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Positive Solutions for Eigenvalue Problem with Time Scales

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Abstract: By exploring the values of the eigenvalue λ , this paper deals with the existence and nonexistence of positive solution for the following three-point boundary value problem with time scales:

$$u^{\Delta\nabla}(t)+\lambda h(t)f(t,u(t))=0, t\in(0,T)\cap\mathbb{T},$$
$$\beta u(0)-\gamma u^{\Delta}(0)=0, \alpha u(\eta)=u(T),$$

where $\mathbb{T}$ is a time scale, $\beta,\gamma\geqslant 0, \beta+\gamma>0, \eta\in(0,\rho(T)), 0<\alpha<T/\eta$, and $d=\beta(T-\alpha\eta)+\gamma(1-\alpha)>0$. Furthermore, two examples are presented for results verification purposes.

Key words: time scales; positive solution; eigenvalues; fixed points

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1 Introduction

The theory of time scales and measure chains was introduced and developed by Hilger as a means of unifying and extending theories from differential and difference equations[1]. Recently, much attention is attracted by questions of existence of positive solutions to boundary value problems, see, e.g. [2-12]. We begin by presenting some basic definitions which can be found in Agarwal and Bohner[2], Atici and Guseinov[4] and Bohner and Peterson[6]. Another excellent source on dynamic systems on measure chains is the book[10].

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T},\sigma(t)=\inf\{\tau\in\mathbb{T}:\tau>t\}$ and $\rho(t)=\sup\{\tau\in\mathbb{T}:\tau>t\}$ (supplemented by $\inf\emptyset:=\sup\mathbb{T}$ and $\sup\emptyset:=\inf\mathbb{T}$) are well-defined. The point $t\in\mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t)=t,\rho(t)<t,\sigma(t)=t,\sigma(t)<t$, respectively. If $\mathbb{T}$ has a right-scattered minimum t_1 , define $\mathbb{T}^k=\mathbb{T}-\{t_1\}$, otherwise, set $\mathbb{T}^k=\mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum t_2 , define $\mathbb{T}^k=\mathbb{T}-\{t_2\}$, otherwise, set $\mathbb{T}^k=\mathbb{T}$.

For $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$, the delta derivative of *f* at *t*, denoted $f^{\Delta}(t)$, is the number (provide it exists) with the property that given any $\varepsilon>0$, there is a neighborhood $U\subset\mathbb{T}$ of *t* such that $|f(\sigma(t))-f(s)-f^{\Delta}(t)[\sigma(t)-s]|\leqslant\varepsilon|\sigma(t)-s|$, for all $s\in U$. For $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$, the nabla derivative of *f* at *t*, denoted by $f^{\Delta}(t)$, is the number (provide it existes) with the property that given any $\varepsilon>0$, there is a neighborhood $U\subset\mathbb{T}$ of *t* such that $|f(\rho(t))-f(s)-f^{\Delta}(t)[\rho(t)-s]|\leqslant\varepsilon|\rho(t)-s|$, for all $s\in U$.

A function $f:\mathbb{T}\to\mathbb{R}$ is ld-continuous (C_{ld}) provided it is continuous at left dense points in $\mathbb{T}$ and its right-sided limit exits at light dense points in $\mathbb{T}$. If $\mathbb{T}=\mathbb{R}$, then *f* is ld-continuous if and only if *f* is continuous. If $\mathbb{T}=\mathbb{Z}$, then function is ld-continuous.

A function $F:\mathbb{T}\to\mathbb{R}$ is called a Δ -antiderivative of $f:\mathbb{T}\to\mathbb{R}$ provided $F^{\Delta}(t)=f(t)$ holds for all $t\in\mathbb{T}^k$. Then the Cauchy Δ -integral from *a* to *t* of *f* is

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defined by $\int_a^t f(s)\Delta s = F(t) - F(a)$ for all $t \in \mathbb{T}$. A function $\Phi : \mathbb{T} \to \mathbb{R}$ we call a ∇-antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $\Phi^{\nabla}(t) = f(t)$ for all $t \in \mathbb{T}_k$. We then define the Cauchy ∇-integral from a to t of f by $\int_a^t f(s)\nabla s = \Phi(t) - \Phi(a)$ for all $t \in \mathbb{T}$.

The following formulas hold:

$$\int_{\rho(t)}^{t} f(s)\nabla s = (t - \rho(t)) f(t), \tag{1}$$

where $f : \mathbb{T} \to \mathbb{R}$ is an arbitrary functional and $t \in \mathbb{T}$;

$$\int_a^b f(t)\nabla t = \begin{cases} \sum_{t\in(a,b]} (t - \rho(t)) f(t), & \text{if } a < b, \\ 0, & \text{if } a = b, \\ -\sum_{t\in(b,a]} [t - \rho(t)) f(t), & \text{if } a < b, \end{cases} \tag{2}$$

where $f \in C_{ld}[a,b]$ and $[a,b]$ consists of only isolated points. Formulas (1) and (2) will be used in the following Example 1 and 2.

In this paper, we are concerned with determining intervals of eigenvalues for the following three-point boundary value problem (BVP in short) on time scales:

$$\begin{cases} u^{\Delta\nabla}(t) + \lambda h(t) f(t, u(t)) = 0, t \in (0,T) \cap \mathbb{T}, \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \alpha u(\eta) = u(T). \end{cases} \tag{I}$$

Where $\mathbb{T}$ is a time scale, $(0, \rho(T)), 0 < \alpha < T/\eta$, $d = \beta(T - \alpha\eta) + \gamma(1-\alpha) > 0, h(t) \in C_{ld}[0,T]$ and $f(t,u) \in C([0,T]\times[0,\infty),[0,\infty))$.

In 2002, Anderson[3] considered the existence of one and three positives solution of the following dynamic equation on time scales:

$$\begin{cases} u^{\Delta\nabla}(t) + a(t) f(u(t)) = 0, t \in (0,T) \cap \mathbb{T}, \\ u(0) = 0, \alpha u(\eta) = u(T), \end{cases} \tag{3}$$

where $a \in C_{ld}[0,T]$ is nonnegative, $f : [0,\infty) \to [0,\infty)$ is continuous, $\eta \in (0, \rho(T))$ and $0 < \alpha < T/\eta$.

Kaufmann[9] discussed the problem (3) and obtained the existence of at least two positive solutions.

Recently, Sun and Li[12] consider the following three-point boundary value problem:

$$\begin{cases} u^{\Delta\nabla}(t) + a(t) f(t, u(t)) = 0, t \in (0,T) \cap \mathbb{T}, \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \alpha u(\eta) = u(T), \end{cases} \tag{4}$$

where $\beta, \gamma \geqslant 0, \beta + \gamma > 0, \eta \in (0, \rho(T)), 0 < \alpha < T/\eta$ and $d = \beta(T - \alpha\eta) + \gamma(1-\alpha) \neq 0$; $h(\cdot)$ and $f(\cdot,\cdot)$ are given functions. They proved that the equation (4) has at least two positive solutions.

In this paper, motivated by the results mentioned above, the fixed-point index theorem is applied to yield positive solutions of the BVP(I) for the eigenvalue λ to an open interval. The conditions of the function f can be relaxed since the function f may not be superlinear case ($f^0 = 0$ and $f_\infty = \infty$, see (C3) below) or sublinear case ($f_0 = 0$ and $f^\infty = 0$, see (C4) below). Furthermore, we will show that values of the function f and eigenvalue λ determine the nonexistence of positive solution for the BVP(I). We organize the paper as follows. In section 2, we state some lemmas that will be required in order to show our main theorems. In section 3, we state and show four theorems for existence and noexistence of positive solution of (I). In section 4, we give two exampls to illustrate our main results.

2 Preliminaries

To discuss the main results in this paper, we will employ several lemmas. These lemmas are based on the linear boundary value problem:

$$\begin{cases} u^{\Delta\nabla}(t) + \lambda h(t) = 0, t \in (0,T) \cap \mathbb{T}, \\ \beta u(0) - \gamma u^{\Delta}(0) = 0, \alpha u(\eta) = u(T). \end{cases} \tag{5}$$

Lemma 1 If $d = \beta(T - \alpha\eta) + \gamma(1-\alpha) \neq 0$. Then for $h \in C_{ld}[0,T]$, the boundary value problem (5) has the unique solution,

$$u(t) = -\lambda \int_0^t (t-s) h(s)\nabla s + \frac{\beta t + \gamma}{d} \cdot \left[\lambda \int_0^T (T-s) h(s)\nabla s - \alpha\lambda \int_0^\eta (\eta - s) h(s)\nabla s\right]. \tag{6}$$

Proof From $u^{\Delta\nabla}(t) + \lambda h(t) = 0$, we have

$$u(t) = u(0) + u^{\Delta}(0)t - \lambda \int_0^t (t-s) h(s)\nabla s. \tag{7}$$

Next we separate the process into two cases: (a) $\beta \neq 0$ and (b) $\beta = 0$.

Case (a): $\beta \neq 0$. By using the first boundary condition of (5), we get $u(0) = \gamma u^{\Delta}(0)/\beta$, and so it follows from (7) that

$$u(t) = \frac{\beta t + \gamma}{\beta} u^{\Delta}(0) - \lambda \int_0^t (t-s) h(s)\nabla s,$$

from the other boundary condition of (5), we find

$$\alpha u(0) + \alpha\eta u^{\Delta}(0) - \alpha\lambda \int_0^\eta (\eta - s) h(s)\nabla s = u(0) + u^{\Delta}(0)T - \lambda \int_0^T (T-s) h(s)\nabla s.$$

Thus,

$$u^{\Delta}(0)=\frac{\beta}{d}[\lambda\int_0^T(T-s)h(s)\nabla s-\alpha\lambda\int_0^{\eta}(\eta-s)h(s)\nabla s],$$

so that

$$u(t):=-\lambda\int_0^t(t-s)h(s)\nabla s+\frac{\beta t+\gamma}{d}\cdot$$
$$[\lambda\int_0^T(T-s)h(s)\nabla s-\alpha\lambda\int_0^{\eta}(\eta-s)h(s)\nabla s].$$

Case (b): $\beta=0$. Since $d=\beta(T-\alpha\eta)+\gamma(1-\alpha)\neq 0$, we have $\gamma(1-\alpha)\neq 0$. From the second boundary condition of (5), it implies that

$$(1-\alpha)u(0)=\lambda\int_0^T(T-s)h(s)\nabla s-\alpha\lambda\int_0^{\eta}(\eta-s)h(s)\nabla s.$$

Therefore, we deduce from (7) that

$$u(t):=-\lambda\int_0^t(t-s)h(s)\nabla s+\frac{\beta t+\gamma}{d}\cdot$$
$$[\lambda\int_0^T(T-s)h(s)\nabla s-\alpha\lambda\int_0^{\eta}(\eta-s)h(s)\nabla s],$$

both cases follow that *u* given in (6) is a solution of (5).

It is easy to verify that boundary value problem $u^{\Delta\nabla}(t)=0,\beta u(0)-\gamma u^{\Delta}(0)=0,\alpha u(\eta)=u(T)$ has only the trivial solution if $d\neq 0$. Thus *u* in (6) is the unique solution of (5).

Lemma 2[3,12] Let $0<\alpha\eta<T$ and $d>0$. If $h\in C_{ld}[0,T]$ and $\lambda h\geqslant 0$, then the unique solution *u* of (5) satisfies $u(t)\geqslant 0$, for $t\in[0,T]\cap\mathbb{T}$.

Lemma 3[3] Let $\alpha\eta>T$. If $h\in C_{ld}[0,T]$ and $\lambda h\geqslant 0$, then the unique solution *u* of (5) has no positive solution.

In view of Lemma 3, we will assume that $\alpha\eta<T$ for the rest of the paper.

Lemma 4[3] Let $0<\alpha\eta<T$ and $d>0$. If $h\in C_{ld}[0,T]$ and $\lambda h\geqslant 0$, then the unique solution *u* of (5) satisfies $\inf_{t\in[\eta,T]}u(t)\geqslant r\|u\|$, where $r=\min\{\alpha(T-\eta)/(T-\alpha\eta),\alpha\eta/T,\eta/T\}$ and $\|u\|=\sup_{t\in[0,T]}|u(t)|$.

Remark 1 Since $\alpha\eta<T$ and $\eta<T$, it follows that $0<r<1$. Therefore, we assume that *r* of the remainder of the paper is the constant in Lemma 4.

Throughout this paper, our Banach space is $E=C_{ld}(\mathbb{T},\mathbb{R})$ with $\|u\|=\sup_{t\in[0,T]\cap\mathbb{T}}|u(t)|$. Define the operator $A_{\lambda}:E\to E$.

$$A_{\lambda}u(t)=-\lambda\int_0^t(t-s)h(s)f(s,u(s))\nabla s+$$
$$\frac{\beta t+\gamma}{d}[\lambda\int_0^T(T-s)h(s)f(s,u(s))\nabla s-$$
$$\alpha\lambda\int_0^{\eta}(\eta-s)h(s)f(s,u(s))\nabla s]. \tag{8}$$

The function *u* is a solution of the boundary value problem (I) if and only if *u* is a fixed point of the operator A_{λ}.

As in paper [3], we define the cone $P=\{u\in B:u(t)\geqslant 0,\inf_{t\in[\eta,T]}u(t)\geqslant r\|u\|\}$, where *r* is the constant in Lemma 4. From Lemma 4, we have $A_{\lambda}:P\to P$. Standard arguments show that the operator A_{λ} is completely continous.

The following three inequalities play important role in our arguments. Since λ,h and *f* are nonnegative, then, for all $u\in E$,

$$A_{\lambda}u(t)\leqslant\lambda\frac{\beta t+\gamma}{d}\int_0^T(T-s)h(s)f(s,u(s))\nabla s. \tag{9}$$

furthermore,

$$A_{\lambda}u(\eta)=-\lambda\int_0^{\eta}(\eta-s)h(s)f(s,u(s))\nabla s+$$
$$\lambda\frac{\beta\eta+\gamma}{d}\int_0^T(T-s)h(s)f(s,u(s))\nabla s-$$
$$\alpha\lambda\frac{\beta\eta+\gamma}{d}\int_0^{\eta}(\eta-s)h(s)f(s,u(s))\nabla s\geqslant$$
$$\lambda\frac{\beta\eta+\gamma}{d}\int_{\eta}^T(T-s)h(s)f(s,u(s))\nabla s. \tag{10}$$

Likewise,

$$A_{\lambda}u(T)=-\lambda\int_0^T(T-s)h(s)f(s,u(s))\nabla s+$$
$$\lambda\frac{\beta T+\gamma}{d}\int_0^T(T-s)h(s)f(s,u(s))\nabla s-$$
$$\alpha\lambda\frac{\beta T+\gamma}{d}\int_0^{\eta}(\eta-s)h(s)f(s,u(s))\nabla s\geqslant$$
$$\lambda\frac{\alpha(\beta\eta+\gamma)}{d}\int_{\eta}^T(T-s)h(s)f(s,u(s))\nabla s. \tag{11}$$

We will also need the following two fixed-pointed index theorems in the sequel.

Lemma 5[13] Let *P* be a cone in Banach space E, $\Omega\subset E$ a bounded set, and $A:\overline{\Omega}\cap P\to P$ a completely continuous operator. If $Au\neq\delta u$, for any $u\in\partial\Omega\cap P$ and $\delta\geqslant 0$, then the fixed point index $i(A,\Omega\cap P,P)=1$.

Lemma 6[14] Let *P* be a cone in Banach space E, $\Omega\subset E$ a bounded set, and $A:\overline{\Omega}\cap P\to P$ a completely continuous operator. If there exists and operator $F:\partial\Omega\cap P\to P$, such that $\inf_{u\in\partial\Omega\cap P}\|Fu\|>0$, and $u-Au\neq\delta Fu$, for any $u\in\partial\Omega\cap P,\delta\geqslant 0$. Then the fixed point index $i(A,\Omega\cap P,P)=0$.

3 Main Results

In order to abbreviate our discussion, we list some assumptions to be used in this paper as follows:

(C1) $h(t):[0,T]\to[0,\infty)$ is ld-continuous such that $0<\int_0^T(T-s)h(s)\nabla s,\int_\eta^T(T-s)h(s)\nabla s<\infty.$

(C2) $f\in C([0,T]\times[0,\infty),[0,\infty)).$

(C3) $0<f^0:=\limsup_{u\to 0^+}\max_{t\in[0,T]}\frac{f(t,u)}{u}<M,$

$$m<f_\infty:=\liminf_{u\to\infty}\min_{t\in[\eta,T]}\frac{f(t,u)}{u}\leqslant\infty.$$

(C4) $0\leqslant f^\infty:=\limsup_{u\to\infty}\max_{t\in[0,T]}\frac{f(t,u)}{u}<M,$

$$m<f_0:=\liminf_{u\to 0^+}\min_{t\in[\eta,T]}\frac{f(t,u)}{u}\leqslant\infty,$$

where

$$M:=\left(\frac{\beta T+\gamma}{d}\int_0^T(T-s)h(s)\nabla s\right)^{-1},$$

$$m:=\left(\min(1,\alpha)\frac{\beta\eta+\gamma}{d}\int_\eta^T(T-s)h(s)\nabla s\right)^{-1}.$$

Theorem 1 Suppose that (C1)~(C3) hold. Then, for each λ satisfying

$$\lambda\in(m/f_\infty,M/f^0),\tag{12}$$

there exists at least one positive solution of the BVP(I).

Proof Since λ satisfies (12), we have

$$0\leqslant mf^0/f_\infty\leqslant\lambda f^0<M,$$
$$m<\lambda f_\infty\leqslant Mf_\infty/f^0\leqslant\infty.\tag{13}$$

According to the first inequality of (13), we see that there exist $r_1>0$ and $\varepsilon_1>0$ such that $\max_{t\in[0,T]}\lambda f(t,u)\leqslant(M-\varepsilon_1)u,0\leqslant u\leqslant r_1$. Consequently, $\lambda f(t,u)\leqslant(M-\varepsilon_1)u,0\leqslant u\leqslant r_1,0\leqslant t\leqslant T$, Let $\Omega_1=\{u\in E:\|u\|<r\}$. We find that, for any $u\in\partial\Omega_1\cap P$,

$$\begin{aligned}\|A_\lambda u\|&=\max_{t\in[0,T]}|A_\lambda u(t)|\leqslant\\ &\lambda\max_{t\in[0,T]}\frac{\beta t+\gamma}{d}\int_0^T(T-s)h(s)f(s,u(s))\nabla s\leqslant\\ &\frac{\beta T+\gamma}{d}(M-\varepsilon_1)r_1\int_0^T(T-s)h(s)\nabla s=\\ &r_1M\frac{\beta T+\gamma}{d}\int_0^T(T-s)h(s)\nabla s-\\ &r_1\varepsilon_1\frac{\beta T+\gamma}{d}\int_0^T(T-s)h(s)\nabla s<r.\end{aligned}\tag{14}$$

We now claim that

$$A_\lambda u\neq\delta_1 u,\forall u\in\partial\Omega\cap P,\delta_1\geqslant 1.\tag{15}$$

In deed, if not, then there exist $u_1\in\partial\Omega\cap P$ and $\delta_1\geqslant 1$ such that $A_\lambda u_1=\delta_1 u_1$. Clearly, $\|A_\lambda u_1\|=\delta_1\|u_1\|\geqslant\|u_1\|=r_1$ is a contradict to (14). Hence (15) holds. Employing (15) and lemma 5, we get

$$i(A_\lambda,\Omega_1\cap P,P)=1.\tag{16}$$

Next we consider the second inequality of (13). From the definition of f_∞, we may choose $\zeta>r_1>0$ and $\varepsilon_2>0$ such that $\min_{t\in[\eta,T]}\lambda f(t,u)\geqslant(m+\varepsilon_2)u,u\geqslant\zeta$. That is

$$\lambda f(t,u)\geqslant(m+\varepsilon_2)u,u\geqslant\zeta,t\in[\eta,T].\tag{17}$$

Denote $r_2>\zeta/r>\zeta>r_1,\Omega_2=\{u\in E:\|u\|<r\}$, and define and operator F by

$$Fu(t)\equiv 1,u\in E.\tag{18}$$

It is easy to verify that $F:\partial\Omega_2\cap P\to P$ is completely continuous. We see that $\inf_{u\in\partial\Omega_2\cap P}\|Fu\|>0$. So, condition (i) of Lemma 6 holds. Next we prove further condition (ii) of Lemma 6.

$$u-A_\lambda u\neq\delta Fu,\forall u\in\partial\Omega_2\cap P,\forall\delta\geqslant 0.\tag{19}$$

Suppose that (19) is not true, then there exist $u_2\in\partial\Omega_2\cap P$ and $\delta_2\geqslant 0$ such that $u_2-A_\lambda u_2=\delta_2 Fu_2$. By Lemma 4, we find $\min\{u_2(t),t\in[\eta,T]\}\geqslant r\|u_2\|>\zeta$. Therefore, we have by (17) that

$$\lambda f(t,u_2(s))\geqslant(m+\varepsilon_2)u_2(s),(t,s)\in[\eta,T]\times[\eta,T],\tag{20}$$

let

$$K_1=\min\{u_2(t),t\in[\eta,T]\}.\tag{21}$$

Then, by lemma 4, $K_1\geqslant r\|u_2\|=rr_2>\zeta$. By (10), (18) and (20), we have

$$\begin{aligned}u_2(\eta)&=A_\lambda u_2(\eta)+\delta_2 Fu_2(\eta)\geqslant\\ &\lambda\frac{\beta\eta+\gamma}{d}\int_\eta^T(T-s)h(s)f(s,u_2(s))\nabla s+\delta_2\geqslant\\ &\frac{\beta\eta+\gamma}{d}\min_{t\in[\eta,T]}u_2(t)\int_\eta^T(T-s)h(s)(m+\varepsilon_2)\nabla s=\\ &K_1 m\frac{\beta\eta+\gamma}{d}\int_\eta^T(T-s)h(s)\nabla s+\\ &K_1\varepsilon_2\frac{\beta\eta+\gamma}{d}\int_\eta^T(T-s)h(s)\nabla s.\end{aligned}$$

Similarly, we can get

$$\begin{aligned}u_2(T)&=A_\lambda u_2(T)+\delta_2 Fu_2(T)\geqslant\\ &\lambda\frac{\alpha(\beta\eta+\gamma)}{d}\int_\eta^T(T-s)h(s)f(s,u_2(s))\nabla s+\delta_2\geqslant\\ &K_1 m\frac{\alpha(\beta\eta+\gamma)}{d}\int_\eta^T(T-s)h(s)\nabla s+\\ &K_1\varepsilon_2\frac{\alpha(\beta\eta+\gamma)}{d}\int_\eta^T(T-s)h(s)\nabla s.\end{aligned}$$

Furthermore, from the process of the proof of paper [3], we obtain

$$\min_{t\in[\eta,T]} u_2(t) = \min_{t\in[\eta,T]}[A_\lambda u_2(t)+\delta_2 Fu_2(t)] \geqslant$$
$$\min\{A_\lambda u_2(\eta), A_\lambda u_2(T)\}+\delta_2 \geqslant$$
$$(K_1+K_1\varepsilon_2 m^{-1})\min\{1,\alpha\}\frac{\beta\eta+\gamma}{d}\cdot$$
$$\int_\eta^T (T-s)h(s)\nabla s = K_1+K_1\varepsilon_2 m^{-1},$$

and so it implies from the second inequality (C1) that

$$u_2(t) > K_1, \forall t\in[\eta,T]. \tag{22}$$

But, (22) is incompatible in light of (21). Thus the assertion (19) holds. Now utilizing lemma 2.6, we get

$$i(A_\lambda, \Omega_2\cap P, P) = 0. \tag{23}$$

Owing to (9), (23) and the fact that $\overline{\Omega_1}\subset\Omega_2$, we have $i(A_\lambda,(\Omega_2\setminus\overline{\Omega_1})\cap P,P) = i(A_\lambda,\Omega_2\cap P,P) - i(A_\lambda,\Omega_1\cap P,P) = 0-1=-1.$ In view of of paper [14], we conclude that the operator A_λ has a fixed point $u^*\in(\Omega_2\setminus\overline{\Omega_1})\cap P$ such that $0<r_1<\|u^*\|<r_2$. It follows that u^* is a positive solution of the BVP(I).

Theorem 2 Assume that (C1, (C2) and (C4) hold. Then, for each λ satisfying

$$\lambda\in(m/f_0, M/f^\infty), \tag{24}$$

There exists at least one positive solution of BVP(I).

Proof For any λ satisfying (24), we have

$$0\leqslant mf^\infty/f_0 = \lambda f^\infty < M,$$
$$m<\lambda f_0 \leqslant Mf_0/f^\infty \leqslant \infty. \tag{25}$$

From the first inequality of (25), we know there exist $r_3>0$ and $\varepsilon_3>0$ such that $\max_{t\in[0,T]}\lambda f(t,u)\leqslant(M-\varepsilon_3)u, u\geqslant r_3.$ Thus,

$$\lambda f(t,u)\leqslant(M-\varepsilon_3)u, u\geqslant r_3, t\in[0,T].$$

Set $\zeta := \max_{(t,u)\in[0,T]\times[0,r_3]}\lambda f(t,u)$. Hence

$$\lambda f(t,u)\leqslant \zeta+(M-\varepsilon_3)u,(t,u)\in[0,T]\times[0,\infty). \tag{26}$$

Let $r_3>\zeta/\varepsilon_3$ and $\Omega_3=\{u\in E:\|u\|<r_3\}$. Then, by (9) and (26), we obtain for any $u\in\partial\Omega_3\cap P$,

$$\|A_\lambda u\| = \max_{t\in[0,T]}|A_\lambda u(t)|\leqslant$$
$$\lambda\max_{t\in[0,T]}\frac{\beta t+\gamma}{d}\int_0^T(T-s)h(s)f(s,u(s))\nabla s\leqslant$$
$$\frac{\beta T+\gamma}{d}\int_0^T(T-s)h(s)[\zeta+(M-\varepsilon_3)u(s)]\nabla s\leqslant$$
$$r_3 M\frac{\beta T+\gamma}{d}\int_0^T(T-s)h(s)\nabla s-(r_3\varepsilon_3-\zeta)\cdot$$
$$\frac{\beta T+\gamma}{d}\int_0^T(T-s)h(s)\nabla s<r_3. \tag{27}$$

We claim that

$$A_\lambda u\neq\delta_3 u, \forall u\in\Omega_3\cap P, \delta_3\geqslant 1. \tag{28}$$

In fact, if (28) is false, there exist $u_3\in\Omega_3\cap P$ and $\delta_3\geqslant 1$ such that $A_\lambda u_3=\delta_3 u_3$. It implies that $|A_\lambda u_3\| = \delta_3\|u_3\|\geqslant\|u_3\| = r_3$ is in contradiction with (27). Hence (28) holds. Applying (28) and Lemma 5, we have

$$i(A_\lambda,\Omega_3\cap P,P)=1. \tag{29}$$

Now we consider the second inequality of (25). We first note that there exist $r_3>\varsigma>0$ and $\varepsilon_4>0$ such that $\min_{t\in[\eta,T]}\lambda f(t,u)\geqslant(m+\varepsilon_4)u, 0\leqslant u\leqslant\varsigma,$ i.e.,

$$\lambda f(t,u)\geqslant(m+\varepsilon_4)u, 0\leqslant u\leqslant\varsigma, t\in[\eta,T].$$

Let $r_4=r\varsigma<r_3$ and $\Omega_4=\{u\in E:\|u\|<r_4\}$. Define an operator F as in (18). Then $F:\partial\Omega_4\cap P\to P$ is completely continuous and $\inf_{u\in\partial\Omega_4\cap P}\|Fu\|>0.$ Therefore, the condition (i) of Lemma 6 holds. Next we will show that condition (ii) of Lemma 6.

$$u-A_\lambda u\neq\delta_4 Fu, \forall u\in\partial\Omega_4\cap P, \forall\delta_4\geqslant 0, \tag{30}$$

is valid. If not, there exist $u_4\in\partial\Omega_4\cap P(\|u_4\|=r_4<\varsigma)$ and $\delta_4\geqslant 0$ such that $u_4-A_\lambda u_4=\delta_4 Fu_4$.

$$K_2=\min\{u_4(t):t\in[\eta,T]\}. \tag{31}$$

Then, from Lemma 4, $K_2\geqslant r\|u_4\|>0$. As in the proof of Theorem 1, we have

$$u_4(\eta)=A_\lambda u_4(\eta)+\delta_4 Fu_4(\eta)\geqslant$$
$$\lambda\frac{\beta\eta+\gamma}{d}\int_\eta^T(T-s)h(s)f(s,u_4(s))\nabla s+\delta_4\geqslant$$
$$K_2 m\frac{\beta\eta+\gamma}{d}\int_\eta^T(T-s)h(s)\nabla s+$$
$$K_2\varepsilon_4\frac{\beta\eta+\gamma}{d}\int_\eta^T(T-s)h(s)\nabla s,$$

and

$$u_4(T)=A_\lambda u_4(T)+\delta_4 Fu_4(T)\geqslant$$
$$\lambda\frac{\alpha(\beta\eta+\gamma)}{d}\int_\eta^T(T-s)h(s)f(s,u_4(s))\nabla s+\delta_4\geqslant$$
$$K_2 m\frac{\alpha(\beta\eta+\gamma)}{d}\int_\eta^T(T-s)h(s)\nabla s+$$
$$K_2\varepsilon_4\frac{\alpha(\beta\eta+\gamma)}{d}\int_\eta^T(T-s)h(s)\nabla s.$$

Likewise, from the process of the proof of paper [3], we have

$$\min_{t\in[\eta,T]}u_4(t)=\min_{t\in[\eta,T]}[A_\lambda u_4(t)+\delta_4 Fu_4(t)]\geqslant$$

$$\min\{A_\lambda u_4(\eta), A_\lambda u_4(T)\} + \delta_4 \geqslant (K_2 + K_2\varepsilon_4 m^{-1}) m \min\{1,\alpha\}\frac{\beta\eta+\gamma}{d}\cdot$$

$$\int_\eta^T (T-s)h(s)\nabla s = K_2 + K_2\varepsilon_4 m^{-1},$$

and so it follows from the second inequality (C1) that

$$u_4(t) > K_2, \forall t \in [\eta, T]. \tag{32}$$

Obviously, (32) contradicts (31). It implies that (30) is valid.

Now utilizing Lemma 6, we obtain

$$i(A_\lambda, \Omega_4 \cap P, P) = 0. \tag{33}$$

According to (29), (33) and the fact that $\overline{\Omega}_4 \subset \Omega_3$, we have $u_4(t) > K_2, \forall t \in [\eta, T]$. $i(A_\lambda, (\Omega_3 \setminus \overline{\Omega}_4) \cap P, P) = i(A_\lambda, \Omega_3 \cap P, P) - i(A_\lambda, \Omega_4 \cap P, P) = 1 - 0 = 1$. Owing to [15], we obtain that the operator A_λ has a fixed point $u_* \in (\Omega_3 \setminus \overline{\Omega}_4) \cap P$ such that $0 < r_4 < \|u_*\| < r_3$. It follows that u_* is a positive solution of the BVP(I).

If we take the eigenvalue $\lambda = 1$ in Theorem 1 and Theorem 2, then we obtain:

Corollary 1 Assume that (C1), (C2) and (C3) (or (C4)) hold. Then the three-point boundary value problem (4) has at least one positive solution.

Remark 2 In Corollary 1, the function f may not be superlinear case ($f^0 = 0, f_\infty = \infty$) or sublinear case ($f_0 = \infty, f^\infty = 0$).

Next, we are concerned with the case of no positive solution for the three-point boundary value problem (I):

Theorem 3 Let (C1) and (C2) hold. In addition let $f^0 < \infty$ and $f^\infty < \infty$ be satisfied. Then there exists $\lambda_1 > 0$ such that for all $0 < \lambda < \lambda_1$, the BVP(I) has no positive solution.

Proof Since $f^0 < \infty$ and $f^\infty < \infty$, there exit $\tau_1 > 0$, $\tau_2 > 0, h_1 > 0$ and $h_2 > 0$ such that $h_1 < h_2$ and for $t \in [0,T], 0 \leqslant u \leqslant h_1$, we have $f(t,u) \leqslant \tau_1 u$, and for $t \in [0,T], u \geqslant h_2$, we get $f(t,u) \leqslant \tau_2 u$. Let

$$\tau^* = \max\{\tau_1, \tau_2, \max\{f(t,u)/u, \forall t \in [0,T], h_1 \leqslant u \leqslant h_2\}\} < \infty. \tag{34}$$

Thus, for $t \in [0,T], u \in \mathbb{R}^+$, we have $f(t,u) \leqslant \tau^* u$. Assume that u is a positive solution BVP(I), we will show that this leads to a contradiction for

$$0 < \lambda < \lambda_1 = \left(\frac{\beta\eta+\gamma}{d}\tau^* \int_0^T (T-s)h(s)\nabla s\right)^{-1}.$$

In fact, if for $0 < \lambda < \lambda_1, t \in [0,T], A_\lambda u(t) = u(t)$, we have

$$\|u\| = \|A_\lambda u(t)\| = \max_{t\in[0,T]} |A_\lambda u(t)| \leqslant \frac{\beta T+\gamma}{d}\lambda\int_0^T (T-s)h(s)f(s,u(s))\nabla s \leqslant \frac{\beta T+\gamma}{d}\lambda\tau^* \max_{t\in[0,T]} u(t)\int_0^T (T-s)h(s)\nabla s \leqslant \frac{\beta T+\gamma}{d}\lambda\tau^*\int_0^T (T-s)h(s)\nabla s\,\|u\| < \|u\|,$$

which is a contradiction.

Theorem 4 Let (C1) and (C2) hold. Assume that

$$\inf_{(t,u)\in[\eta,T]\times[0,\infty)} f(t,u)/u > 0,$$

is satisfied. Then there exist $\lambda_2 > 0$, the BVP(I) has no positive solution.

Proof Since $\inf_{t\in[\eta,T]} f(t,u)/u > 0$, for all $u \in \mathbb{R}^+$, we have

$$\tau_* = \inf\{f(t,u)/u, \forall t \in [0,T], u \in \mathbb{R}^+\} > 0 \tag{35}$$

Thus, for $t \in [0,T], u \in \mathbb{R}^+$, we deduce $f(t,u) \geqslant \tau_* u$. Suppose u is a positive solution of the BVP(I), we will verify that this lead to a contradiction for

$$\left(\min\{1,\alpha\} r\tau_* \frac{\beta T+\gamma}{d}\int_\eta^T (T-s)h(s)\nabla s\right)^{-1},$$

indeed, for $\lambda > \lambda_2, t \in [\eta, T]$, if $A_\lambda u(t) = u(t)$, we have

$$\|u\| = \|A_\lambda u(t)\| \geqslant \min_{t\in[\eta,T]} |A_\lambda u(t)| = \min\{A_\lambda u(\eta), A_\lambda u(T)\} \geqslant \min\{1,\alpha\}\lambda\frac{\beta\eta+\gamma}{d}\int_\eta^T (T-s)h(s)f(s,u(s))\nabla s \geqslant \min\{1,\alpha\}\lambda \min_{t\in[\eta,T]} u(t)\frac{\beta\eta+\gamma}{d}\cdot \int_0^T (T-s)h(s)\frac{f(s,u(s))\nabla s)}{u(s)}\nabla s \geqslant \min\{1,\alpha\}\lambda\cdot \tau_* r\|u\|\frac{\beta\eta+\gamma}{d}\int_0^T (T-s)h(s)\nabla s > \|u\|,$$

which is a contradiction.

4 Two Examples

Example 1 Set $\mathbb{T} = \{0\} \cup \{i/4 : i \subset \mathbb{N}\}$. We consider the following BVP:

$$\begin{cases} u^{\Delta\nabla}(t) + \lambda \dfrac{t}{200}\dfrac{1+100u}{1+u} = 0, [0,3] \cap \mathbb{T}, \\ u(0) - u^\Delta(0) = 0, u(1) = u(3). \end{cases} \tag{II}$$

From the BVP(II), we have $\beta=1,\gamma=1,\alpha=1,\eta=1,$ $T=3$. Then $d=\beta(T-\alpha\eta)+\gamma(1-\alpha)=2,\beta T+\gamma=4,\beta\eta+\gamma=2$. Denote

$$f(t,u)=f(u):=\frac{1+100u}{1+u},h(t):=\frac{t}{200},$$

consequently, $f(t,u)$ is continuous in respect with $(t,u)\in[0,T]\times[0,\infty)$ and $h(t)$ is continuous. Applying the formula (1) and (2), we have

$$\int_0^T(T-s)h(s)\nabla s=\int_0^3(3-s)\frac{s}{200}\nabla s=\frac{283}{12\,800},$$

$$\int_\eta^T(T-s)h(s)\nabla s=\int_1^3(3-s)\frac{s}{200}\nabla s=\frac{193}{12\,800},$$

Therefore, condition (C1) of Theorem 1 is satisfied. Furthermore, we compute

$$M=(\frac{\beta T+\gamma}{d}\int_0^T(T-s)h(s)\nabla s)^{-1}=\frac{6\,400}{283}\approx 22.61,$$

$$m=(\min\{1,\alpha\}\frac{\beta\eta+\gamma}{d}\int_\eta^T(T-s)h(s)\nabla s)^{-1}=\frac{12\,800}{193}\approx 66.32,$$

$$f^0=\limsup_{u\to0^+}\max_{t\in[0,T]}\frac{f(t,u)}{u}=1<M,$$

$$m<f_\infty=\liminf_{u\to\infty}\min_{t\in[0,T]}\frac{f(t,u)}{u}=100.$$

So, condition (C3) of Theorem 1 is met. Clearly, $m/f_\infty\approx 0.663,M/f^0\approx 22.61$. Taking $\lambda\in(0.663\,2,22.612)\subset(m/f_\infty,M/f^0)$, then, by Theorem 3 and Theorem 4, the BVP(II) has no positive solution.

On the other hand, we consider that BVP has the case of no positive solution. Now, we know, by Lemma 4, (34) and (35),

$$r=\min\{\frac{\alpha(T-\eta)}{T-\alpha\eta},\frac{\alpha\eta}{T},\frac{\eta}{T}\}=\frac{1}{3},$$

$$\tau^*=100,\lambda_1=(\frac{\beta T+\gamma}{d}\tau^*\int_0^T(T-s)h(s)\nabla s)^{-1}\approx 0.226\,2,$$

$$\tau_*=1,\lambda_2=(\min\{1,\alpha\}r\tau_*\frac{\beta\eta+\gamma}{d}\cdot\int_\eta^T(T-s)h(s)\nabla s)^{-1}\approx 198.97.$$

therefore, if $\lambda\in(0,0.226\,2)$ or $\lambda\in(198.97,\infty)$, then, by Theorem 3 and Theorem 4, the BVP(II) has no positive solution.

Example 2 Set $\mathbb{T}=\{0,1\}\cup\{1+i/4:i\subset\mathbb{N}\}$. We consider the following BVP:

$$\begin{cases}u^{\Delta\nabla}(t)+\lambda\frac{t}{200}\frac{1+100u}{1+u}=0,[0,3]\cap\mathbb{T},\\ u(0)-u^{\Delta}(0)=0,u(1)=u(3).\end{cases}\tag{III}$$

Clearly, from the BVP(III), we know $\beta=1,\gamma=1,$ $\alpha=1,\eta=1,T=3,d=2,\beta T+\gamma=4,\beta\eta+\gamma=2$. Denote

$$f(t,u)=f(u):=\frac{1+100u}{1+u},h(t):=\frac{t}{200},$$

Consequently, $f(t,u)$ is continuous in respect with $(t,u)\in[0,T]\times[0,\infty)$ and $h(t)$ is continuous. Employing (1) and (2), we have

$$\int_0^T(T-s)h(s)\nabla s=\int_0^1(3-s)\frac{s}{200}\nabla s+\int_1^3(3-s)\frac{s}{200}\nabla s=\frac{803}{38\,400},$$

$$\int_\eta^T(T-s)h(s)\nabla s=\int_1^3(3-s)\frac{s}{200}\nabla s=\frac{193}{12\,800}.$$

Therefore, condition (C1) of Theorem 2 is satisfied. Moreover, we compute

$$M=(\frac{\beta T+\gamma}{d}\int_0^T(T-s)h(s)\nabla s)^{-1}=\frac{19\,200}{803}\approx 23.911,$$

$$m=(\min\{1,\alpha\}\frac{\beta\eta+\gamma}{d}\int_\eta^T(T-s)h(s)\nabla s)^{-1}=\frac{12\,800}{193}\approx 66.32,$$

$$f^\infty=\limsup_{u\to\infty}\max_{t\in[0,T]}\frac{f(t,u)}{u}=1<M,$$

$$m<f_0=\liminf_{u\to0^+}\min_{t\in[0,T]}\frac{f(t,u)}{u}=100.$$

So, condition (C3) of Theorem 2 is fulfilled. Clearly, $m/f_0\approx 0.663,M/f^\infty\approx 23.91$. Taking $\lambda\in(0.663\,2,23.912)\subset(m/f_0,M/f^\infty)$. Then conditions of Theorem 3.2 are met. So the BVP (III) has at least one positive solution.

On the other hand, as in the computation of Example 1, we have

$$r=\min\{\frac{\alpha(T-\eta)}{T-\alpha\eta},\frac{\alpha\eta}{T},\frac{\eta}{T}\}=\frac{1}{3},$$

$$\tau^*=100,\lambda_1=(\frac{\beta T+\gamma}{d}\tau^*\int_0^T(T-s)h(s)\nabla s)^{-1}\approx 0.239\,1,$$

$$\tau_*=1,\lambda_2=(\min\{1,\alpha\}r\tau_*\frac{\beta T+\gamma}{d}\cdot\int_\eta^T(T-s)h(s)\nabla s)^{-1}\approx 198.96.$$

Therefore, if $\lambda\in(0,0.239\,2)$ or $\lambda\in(198.967,\infty)$, then, by Theorem 3 and Theorem 4, the BVP(III) has no positive solution.

References:

[1] Hilger S. Analysis on measure chains: A unified approach to continuous and discrete calculus[J]. Results Math, 1990, 18:18-56.

[2] Agarwal R P, Bohner M. Basic calculus on time scales and some of its applications[J]. Results Math, 1999, 35:3-22.

[3] Anderson D R. Solutions to second-order three-point porblems on time scales[J]. J Diff Equat Appl, 2002, 8:673-688.

[4] Atici F M, Guseinov G S. On Green's functions and positive solutions for boundary value problems on time scales[J]. J Comp Appl Math, 2002, 14:75-99.

[5] Avery R I, Anderson D R. Existence of three positive solution to a second-order boundary value problem on a measure chain[J]. J Comp Appl Math, 2002, 14:65-73.

[6] Bohner M, Peterson A. Advances in dynamic equations on time scales[M]. Cambridge: Birkauser Boston, 2001.

[7] Chyan C J, Henderson J. Eigenvalue problems for nonlinear differential equations on a measure chain[J]. J Math Anal Appl, 2000, 245:547-559.

[8] Hong C H, Yeh C C. Positive solution for eigenvalue problems on a measure chain[J]. Nonlinear Anal, 2002, 51:499-507.

[9] Kaufmann E R. Positive solutions of a three-point boundary-value problem on a time scale[J]. Electron J Diff Equ, 2003, 82:1-11.

[10] Lakshmihantham V, Sivasundaram S, Kaaymakcalan B. Dynamical systems on Measure chains[M]. Boston: Kluwer Acad, 1996.

[11] Lan K Q. Multiple positive solutions of semilinear differential equations with singularities[J]. J London Math Soc, 2001, 63:690-704.

[12] Sun H R, Li W T. Positive solutions of a nonlinear three-point boundary value problems on time scales[J]. J Math Anal Appl, 2004, 299:508-524.

[13] Guo D J, Sun J X. Calculation and application of topologic degree[J]. J Math Resear Exp, 1988, 8:469-480.

[14] Guo D J. Some fixed point theorems on cone maps[J]. Kexue Tongbao, 1984, 19:575-578.

[15] Guo D J, Lakshmikatham V. Nonlinear problems in abstract cones[M]. New York: Academic Press, 1998.

时间尺标上本征值问题的正解

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摘要：通过讨论本征值λ，考虑了时间尺度上三点边值问题正解的存在性与非存在性:

$$u^{\Delta\nabla}(t)+\lambda h(t)f(t,u(t))=0, t\in(0,T)\cap\mathbb{T},$$
$$\beta u(0)-\gamma u^{\Delta}(0)=0, \alpha u(\eta)=u(T),$$

其中，$\mathbb{T}$是时间尺度，$\beta,\gamma\geqslant 0,\beta+\gamma>0,\eta\in(0,\rho(T)),0<\alpha<T/\eta$和$d=\beta(T-\alpha\eta)+\gamma(1-\alpha)>0$．此外，使用 2 个例子说明其结果.

关键词：时间尺度；正解；本征值；不动点

中图分类号：O175.8 文献标识码：A

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