Approaching Capacity at High Rates with Iterative Hard-Decision Decoding

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***Abstract*—A variety of low-density parity-check (LDPC) ensembles have now been observed to approach capacity with message-passing decoding. However, all of them use soft (i.e., non-binary) messages and a posteriori probability (APP) decoding of their component codes. In this paper, we analyze a class of spatially-coupled generalized LDPC codes and observe that, in the high-rate regime, they can approach capacity under iterative hard-decision decoding. These codes can be seen as generalized product codes and are closely related to braided block codes.**

***Index Terms*—GLDPC codes, density evolution, product codes, braided codes, syndrome decoding**

I. Introduction

In his groundbreaking 1948 paper, Shannon defined the capacity of a noisy channel as the largest information rate for which reliable communication is possible [1]. Since then, researchers have spent countless hours looking for ways to achieve this rate in practical systems. In the 1990s, the problem was essentially solved by the introduction of iterative soft decoding for turbo and low-density parity-check (LDPC) codes [2], [3], [4]. Although their decoding complexity is significant, these new codes were adopted quickly in wireless communication systems where the data rates were not too large [5], [6]. In contrast, complexity issues have slowed their adoption in very high-speed systems, such as those used in optical and wireline communication.

In this paper, we consider an ensemble of spatially-coupled generalized LDPC (GLDPC) codes based on t-error correcting block codes. For the binary symmetric channel (BSC), we show that the redundancy-threshold tradeoff of this ensemble, under iterative *hard-decision decoding*, scales optimally in the high-rate regime. To the best of our knowledge, this is the first example of an iterative hard-decision decoding (HDD) system that can approach capacity. It is interesting to note that iterative HDD of product codes was first proposed well before the recent revolution in iterative decoding but the performance gains were limited [7]. Iterative decoding of product codes became competitive only after the advent of iterative soft decoding based on the turbo principle [8], [9].

Our choice of ensemble is motivated by the generalized product codes now used in optical communications [10] and their similarity to braided block codes [11], [12]. In particular, we consider iterative HDD of generalized product codes with t-error correcting component codes. This is similar to other recent work on coding system for optical communication systems [13], [14], [15]. The main difference is that HDD of our spatially-coupled GLDPC ensemble can be rigorously analyzed via density evolution (DE) even when miscorrection occurs. The DE analysis also allows us to show that iterative HDD can approach capacity in the high-rate regime. In [16], counting arguments are used to analyze the iterative HDD of GLDPC codes (without spatial coupling) for adversarial error patterns and therefore somewhat lower thresholds are reported.

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II. Basic Ensemble

Let $\mathcal{C}$ be an (n,k,d) binary linear code that can correct all error patterns of weight at most t (i.e., $d \geq 2t+1$). For example, one might choose $\mathcal{C}$ to be a primitive BCH code with parameters $(2^\nu-1, 2^\nu-\nu t-1, 2t+1)$. Now, we consider a GLDPC ensemble where every bit node satisfies two code constraints defined by $\mathcal{C}$.

Definition 1. Each element of the $(\mathcal{C}, m)$ GLDPC ensemble is defined by a Tanner graph, denoted by $\mathcal{G} = (\mathcal{I} \cup \mathcal{J}, \mathcal{E})$, with a set $\mathcal{I}$ of $N = \frac{mn}{2}$ degree-2 bit nodes and a set $\mathcal{J}$ of m degree-n code-constraint nodes defined by $\mathcal{C}$. A random element from the ensemble is constructed by using an uniform random permutation for the mn edges from the bit nodes to the constraint nodes.

A. Decoder

It is well-known that GLDPC codes perform well under iterative soft decoding [8], [9]. The main drawback is that a posteriori probability (APP) decoding of the component codes can require significant computation. For this reason, we consider message-passing decoding based on bounded-distance decoding (BDD) of the component codes. Let the bit-level mapping implied by BDD, $D_i : \{0,1\}^n \to \{0,1\}$, map the received vector $\boldsymbol{v} \in \{0,1\}^n$ to the i-th decoded bit according to the rule

$$D_i(\boldsymbol{v}) = \begin{cases} c_i & \text{if } \boldsymbol{c} \in \mathcal{C} \text{ satisfies } d_H(\boldsymbol{c},\boldsymbol{v}) \leq t \\ v_i & \text{if } d_H(\boldsymbol{c},\boldsymbol{v}) > t \text{ for all } \boldsymbol{c} \in \mathcal{C}, \end{cases}$$

where $d_H(\cdot,\cdot)$ be the Hamming distance. The same analysis can be applied to symmetric decoders (e.g., all syndrome decoders) where the mapping $D_i(\cdot)$ satisfies $D_i(\boldsymbol{v} \oplus \boldsymbol{c}) = D_i(\boldsymbol{v}) \oplus c_i$ for all $\boldsymbol{c} \in \mathcal{C}$ and $i = 1, \ldots, n$.

Decoding proceeds by passing binary messages along the edges connecting the variable and constraint nodes and is best understood from the implied computation graph. Let $r_i \in \{0,1\}$ denote the received channel value for variable node i and $\mu_{i\to j}^{(\ell)}$ be the binary message from the i-th variable node to the j-th constraint node in the ℓ-th iteration. For simplicity, we assume no bit appears twice in a constraint and let $\sigma_j(k)$ be the index of the variable node connected to the k-th socket of the j-th constraint. Let j' be the other neighbor of the i-th variable node and let $\sigma_j(k) = i$. Then, the iterative decoder is defined by the recursion

$$\mu_{i\to j'}^{(\ell+1)} \triangleq D_i\left(\boldsymbol{v}_{j\to i}^{(\ell)}\right), \tag{1}$$

where, for the i-th variable node and the j-th constraint node, the candidate decoding vector is $\boldsymbol{v}_{j\to i}^{(\ell)} = (\mu_{\sigma_j(1)\to j}^{(\ell)}, \mu_{\sigma_j(2)\to j}^{(\ell)}, \ldots, \mu_{\sigma_j(n)\to j}^{(\ell)})$ except that the k-th entry is replaced by r_i. It is very important to note that the above decoder passes extrinsic messages and *is not the same* as simply iterating between row and column decoders. This allows one to rigorously apply DE.

B. Density Evolution

The iterative decoding performance of GLDPC codes can be analyzed via density evolution (DE) because, for a randomly chosen bit node, any fixed-depth neighborhood in the Tanner graph is a tree with high probability as $m \to \infty$. For HDD of the component codes, this DE can be written as a one-dimensional recursion.

If we assume that the component decoders are symmetric, then it suffices to consider the case where the all-zero codeword is transmitted over a BSC with error probability p [17, pp. 188–191]. Let $x^{(\ell)}$ be the error probability of the hard-decision messages passed from the variable nodes to the constraint nodes after ℓ iterations. For an arbitrary symmetric decoder, let $P_n(i)$ be the probability that a randomly chosen bit is decoded incorrectly when it is initially incorrect and there are i random errors in the other $n-1$ positions. Likewise, let $Q_n(i)$ be the probability that a randomly chosen bit is decoded incorrectly when it is initially correct and there are i random errors in the other $n-1$ positions. Then, for the $(\mathcal{C}, m)$ GLDPC ensemble, the DE recursion implied by (1) is defined by $x^{(0)} = p$, $x^{(\ell+1)} = f_n\left(x^{(\ell)}; p\right)$, and (with $\overline{p} \triangleq 1-p$)

$$f_n(x;p) \triangleq \sum_{i=0}^{n-1} \binom{n-1}{i} x^i (1-x)^{n-i-1} \left(pP_n(i) + \overline{p}Q_n(i)\right). \tag{2}$$

For the iterative BDD described above, the quantities $P(i)$ and $Q(i)$ can be written in terms of the number of codewords, A_l, of weight l in $\mathcal{C}$ [18]. Let us define $l(i,\delta,j) = i-\delta+2j+1$ and $V(n,i,\delta,j) = \binom{l(i,\delta,j)}{l(i,\delta,j)-j}\binom{n-l(i,\delta,j)-1}{\delta-1-j}\binom{n-1}{i}^{-1}$. Since all decoding regions are disjoint, one can compute

$$P_n(i) = 1 - \sum_{\delta=1}^{t}\sum_{j=0}^{\delta-1} \frac{n-l(i,\delta,j)}{n} A_{l(i,\delta,j)} V(n,i,\delta,j) \tag{3}$$

for $t \le i \le n-t-2$ and $P_n(i) = 0$ for $0 \le i \le t-1$. Similarly,

$$Q_n(i) = \sum_{\delta=1}^{t}\sum_{j=0}^{\delta-1} \frac{l(i,\delta,j)+1}{n} A_{l(i,\delta,j)+1} V(n,i,\delta,j) \tag{4}$$

for $t+1 \le i \le n-t-1$, and $Q_n(i) = 0$ for $0 \le i \le t$. Note that, when the code contains the all-one codeword, $P_n(i) = 1$ for $n-t-1 \le i \le n-1$, and $Q_n(i) = 1$ for $n-t \le i \le n-1$.

Similar to DE for LDPC codes on the BEC [17, pp. 95–96], there is a compact characterization of the hard-decision decoding threshold p^*. The successful decoding condition $f_n(x;p) < x$ provides a natural lower bound on the noise threshold and it can be rewritten as $p\left[f_n(x;1) - f_n(x;0)\right] + f_n(x;0) < x$ to show that

$$p^* = \inf_{x\in(0,1)} \frac{x - f_n(x;0)}{f_n(x;1) - f_n(x;0)}.$$

It is also worth noting that essentially the same recursion can be used for a BEC with erasure probability p. In this case, $Q_n(i) = 0$ and one redefines $P_n(i)$ to be the probability that a randomly chosen bit is not recovered when it is initially erased and there are i random erasures in the other $n-1$ positions.

III. Spatially-Coupled Ensemble

Now, we consider a spatially-coupled GLDPC ensemble where every bit node satisfies two code constraints defined by $\mathcal{C}$. Similar to the definition introduced in [19], the spatially-coupled GLDPC ensemble $(\mathcal{C}, m, L, w)$ is defined as follows.

Definition 2. The Tanner graph of an element of the $(\mathcal{C}, m, L, w)$ spatially-coupled GLDPC contains L positions, $[1, L]$, of bit nodes and $L+w-1$ positions, $[1, L+w-1]$, of code-constraint nodes defined by $\mathcal{C}$. Let m be chosen such that mn is divisible by both 2 and w. At each position, there are $\frac{mn}{2}$ degree-2 bit nodes and m degree-n code-constraint nodes. A random element of the $(\mathcal{C}, m, L, w)$ spatially-coupled GLDPC ensemble is constructed as follows. At each bit position and code-constraint position, the mn sockets are partitioned into w groups of $\frac{mn}{w}$ sockets via a uniform random permutation. Let $\mathcal{S}_{i,j}^{(b)}$ and $\mathcal{S}_{i,j}^{(c)}$ be, respectively, the j-th group at the i-th bit position and the j-th group at i-th code-constraint position, where $j \in [0, w-1]$. The Tanner graph is constructed by using a uniform random permutation to connect $\mathcal{S}_{i,j}^{(b)}$ to $\mathcal{S}_{i+j,w-j-1}^{(c)}$ by mapping the $\frac{mn}{w}$ edges between the two groups.

A. Density Evolution

To derive the DE update equation of the $(\mathcal{C}, m, L, w)$ spatially-coupled GLDPC ensemble, we let $x_i^{(\ell)}$ be the average error probability of hard-decision messages emitted by bit nodes at position i after the ℓ-th iteration. According to Definition 2, the average error probability of all inputs to a code-constraint node at position i is $\overline{x}_i^{(\ell)} = \frac{1}{w}\sum_{j=0}^{w-1} x_{i-j}^{(\ell)}$. Also, it follows that $x_i^{(\ell)} = \frac{1}{w}\sum_{k=0}^{w-1} f_n\left(\overline{x}_{i+k}^{(\ell-1)}; p\right)$, where $f_n(x;p)$ is defined in (2) and $x_i^{(\ell)} = 0$ for $i \notin [1, L]$. Therefore, the DE update for this ensemble is given by

$$x_i^{(\ell+1)} = \frac{1}{w}\sum_{k=0}^{w-1} f_n\left(\frac{1}{w}\sum_{j=0}^{w-1} x_{i-j+k}^{(\ell)}; p\right). \tag{5}$$

IV. BCH Component Codes

In the remainder of this paper, an $(n,k,2t+1)$ binary primitive BCH code (or its $(n,k-1,2t+2)$ even-weight subcode) will be used as the component code for both the $(\mathcal{C},m)$ GLDPC and $(\mathcal{C},m,L,w)$ spatially-coupled GLDPC ensembles. When the exact weight spectrum is known, one can compute $P_n(i)$ and $Q_n(i)$ using (3) and (4), respectively. Otherwise, one can use the asymptotically-tight binomial approximation

$$A_l = \begin{cases} 2^{-\nu t}\binom{n}{l}(1+o(1)) & \text{if } d\le l\le n-d,\\ 1, & \text{if } l=0,\ l=n,\\ 0, & \text{otherwise,}\end{cases} \tag{6}$$

where $d = 2t+1$ and $n = 2^\nu - 1$ [20][21].

For the $(n,k-1,2t+2)$ even-weight subcode of an $(n,k,2t+1)$ primitive BCH code, the approximate number of codewords is denoted by $\tilde{A}_l$ where $\tilde{A}_l = A_l$ when l is even and $\tilde{A}_l = 0$ when l is odd. Let $\tilde{P}_n(i)$ and $\tilde{Q}_n(i)$ be the miscorrection probabilities implied by $\tilde{A}_l$ for the even-weight subcode. Similar to $P_n(i)$ and $Q_n(i)$ in the $(n,k,2t+1)$ primitive BCH code, it can be shown that $\tilde{P}_n(i) = 0$ for $0\le i\le t-1$ and $\tilde{Q}_n(i) = 0$ for $0\le i\le t+1$. Then, the DE recursions for the $(\mathcal{C},m)$ GLDPC ensemble and the $(\mathcal{C},m,L,w)$ spatially-coupled GLDPC ensemble can be obtained from (2) and (5), respectively.

A. High-Rate Scaling

In [14], Justesen analyzes the asymptotic performance of long product codes under the assumption that the component decoders have no miscorrection. Using the random graph argument, a recursion for the "Poisson parameter" is obtained. That recursion leads to a threshold, for successful decoding, on the average number of error bits attached to a code-constraint node. In this section, we obtain a similar recursion as the scaled limit, as $n\to\infty$, of our DE analysis. The main contribution is that our approach rigorously accounts for miscorrection.

We first introduce some notation and a few lemmas that simplify the development. Consider the Poisson distribution with mean λ. Let $\phi(\lambda;k)$, $\psi(\lambda;k)$ and $\varphi(\lambda;k)$ be, respectively, the tail probability, the tail probability for the even terms, and the tail probability for the odd terms. Then, we have

$$\phi(\lambda;k) \triangleq \textstyle\sum_{i=k+1}^{\infty}\frac{\lambda^i}{i!}e^{-\lambda} = 1-\frac{\Gamma(k+1,\lambda)}{\Gamma(k+1)}$$
$$\psi(\lambda;k) \triangleq \tfrac{1+e^{-2\lambda}}{2} - \textstyle\sum_{i=0}^{\lfloor k/2\rfloor}\frac{\lambda^{2i}}{(2i)!}e^{-\lambda}$$
$$\varphi(\lambda;k) \triangleq \tfrac{1-e^{-2\lambda}}{2} - \textstyle\sum_{i=0}^{\lfloor k/2\rfloor}\frac{\lambda^{(2i+1)}}{(2i+1)!}e^{-\lambda}.$$

Lemma 3. *For the codes described above and $t\le i\le n-1$,*

$$\lim_{n\to\infty} P_n(i) = \lim_{n\to\infty}\tilde{P}_n(i) = 1.$$

Lemma 4. *For the codes described above and $t+1\le i\le n-t-1$, we have $Q_n(i) = \frac{1}{(t-1)!\,n}(1+o(1))$. Similarly,*

$$\tilde{Q}_n(i) = \begin{cases}\frac{1}{(t-2)!\,n^2}(1+o(1)) & \textit{if } i+t \textit{ is odd},\\ \frac{1}{(t-1)!\,n}(1+o(1)) & \textit{if } i+t \textit{ is even},\end{cases}$$

for $t+2\le i\le n-t-2$.

Consider the DE recursion (2) for the $(\mathcal{C},m)$ GLDPC ensemble. For fixed ρ, let $p = \frac{\rho}{n-1}$ scale with n and $\lambda_n^{(\ell)} = (n-1)x^{(\ell)}$. From (2), the recursion for $\lambda_n^{(\ell)}$ equals

$$\begin{aligned}\lambda_n^{(\ell+1)} &= (n-1)f_n\left(\frac{\lambda_n^{(\ell)}}{n-1};\frac{\rho}{n-1}\right)\\ &= \sum_{i=t}^{n-1}\binom{n-1}{i}\left(\frac{\lambda_n^{(\ell)}}{n-1}\right)^i\left(1-\frac{\lambda_n^{(\ell)}}{n-1}\right)^{n-1-i}\\ &\quad\times\left(\rho\left(P_n(i)-Q_n(i)\right)+(n-1)\,Q_n(i)\right),\end{aligned} \tag{7}$$

with initial value $\lambda_n^{(0)} = \rho$ for all n.

Using Lemma 3 and Lemma 4, we can find a simpler recursion for $\lambda^{(\ell)} \triangleq \lim_{n\to\infty}\lambda_n^{(\ell)}$.

Lemma 5. *Since the limit of the RHS of (7) exists for all $\lambda_n^{(\ell)}\ge 0$, the recursion for $\lambda^{(\ell)}$ is given by $\lambda^{(0)} = \rho$ and*

$$\lambda^{(\ell+1)} = f\left(\lambda^{(\ell)};\rho\right) \triangleq \rho\phi\left(\lambda^{(\ell)};t-1\right) + \frac{1}{(t-1)!}\phi\left(\lambda^{(\ell)};t\right). \tag{8}$$

Remark 6. For any $n<\infty$, $\frac{n}{n-1}\rho$ can be seen as the average number of initial error bits attached to a code-constraint node, and $\frac{n}{n-1}\lambda_n^{(\ell)}$ can be viewed as the average number of error messages passed to a code-constraint node after the ℓ-th iteration. Since $\frac{n}{n-1}\lambda_n^{(\ell)} \to \lambda^{(\ell)}$, it follows that the recursion (8) tracks the evolution of the average number of error messages passed to a code-constraint node.

The DE recursion of the GLDPC ensemble with the even-weight BCH subcode can be obtained by modifying (8).

Lemma 7. *For the GLDPC ensemble based on the even-weight BCH subcode, if t is even, the recursion for $\lambda^{(\ell)}$ is $\lambda^{(\ell+1)} = \tilde{f}_e\left(\lambda^{(\ell)};\rho\right)$ with*

$$\tilde{f}_e\left(\lambda^{(\ell)};\rho\right) \triangleq \rho\phi\left(\lambda^{(\ell)};t-1\right) + \frac{1}{(t-1)!}\psi\left(\lambda^{(\ell)};t\right).$$

If t is odd, the recursion is $\lambda^{(\ell+1)} = \tilde{f}_o\left(\lambda^{(\ell)};\rho\right)$ with

$$\tilde{f}_o\left(\lambda^{(\ell)};\rho\right) \triangleq \rho\phi\left(\lambda^{(\ell)};t-1\right) + \frac{1}{(t-1)!}\varphi\left(\lambda^{(\ell)};t\right).$$

For the spatially-coupled GLDPC ensemble, let $\lambda_i^{(\ell)}$ with $i\in[1,L]$ be the average number of error messages passed to code-constraint nodes averaged over all code-constraint nodes at positions $j\in[i,i+w-1]$ after the ℓ-th iteration. We set $\lambda_i^{(0)} = \rho$ for all $i\in[1,L]$, and set $\lambda_i^{(\ell)} = 0$ for all $i\notin[1,L]$ and $\ell\ge 0$. Similar to (5), the recursion for spatially-coupled ensemble is

$$\lambda_i^{(\ell+1)} = \frac{1}{w}\sum_{k=0}^{w-1} f\left(\frac{1}{w}\sum_{j=0}^{w-1}\lambda_{i-j+k}^{(\ell)};\rho\right) \tag{9}$$

for $i \in [1, L]$. When the even-weight BCH subcode is used as a component code in the spatially-coupled GLDPC ensemble, the recursion becomes

$$\lambda_i^{(\ell+1)} = \begin{cases} \frac{1}{w}\sum_{k=0}^{w-1} \tilde{f}_e\left(\frac{1}{w}\sum_{j=0}^{w-1}\lambda_{i-j+k}^{(\ell)};\rho\right) & \text{if } t \text{ is even,} \\ \frac{1}{w}\sum_{k=0}^{w-1} \tilde{f}_o\left(\frac{1}{w}\sum_{j=0}^{w-1}\lambda_{i-j+k}^{(\ell)};\rho\right) & \text{if } t \text{ is odd.} \end{cases}$$

V. Bounds on the Noise Threshold

Suppose that one ignores the effect of miscorrection and considers the natural hard-decision peeling decoder for the $(\mathcal{C}, m)$ ensemble based on BCH codes, then it is easy to see that at most mt errors can be corrected using BDD. To achieve this upper bound, it must happen that each code corrects exactly t errors. If some codes decode with fewer than t errors, then there is an irreversible loss of error-correcting potential. Since there are $\frac{nm}{2}$ code bits per code constraint, normalizing this number shows that the noise threshold is upper bounded by $\frac{2t}{n}$. In terms of the average number of errors in each code constraint, the threshold is upper bounded by $2t$ because each code involves n bits.

Consider the iterative HDD without miscorrection and let $\hat{f}_n(x;p)$ denote (2) with $Q_n(i) = 0$. To find the noise threshold of the spatially-coupled system, one can apply the results of [22] by observing that the DE update for the $(\mathcal{C}, m)$ GLDPC ensemble defines an scalar admissible system where f, g in [22] are defined by $f(x;\epsilon) = \hat{f}_n(x;p)\big|_{p=\epsilon}$ and $g(x) = x$. In this case, the associated potential function is given by

$$U_n(x;p) = \int_0^x \left(z - \hat{f}_n(z;p)\right) \mathrm{d}z$$

and the potential threshold of iterative HDD without miscorrection is given by

$$\hat{p}_n^{**} = \sup\left\{p \in [0,1] \mid \min_{x\in[0,1]} U_n(x;p) \geq 0\right\}. \tag{10}$$

This threshold for iterative HDD without miscorrection is achieved by $(\mathcal{C}, m, L, w)$ spatially-coupled GLDPC ensembles in the limit where $m \gg L \gg w$ as $w \to \infty$.

When miscorrection is ignored, the recursion $x^{(\ell+1)} = \hat{f}_n(x^{(\ell)};p)$ is equivalent to the DE recursion for the erasure channel with component codes that correct t erasures. Using this setup, one can derive rigorous upper bounds on the maximum a posteriori (MAP) threshold by the theory of extrinsic information transfer (EXIT) functions [17], [23]. Since the component decoder is suboptimal and fails to recover correctable erasure patterns with more than t erasures, the threshold $\hat{p}_n^{**}$ in (10) is not equal to the aforementioned upper bound on the MAP threshold of the scalar admissible system.

Using (8), the high-rate scaling limit of the potential function can be written as

$$\begin{aligned} U(\lambda;\rho) &= \lim_{n\to\infty}(n-1)^2 U_n\left(\tfrac{\lambda}{n-1};\tfrac{\rho}{n-1}\right) \\ &= \int_0^\lambda (z - \rho\phi(\lambda;\rho))\,\mathrm{d}z. \end{aligned}$$

For GLDPC ensembles based on primitive t-error correcting BCH codes, the high-rate limit of the potential threshold, $\hat{\rho}_t^{**}$, for iterative HDD without miscorrection is given by

$$\hat{\rho}_t^{**} = \sup\left\{\rho \in [0,\infty) \mid \min_{\lambda\geq 0} U(\lambda;\rho) \geq 0\right\}. \tag{11}$$

Lemma 8. *For the high-rate scaling limit without miscorrection, the noise threshold in terms of the average number of errors in a code constraint satisfies $\hat{\rho}_t^{**} \geq 2t-2$ for all $t \geq 2$.*

From the threshold of iterative HDD without miscorrection, $\hat{\rho}_t^{**}$, one can bound the threshold of the iterative HDD *with miscorrection* using the following lemma.

Lemma 9. *In the high-rate scaling limit, the noise threshold of iterative HDD with miscorrection, ρ_t^{**}, satisfies $\hat{\rho}_t^{**} - \frac{1}{(t-1)!} \leq \rho_t^{**} \leq \hat{\rho}_t^{**}$ for $t \geq 2$.*

Now, we introduce the notion of ϵ-redundancy.

Definition 10. Let $C(p)$ be the capacity of a BSC(p). For an $\epsilon > 0$, a code ensemble with rate R and threshold p^* is called *ϵ-redundancy achieving* if

$$\frac{1 - C(p^*)}{1 - R} \geq 1 - \epsilon.$$

Let $n_\nu \triangleq 2^\nu - 1$. The following lemma shows that, for any $\epsilon > 0$, a sequence of ensembles with rate $R_\nu = 1 - \frac{2\nu t}{n_\nu}$ and threshold $p_\nu^* = \frac{2t}{n_\nu}$ is ϵ-redundancy achieving over BSC channels when $\nu \in \mathbb{Z}_+$ is large. That is, for any $\epsilon > 0$, there exists a $V \in \mathbb{Z}_+$ such that, for all $\nu \geq V$, one has

$$\frac{1 - C(2tn_\nu^{-1})}{2t\nu n_\nu^{-1}} \geq 1 - \epsilon.$$

Lemma 11. *Consider a sequence of BSCs with error probability $2tn_\nu^{-1}$ for fixed t and increasing ν. Then, the ratio of $1 - C\left(2tn_\nu^{-1}\right)$ to $2t\nu n_\nu^{-1}$ goes to 1 as $\nu \to \infty$. That is*

$$\lim_{\nu\to\infty} \frac{1 - C\left(2tn_\nu^{-1}\right)}{2t\nu n_\nu^{-1}} = 1. \tag{12}$$

Proof: Recall that the capacity of the BSC(p) is $1 - H(p)$, where $H(p) = -p\log_2(p) - (1-p)\log_2(1-p)$ is the binary entropy function. The numerator of the LHS of (12) can be written as

$$H\left(\frac{2t}{n_\nu}\right) = \frac{2t\log_2 n_\nu}{n_\nu}\left(1 - \frac{\log_2\left(\frac{2t}{e}\right)}{\log_2 n_\nu} - O\left(n_\nu^{-1}\right)\right). \tag{13}$$

By substituting (13) into the LHS of (12), we have

$$\frac{1 - C\left(2tn_\nu^{-1}\right)}{2t\nu n_\nu^{-1}} = \frac{2tn_\nu^{-1}\log_2(n_\nu)}{2t\nu n_\nu^{-1}}\left(1 - O\left(\nu^{-1}\right)\right).$$

Then, the equality (12) follows since $\log_2(n_\nu) = \nu + o(1)$. ■

Thus, the following Theorem shows that iterative HDD of the spatially-coupled GLDPC ensemble can approach capacity in high-rate regime.

Theorem 12. *For any $\epsilon > 0$, there exist a tuple (t, n, L, w) such that iterative HDD of the $(\mathcal{C}, \infty, L, w)$ GLDPC spatially-coupled ensemble is ϵ-redundancy achieving when $\mathcal{C}$ is a t-error correcting BCH code of length n .*

Sketch of proof : The proof follows from combining Lemmas 8, 9, and 11 with standard construction arguments for sparse graph codes. ■

VI. Numerical Results and Comparison

In the following numerical results, the iterative HDD threshold of $(\mathcal{C}, m, L, w)$ spatially-coupled GLDPC ensemble with $L = 1025$, and $w = 16$ are considered. In Table I, the thresholds of the ensembles are shown in terms of the average number of error bits attached to a code-constraint node. Let $p^*_{n,t}$ be the iterative HDD threshold of the spatially-coupled GLDPC ensemble based on a $(n, k, 2t+1)$ binary primitive BCH component code, and $\tilde{p}^*_{n,t}$ be the iterative HDD threshold of the spatially-coupled GLDPC ensemble based on the $(n, k-1, 2t+2)$ even-weight subcode. Then, we define $a^*_{n,t} \triangleq np^*_{n,t}$ and $\tilde{a}^*_{n,t} \triangleq n\tilde{p}^*_{n,t}$ to be the thresholds in terms of the average number of error bits attached to a component code. In the high-rate scaling limit, we let ρ^*_t and $\tilde{\rho}^*_t$ denote the iterative HDD thresholds of the ensembles based on primitive BCH component codes and their even-weight subcodes, respectively. Moreover, the threshold of HDD without miscorrection, $\hat{\rho}^*_t$, is shown in Table I along with the potential threshold, $\hat{\rho}^{**}_t$, of iterative HDD without miscorrection from (11).

From Table I, one can observe that the thresholds (ρ^*_t, $\tilde{\rho}^*_t$ and $\hat{\rho}^*_t$) of the spatially-coupled ensemble with primitive BCH component codes or the even-weight subcodes approach to $2t$ as t increases. This verifies the results predicted by Lemma 8 and the vanishing impact of miscorrection predicted by Lemma 11.

VII. Conclusion

The iterative HDD of GLDPC ensembles, based on on t-error correcting block codes, is analyzed with and without spatial coupling. Using DE analysis, noise thresholds are computed for a variety of component codes and decoding assumptions. In particular, the case of binary primitive BCH component-codes is considered along with their even-weight subcodes. For these codes, the miscorrection probability is characterized and included in the DE analysis. Scaled DE recursions are also computed for the high-rate limit. When miscorrection is neglected, the resulting recursion for the basic ensemble matches the results of [13], [14]. It is also proven that iterative HDD threshold of the spatially-coupled GLDPC ensemble can approach capacity in high-rate regime. Finally, numerical results are presented that both verify the theoretical results and demonstrate the effectiveness of these codes for high-speed communication systems.

Table I
ITERATIVE HDD THRESHOLDS OF $(\mathcal{C}, \infty, 1025, 16)$ SPATIALLY-COUPLED GLDPC ENSEMBLE WITH BINARY PRIMITIVE BCH CODES

t	3	4	5	6	7
$a^*_{255,t}$	5.432	7.701	9.818	11.86	13.87
$a^*_{511,t}$	5.417	7.665	9.811	11.86	13.85
$a^*_{1023,t}$	5.401	7.693	9.821	11.87	13.88
ρ^*_t	5.390	7.688	9.822	11.91	13.93
$\tilde{a}^*_{255,t}$	5.610	7.752	9.843	11.88	13.87
$\tilde{a}^*_{511,t}$	5.570	7.767	9.811	11.86	13.85
$\tilde{a}^*_{1023,t}$	5.606	7.765	9.841	11.88	13.88
$\tilde{\rho}^*_t$	5.605	7.761	9.840	11.91	13.93
$\hat{\rho}^*_t$	5.735	7.813	9.855	11.91	13.93
$\hat{\rho}^{**}_t$	5.754	7.843	9.896	11.93	13.95

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