i \in \{1, \cdots, t\}$, where $a_i(x) z_i = y_i$ and $y_j = 0$ for every $j \neq i$. Substituting the identities obtained in Lemma 9 and Lemma 10 into Equation (12) it follows that

$$\begin{aligned}|\mathcal{C}|W_{\mathcal{C}^\perp,(\overline{P},\pi)}(x;z_{t+1},\cdots,z_1) &= |\mathcal{C}|z_{t+1}\\ &+\left(\frac{x}{1-x}\right)^m W_{\mathcal{C},(P,\pi)}(Q_1(x);0,a_1(x)z_1,0,\cdots,0)\\ &+\left(\frac{x}{1-x}\right)^m W_{\mathcal{C},(P,\pi)}(Q_2(x);0,0,a_2(x)z_2,0,\cdots,0)\\ &+\cdots+\left(\frac{x}{1-x}\right)^m W_{\mathcal{C},(P,\pi)}(Q_t(x);0,\cdots,0,a_t(x)z_t)\\ &+W_{\mathcal{C},(P,\pi)}(1;g_0,\cdots,g_t)-W_{\mathcal{C},(P,\pi)}(1;h_0,\cdots,h_t).\end{aligned}$$

On the left side of the above equality we have the leveled weight enumerator of $\mathcal{C}^\perp$ (the dual code of $\mathcal{C}$). On the right side we have an expression that depends not on the code itself but only on the leveled weight enumerator of $\mathcal{C}$. Hence, if $\mathcal{C}_1$ is a linear (P,π)-code that has the same (P,π)-polynomial as $\mathcal{C}$, since $W_{\mathcal{C}_1^\perp,(\overline{P},\pi)}(x;1,\cdots,1)$ is the $(\overline{P},\pi)$-polynomial of $\mathcal{C}_1^\perp$, it follows that

$$W_{\mathcal{C}_1^\perp,(\overline{P},\pi)}(x;1,\cdots,1)=W_{\mathcal{C}^\perp,(\overline{P},\pi)}(x;1,\cdots,1),$$

ie, the $(\overline{P},\pi)$-polynomial of $\mathcal{C}^\perp$ is uniquely determined by (P,π)-polynomial of $\mathcal{C}$ for every code $\mathcal{C}$, hence the poset-block structure admits a MW-I. ■

C. Relationship between Weight Distributions

In this section, we will use the same conditions and notations stated before Theorem 2 in the previous section. For every $k\in\{0,\cdots,n\}$, let

$$P_k^{\gamma_i}(x:n)=\sum_{l=0}^{k}(-1)^l\gamma_i^{k-l}\binom{x}{l}\binom{n-x}{k-l}$$

be the Krawtchouk polynomial whose generator function is given by

$$(1+\gamma_i z)^{n-x}(1-z)^x=\sum_{k=0}^{\infty}P_k^{\gamma_i}(x:n)z^k. \tag{18}$$

If $x\in\{0,\cdots,n\}$, we can switch the upper limit of summation by n. This generator functions arise naturally when we are setting a relationship between the (P,π)-polynomial coefficients of $\mathcal{C}$ and the $(\overline{P},\pi)$-polynomial coefficients of $\mathcal{C}^\perp$ (for details about the Krawtchouk polynomials in coding theory, see [12]).

Lemma 11: Let (P,π) be a poset-block over $[m]$ that admits MW-I and $\mathcal{C}$ a linear (P,π)-code with length n. Then

$$|\mathcal{C}|W_{\mathcal{C}^\perp,(\overline{P},\pi)}(x)=|\mathcal{C}|+ \sum_{i=1}^{t}q^{\widehat{b_i}}x^{\widehat{m_i}}\left[\sum_{k=1}^{m_i}\left(a_k(j:m_i)+\binom{m_i}{k}\gamma_i^k|\mathcal{C}_{i-1}|\right)x^k\right] \tag{19}$$

where $a_k(j:m_i)=\sum_{j=1}^{m_i}A_{r_i+j}P_k^{\gamma_i}(j:m_i)$.

Proof: Set

$$E_1(x)= \sum_{i=1}^{t}q^{\widehat{b_i}}\left(\frac{1+\gamma_i x}{x}\right)^{m-\widehat{m_i}}(1-x)^{\widehat{m_{i-1}}}LW^{(i)}_{\mathcal{C},(P,\pi)}(Q_i(x))$$

and

$$E_2(x)=\sum_{i=1}^{t}x^{\widehat{m_i}}q^{\widehat{b_i}}\left(\left(\frac{1-x}{Q_i(x)}\right)^{m_i}|\mathcal{C}_{i-1}|-|\mathcal{C}_i|\right).$$

Putting $z_1=\cdots=z_{t+1}=1$ and replacing (12) in (17), it follows that

$$W_{\mathcal{C}^\perp,(\overline{P},\pi)}(x)=\ 1+F_1(x)+\frac{1}{|\mathcal{C}|}E_2(x) \tag{20}$$

where $F_1(x)=\frac{1}{|\mathcal{C}|}\frac{x^m}{(1-x)^m}E_1(x)$. Using the Identity (3) in $E_1(x)$ and recalling that $r_i=m-\widehat{m_{i-1}}$ and $\widehat{m_i}-\widehat{m_{i-1}}=m_i$, it follows that

$$\begin{aligned}E_1(x)&=\sum_{i=1}^{t}q^{\widehat{b_i}}\left(\frac{1+\gamma_i x}{x}\right)^{m-\widehat{m_i}}(1-x)^{\widehat{m_{i-1}}}\times\\ &\qquad\sum_{j=1}^{m_i}A_{r_i+j}\left(\frac{1-x}{1+\gamma_i x}\right)^{r_i+j}\\ &=\sum_{i=1}^{t}\frac{q^{\widehat{b_i}}}{x^{m-\widehat{m_i}}}\sum_{j=1}^{m_i}A_{r_i+j}(1+\gamma_i x)^{m_i-j}(1-x)^{m+j},\end{aligned}$$

and therefore

$$\begin{aligned}F_1(x)&=\frac{1}{|\mathcal{C}|}\sum_{i=1}^{t}q^{\widehat{b_i}}x^{\widehat{m_i}}\sum_{j=1}^{m_i}A_{r_i+j}(1+\gamma_i x)^{m_i-j}(1-x)^j\\ &=\frac{1}{|\mathcal{C}|}\sum_{i=1}^{t}q^{\widehat{b_i}}x^{\widehat{m_i}}\sum_{j=1}^{m_i}A_{r_i+j}\sum_{k=0}^{m_i}P_k^{\gamma_i}(j:m_i)x^k\end{aligned} \tag{21}$$

where the second equality follows from (18). Hence if

$$a_k(j:m_i)=\sum_{j=1}^{m_i}A_{r_i+j}P_k^{\gamma_i}(j:m_i),$$

since $P_0^{\gamma_i}=1$, and then $a_0(j:m_i)=|\mathcal{C}_i|-|\mathcal{C}_{i-1}|$ by (16). Therefore

$$\begin{aligned}&|\mathcal{C}|F_1(x)=\\ &\sum_{i=1}^{t}q^{\widehat{b_i}}x^{\widehat{m_i}}\sum_{k=0}^{m_i}\left(\sum_{j=1}^{m_i}A_{r_i+j}P_k^{\gamma_i}(j:m_i)\right)x^k\\ &=\sum_{i=1}^{t}q^{\widehat{b_i}}x^{\widehat{m_i}}\left(|\mathcal{C}_i|-|\mathcal{C}_{i-1}|+\sum_{k=1}^{m_i}a_k(j:m_i)x^k\right).\end{aligned} \tag{22}$$

From Newton's binomial theorem we have that

$$\begin{aligned}E_2(x)&=\sum_{i=1}^{t}x^{\widehat{m_i}}q^{\widehat{b_i}}\left[\left(1+\sum_{k=1}^{m_i}\binom{m_i}{k}\gamma_i^k x^k\right)|\mathcal{C}_{i-1}|-|\mathcal{C}_i|\right]\\ &=\sum_{i=1}^{t}x^{\widehat{m_i}}q^{\widehat{b_i}}\left(|\mathcal{C}_{i-1}|-|\mathcal{C}_i|+\sum_{k=1}^{m_i}\binom{m_i}{k}\gamma_i^k|\mathcal{C}_{i-1}|x^k\right)\end{aligned} \tag{23}$$

and the result follows from (20), (22) and (23). ■

In the conditions stated in Lemma (11) we have that

$$\begin{aligned}
&W_{\mathcal{C}^\perp,(\overline{P},\pi)}(x) = \\
&\quad \overline{A}_0 + (\overline{A}_1 x + \cdots + \overline{A}_{m_t} x^{m_t}) \\
&\quad + (\overline{A}_{m_t+1} x + \cdots + \overline{A}_{m_t+m_{t-1}} x^{m_{t-1}}) x^{m_t} \\
&\quad + \cdots + \\
&\quad + (\overline{A}_{m_t+\cdots+m_2+1} x + \cdots + \overline{A}_{m_t+\cdots+m_1} x^{m_1}) x^{m_t+\cdots+m_2} \\
&\quad = 1 + \sum_{i=1}^{t} x^{\widehat{m_i}} \sum_{k=1}^{m_i} \overline{A}_{\widehat{m_i}+k} x^k \qquad (24)
\end{aligned}$$

therefore from (19) and (24) follows the next theorem, that characterizes the weight distribution of $\mathcal{C}^\perp$ in terms of the distribution of $\mathcal{C}$.

Theorem 3: Let (P,π) be a hierarchical poset-block over $[m]$ with t levels satisfying MW-I and $\mathcal{C}$ a linear (P,π)-code with length n over $\mathbb{F}_q$. Being $\gamma_i = (q^{d_i} - 1)$ and b_j the dimension of Γ_P^j, for any given $i \in [t]$ and $k \in [m_i]$ we have that

$$\begin{aligned}
\overline{A}_{\widehat{m_i}+k} = & \frac{q^{\widehat{b_i}}}{|\mathcal{C}|} \sum_{j=1}^{m_i} \left(A_{r_i+j} P_k^{\gamma_i}(j : m_i) \right) \\
& + \frac{q^{\widehat{b_i}}}{|\mathcal{C}|} \binom{m_i}{k} \gamma_i^k \sum_{j=0}^{r_i} A_j .
\end{aligned}$$

We remark that when we consider a trivial structure of blocks, $b_j = m_j$ and $d_j = 1$ for all $j \in [t]$, then we have the result obtained in Theorem 4.4 from [5]. On the other hand, when considering a trivial poset structure (an antichain poset where none of elements are comparable), then $t = 1$ and $m = m_1$, hence given $k \in [m_1]$ we have that

$$\overline{A}_k = \frac{1}{|\mathcal{C}|} \sum_{j=0}^{m} A_j P_k^{\gamma_1}(j : m).$$

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