Symmetry Groups and Exact Solutions of a (3+1)-dimensional Nonlinear Evolution Equation and Maccari's System

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Abstract: Based on the generalized symmetry group method and symbolic computation, both the Lie point groups and the non-Lie symmetry groups of a (3+1)-dimensional nonlinear evolution equation as well as the Maccari's system are firstly obtained. Furthermore, some exact solutions of the two equations are derived from some simple solutions by the symmetry groups.

Key words: symmetry group; (3+1)-dimensional nonlinear evolution equation; Maccari's system

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As we know, it is important to investigate the exact explicit solutions of nonlinear partial differential equations (NLPDEs). Unfortunately, it is almost impossible to find all the solutions of NLPDEs. Finding solutions of NLPDEs is an arduous task and only in certain special cases one can write down the solutions explicitly. Despite of this fact, in recent years some progress has been made and many effective methods for obtaining exact solutions of NLPDEs have been proposed, such as the inverse scattering method[1], Darboux transformation[2], the formal variable separation approach[3-6], the tanh method[7], the generalized projective Riccati equation method[8-9], the sub-equation expansion method[10] and so on.

In particularly, the group theory method is also an important method for studying solutions of NLPDEs. So far, many powerful methods to obtain symmetries, symmetry groups, symmetry reductions and group invariant solutions of NLPDEs have been developed by mathematicians and physicists[11-14]. But for some NLPDEs, the final expression obtained by the classical Lie point symmetry group may be quite complicated and difficult for real applications especially for physicists and non-mathematical scientists. Fortunately, Clarkson and Kruskal (CK) introduced a direct method to derive the symmetry reductions of nonlinear system without covering any group theory[15]. Some important developments about this method can be found in refs[16-18]. Recently, on the basis of the classical Lie group and CK method, Lou proposed a new direct method to obtain both the Lie symmetry group and non-Lie symmetry group[19]. Furth more, the final expressions of the exact finite transformation of the Lie groups are much simpler than those obtained via the standard approach for many NLPDEs[20-23]. Sometimes it is difficult for us to obtain the Lie symmetry group of NLPDEs by integrating the vector fields involving many arbitrary functions, so we turn to the symmetry group direct method. By the

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symmetry group direct method, the problems of seeking for the full symmetry groups of given NLPDEs are changed into solving a set of overdetermined NLPDEs. Due to the nonlinearity of the overdetermined NLPDEs, it is usually difficult for us to obtain the solutions of them. However, once the full symmetry group of a given system is given, the related Lie point symmetries can be obtained simply by restricting the arbitrary functions or arbitrary constants in infinitesimal forms. At the same time, the related group of discrete transformations of the given system can also be derived from the full symmetry groups. Finally, using the obtained full symmetry groups and a known simple solution, one can obtain a type of group invariant solutions.

The main idea of the symmetry direct method is to seek full symmetry groups of given PDEs in the form

$$u(x_1,x_2,\cdots,x_n)=W(x_1,x_2,\cdots,x_n,U(X_1,X_2,\cdots,X_n)), \tag{1}$$

where $X_i = X_i(x_1,x_2,\cdots,x_n), i=1,2,\cdots,n.$

For a given PDE

$$F(x_i,u,u_{x_i},u_{x_ix_j},\cdots,i,j=1,2,\cdots,n)=0, \tag{2}$$

substituting (1) into (2) and demanding U satisfies the same PDE imposes conditions upon W, X_i and their derivatives under the transformation $\{u,x_1,x_2,\cdots,x_n\}\rightarrow\{U,X_1,X_2,\cdots,X_n\}$ that enable one to solve for W, X_i. It's interesting that for many real physical systems, it's enough to seek the symmetry groups transformation in a simple form

$$u(x_1,x_2,\cdots,x_n)=\alpha(x_1,x_2,\cdots,x_n)+\beta(x_1,x_2,\cdots,x_n)\,U(X_1,X_2,\cdots,X_n). \tag{3}$$

In the paper, we devote to investigating the symmetry group method to the full symmetry groups for two NLPDEs: a (3+1)-dimensional nonlinear evolution equation (NEE) and Maccari's system, which have widespread applications in physics, biology, electrical networks as well as applied science. In the following sections, firstly we obtain their relative symmetry groups, and then some general solutions of them are derived by the finite symmetry group transformation and two simple solutions.

1 Transformation group and some solutions of a (3+1)-NEE

So far as we know, a lot of research on the (3+1)-dimensional soliton equation

$$3w_{xz}-(2w_t+w_{xxx}-2ww_x)_y+2(w_x\partial_x^{-1}w_y)_x=0 \tag{4}$$

has been conducted. In ref.[24], Eq. (4) was decomposed into systems of solvable ordinary differential equations with the help of the (1+1)-dimensional AKNS equations. The Abel-Jacobi coordinates are introduced to straighten out the associated flows, from which algebraic-geometrical solutions of the (3+1)-dimensional evolution equation are explicitly given in terms of the Riemann theta functions. In ref.[25], an N-soliton solution for Eq. (4) and its Wronskian form were derived using the Hirota method and Wronskian technique. Wu[26] has given a bilinear Bäcklund transformation and some soliton solutions for Eq. (4) the Grammian determinant solution and the Pfaffianization of Eq. (4) was derived. However, the finite symmetry transformation group has not been revealed in previous articles.

Now, we let

$$w_y=u_x. \tag{5}$$

Substituting Eq. (5) into Eq. (4), then Eq. (4) becomes:

$$3w_{xz}-2(w_t+w_{xxx}-2ww_x)_y+2(w_xu)_x=0,\quad w_y=u_x. \tag{6}$$

In order to apply the standard infinitesimal procedure to find the infinitesimals of x, y, z, t, u, w and hence the symmetry group of Eqs. (6), we write Eqs. (6) as

$$\Delta_1(x,y,z,t,u,w)=0,\quad \Delta_2(x,y,z,t,u,w)=0.$$

We generally give the infinitesimal operator

$$v=A\partial_x+B\partial_y+C\partial_z+D\partial_t+\phi_1\partial_u+\phi_2\partial_w, \tag{7}$$

where the coefficients A, B, C, D, ϕ_1, ϕ_2 are functions of x, y, z, t, u, w. According to the general theory for symmetries of differential equations, to find these functions we prolong the vector to the fourth order derivatives and require that the fourth prolonged vector field annihilates Δ_1, Δ_2, on the solution manifold of Eqs. (6) respectively, this condition provides us with quite a complicated system of determining equations for coefficients. With the help of symbolic computation system *Maple*, we can obtain the following results:

$$A=(C_1/3)x+F_2(z,t),$$
$$B=(C_3-2C_1/3)y+F_1(z),$$
$$C=C_3z+C_4,$$
$$D=C_1t+C_2,$$
$$\phi_1=(-C_3+C_1/3)u+3/2\frac{\partial F_2(z,t)}{\partial z},$$
$$\phi_2=(-2C_1/3)w+3/2\frac{dF_1(z)}{dz}-\frac{\partial F_2(z,t)}{\partial t}. \quad (8)$$

To obtain the similarity reductions and some exact solution of the (3+1)-dimensional soliton equation, we have to solve the characteristic line equations

$$\frac{\mathrm{d}x}{A}=\frac{\mathrm{d}y}{B}=\frac{\mathrm{d}z}{C}=\frac{\mathrm{d}t}{D}=\frac{\mathrm{d}\varphi_1}{u}=\frac{\mathrm{d}\varphi_2}{w}. \quad (9)$$

However, it's difficult for us to give out the general solutions of Eq. (9). So we shall turn to the symmetry group direct method recently developed by Lou et al for the general symmetry groups of Eqs. (6). According to the symmetry group direct method developed by Lou et al, one can find that the symmetry transformation should have the following form:

$$u=\alpha_1+\beta_1U(\xi,\eta,\kappa,\tau)+\gamma_1W(\xi,\eta,\kappa,\tau),$$
$$w=\alpha_2+\beta_2W(\xi,\eta,\kappa,\tau)+\gamma_2U(\xi,\eta,\kappa,\tau), \quad (10)$$

where $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\xi,\eta,\kappa,\tau$ are all functions of $\{x,y,z,t\}$.

Restricting $U(\xi,\eta,\kappa,\tau)=U,\ W(\xi,\eta,\kappa,\tau)=W$, they should satisfy the same form as Eqs. (6) but with new independent variables $\{\xi,\eta,\kappa,\tau\}$,

$$3W_{\xi\kappa}-(2W_\tau+W_{\xi\xi\xi}-2WW_\xi)_\eta+2(W_\xi U)_\xi=0,$$
$$W_\eta=U_\xi. \quad (11)$$

Substituting Eqs. (10) into Eqs. (6), and eliminating $U_{\xi\xi\xi\eta}$ and W_η by using Eqs. (11), from that, the remained determining equations of the functions $\xi,\eta,\tau,\kappa,\alpha,\beta$ can be got by vanishing the coefficients of U and its derivatives, then we find out the general solution of the determining equations by tedious calculations. The result reads:

$$\xi=\delta C_1^{1/3}x+\xi_1,\eta=(C_3/C_1^{2/3})\delta y+\eta_1,$$
$$\kappa=C_3z+C_4,\tau=C_1t+C_2,\gamma_1=\gamma_2=0. \quad (12)$$
$$\alpha_1=-\frac{3}{2}\frac{\delta^2}{C_1^{1/3}}\xi_{1z},\alpha_2=\frac{\delta^2}{C_1^{1/3}}\xi_{1t}-\frac{3}{2}\frac{\delta^2C_1^{2/3}}{C_3}\eta_{1z},$$
$$\beta_1=\frac{\delta^2C_3}{C_1^{1/3}},\beta_2=\delta^2C_1^{2/3},$$

where $\xi_1=\xi_1(z,t)$, $\eta_1=\eta_1(z)$, the subscripts denote derivatives of z and t respectively, C_1,C_2,C_3,C_4 are all arbitrary constant while the constant δ possess discrete values determined by

$$\delta=1,-(1\pm i\sqrt{3})/2,(i=\sqrt{-1}). \quad (13)$$

From the above results, one can get the following theorem for Eqs. (6).

Theorem 1 If U=U(x, y, t, z), W=W(x, y, t, z) is a solution of Eqs. (6), then so is

$$u'=-\frac{3}{2}\frac{\delta^2}{C_1^{1/3}}\xi_{1z}+\frac{\delta^2C_3}{C_1^{1/3}}U(\xi,\eta,\kappa,\tau),$$
$$w'=\frac{\delta^2}{C_1^{1/3}}\xi_{1t}-\frac{3}{2}\frac{\delta^2C_1^{2/3}}{C_3}\eta_{1z}+\delta^2C_1^{2/3}W(\xi,\eta,\kappa,\tau), \quad (14)$$

where ξ,η,κ,τ are given by Eqs. (12).

Remark 1 From Theorem 1, it is easy to see that the symmetry group G_S of the (3+1)-dimensional soliton equation is divided into three sectors which can be considered as a product of the discrete D_S group and the usual Lie point symmetry group S which is related to transformations $\{u,w\}\to\{u',w'\}$ with $\delta=1$. That means

$$G_S=D_S\otimes S,D_S=\{I,R_1,R_2\}, \quad (15)$$

where I is the identity transformation, R_1, R_2 are as follows:

$$R_1=\{x,y,z,t,u,w\}\to\{\delta_1x,\delta_1y,z,t,\delta_1^2u,\delta_1^2w\},$$
$$\delta_1=-(1+i\sqrt{3})/2,$$
$$R_2=\{x,y,z,t,u,w\}\to\{\delta_2x,\delta_2y,z,t,\delta_2^2u,\delta_2^2w\},$$
$$\delta_2=-(1-i\sqrt{3})/2. \quad (16)$$

To find some types of exact solutions in high dimensions is one of the most important and difficult work. In this section, we just write down some exact solutions with help of the group transformation theorem and travelling solution for the (3+1)-dimensional soliton equation.

By considering the wave transformations $u=U(\chi)$, $w=W(\chi)$, $\chi=x+qy+nz+pt$, we change Eqs. (6) into the form

$$(3n-2pq)W''-qW^{(4)}+2qW'^2+2qWW''+$$

$$2UW'' + 2W'U' = 0,$$
$$U' = qW'. \quad (17)$$

According to the subequation expansion method, we use the following series expansion as solutions of Eqs. (17):

$$W = \sum_{i=0}^{n} a_i \varphi^i, \varphi = \varphi(\chi), \varphi' = \varepsilon(\sum_{j=0}^{n} c_j \varphi^j)^{1/2} (\varepsilon = \pm 1), \quad (18)$$

Balancing the highest derivative terms with non-linear terms in Eqs. (18) gives: $r = n + 2$.

Therefore, we may choose $n = 2$, $r = 4$, and then substitute Eqs. (18) into Eqs. (17), we obtain a system of algebraic equations with *Maple*. Then we can then obtain all the possible solutions eliminating ε from the above system and solving the system. For simplification, we just list one traveling wave solution as following:

$$W_1 = \frac{1}{4}c_2 - \frac{3n}{4q} + \frac{p}{2} \pm \frac{3}{2}\frac{c_2 \sec h^2(\sqrt{c_2/2}\chi)}{\tanh(\sqrt{c_2/2}\chi)+\varepsilon} + \frac{3}{4}\frac{c_2 \sec h^4(\sqrt{c_2/2}\chi)}{[\tanh(\sqrt{c_2/2}\chi)+\varepsilon]^2}, U_1 = qW_1, \quad (19)$$

where $\chi = x + qy + nz + pt$, $\varepsilon = \pm 1$, and n, p, q are arbitrary constants while the constant $c_2 > 0$.

According to the symmetry group transformation Theorem 1 and the traveling wave solution Eqs. (19), we can obtain a new type of group invariant solution for Eqs. (6) as follows:

$$w' = \frac{\delta^2}{C_1^{1/3}}\xi_{1t} - \frac{3}{2}\frac{\delta^2 C_1^{2/3}}{C_3}\eta_{1z} + \delta^2 C_1^{2/3}\varOmega,$$
$$u' = -\frac{3}{2}\frac{\delta^2}{C_1^{1/3}}\xi_{1z} + q\frac{\delta^2 C_3}{C_1^{1/3}}\varOmega. \quad (20)$$

where

$$\varOmega = \frac{1}{4}c_2 - \frac{3n}{4q} + \frac{p}{2} \pm \frac{3}{2}\frac{c_2 \sec h^2(\sqrt{c_2/2}\varPhi)}{\tanh(\sqrt{c_2/2}\varPhi)+\varepsilon} + \quad (21)$$
$$\frac{3}{4}\frac{c_2 \sec h^4(\sqrt{c_2/2}\varPhi)}{[\tanh(\sqrt{c_2/2}\varPhi)+\varepsilon]^2},$$
$$\varPhi = \delta C_1^{1/3}x + \xi_1 + q(\frac{\delta C_3}{C_1^{2/3}}y + \eta_1) + n(C_3 z + C_4) + p(C_1 t + C_2),$$

$\xi_1 = \xi_1(z,t), \eta_1 = \eta_1(z)$, $C_1, C_2, C_3, C_4, n, p, q$ are all arbitrary constant while the constant δ possess discrete values determined by Eq. (13).

2 Transformation group and some solutions of Maccari's system

By using asymptotically exact reduction method based on Fourier expansion and spatiotemporal rescaling, Maccari derived the Maccari's system from the KP equation, and discussed the construction of Lax pairs. Several periodic wave solutions and soliton solutions have recently been reported based on an *F*-expansion method and hyperbolic function method. Maccari's system reads:

$$iq_t + q_{xx} + qh = 0,$$
$$h_t + h_y + (|q|^2)_x = 0, \quad (22)$$

with $q \equiv q(x,y,t)$ complex and $h \equiv h(x,y,t)$ real.

Here we let $q = u\mathrm{e}^{iv}$, where $u = u(x,y,t)$, $v = v(x,y,t)$ are real functions, and separate the real and imaginary part, then Eqs. (22) become:

$$uv_{xx} + 2u_x v_x + u_t = 0,,$$
$$uv_t - u_{xx} + uv_x^2 - uh = 0,$$
$$h_t + h_y + (u^2)_x = 0. \quad (23)$$

We apply the standard infinitesimal procedure to find the infinitesimals of x, y, t, u, v, h and hence the symmetry group of Eqs. (23). we generally give the infinitesimal operator

$$v = \xi\partial_x + \eta\partial_y + \tau\partial_t + \phi_1\partial_u + \phi_2\partial_v + \phi_3\partial_h, \quad (24)$$

where the coefficients $\xi, \eta, \tau, \phi_1, \phi_2, \phi_3$ are functions of $\{x, y, t, u, v, h\}$.

According to the general theory for symmetries of differential equations, to find these functions we prolong the vector to the third order derivatives and require that the third prolonged vector field annihilates Eq. (23) on the solution manifold respectively, this condition provides us with quite a complicated system of determining equations for coefficients. With the help of symbolic computation system *Maple*, we can obtain the following results:

$$\xi = (F_2'/2)x + F_3 y, \eta = F_1, \tau = F_1 + F_2,$$
$$\phi_1 = -1/4U(2F_1' + F_2'),$$
$$\phi_2 = 1/8F_2''x^2 + 1/2F_3'x + F_4 + F_5,$$
$$\phi_3 = 1/8F_2'''x^2 + 1/2F_3''x + F_5' - hF_2', \quad (25)$$

where

$$F_1=F_1(y),F_4=F_4(y),F_2=F_2(\rho),$$
$$F_3=F_3(\rho),F_5=F_5(\rho),(\rho=t-y),$$

and the superscripts denote derivatives of y and ρ. To obtain the similarity reductions and some exact solution of the Maccari's system, we have to solve the characteristic line equations

$$\frac{\mathrm{d}x}{\xi}=\frac{\mathrm{d}y}{\eta}=\frac{\mathrm{d}t}{\tau}=\frac{\mathrm{d}u}{\phi_1}=\frac{\mathrm{d}v}{\phi_2}=\frac{\mathrm{d}h}{\phi_3}. \tag{26}$$

However, it's difficult for us to give out the general solutions of Eq. (23). So we shall turn to the symmetry group direct method for the general symmetry groups of Eqs. (28).

We let

$$u=\alpha_1+\beta_1U+\gamma_1V+\theta_1H,$$
$$v=\alpha_2+\beta_2U+\gamma_2V+\theta_2H,$$
$$h=\alpha_3+\beta_3U+\gamma_3V+\theta_3H, \tag{27}$$

where $\alpha_i,\beta_i,\gamma_i,\theta_i,(i=1,2,3),\xi,\eta,\tau$ are all functions of $\{x,y,t\}$, while $U\equiv U(\xi,\eta,\tau)$, $V\equiv V(\xi,\eta,\tau)$, $H\equiv H(\xi,\eta,\tau)$ satisfy the Maccari's system but with new independent variables $\{\xi,\eta,\tau\}$, i.e.,

$$UV_{\xi\xi}+2U_\xi V_\xi+U_\tau=0,$$
$$UV_\tau-U_{\xi\xi}+UV_\xi^2-UH=0,$$
$$H_\tau+H_\eta+(U^2)_\xi=0. \tag{28}$$

Substituting Eqs. (27) into Eqs. (23) then vanishing $V_{\xi\xi},U_{\xi\xi}$ and H_τ by using Eqs. (28), from that, the remained determining equations of the functions of $\alpha_i,\beta_i,\gamma_i,\theta_i$ $(i=1,2,3),\xi,\eta,\tau$ can be read off by eliminating the coefficients of the polynomials of $\{U,V,H\}$ and its derivatives, then after tedious calculation, it is straightforward to figure out the general solution of the determining equations. The result as follows,

$$\xi=\delta_2\sqrt{\delta_1F'}x,\eta=\eta_0,\tau=\eta_0+F+C,\gamma_2=\delta_1,$$
$$\alpha_2=-[F''/(8F')]x^2+A_1+A_2,$$
$$\alpha_3=[-F'''/(8F')+3F''^2/(16F'^2)]x^2+A_2',$$
$$\beta_1=\delta_3\sqrt{\delta_2\sqrt{\delta_1F'}\eta_0'},\theta_3=\delta_1F',$$
$$\alpha_1=\beta_2=\beta_3=\gamma_1=\gamma_3=\theta_1=\theta_2=0, \tag{29}$$

Where $\eta=\eta_0(y),A_1=A_1(y),A_2=A_2(\rho)$ $(\rho=t-y)$, and the C is the arbitrary constant, and the constant $\delta_1,\delta_2,\delta_3$ possess discrete values determined by $\delta_1=\pm1$, $\delta_2=\pm1,\delta_3=\pm1$.

From the above results, noticing that q is complex while h is real at the same time, one can get the theorem for Eqs. (23) as follows:

Theorem 2 If $U=U(x,y,t),V=V(x,y,t)$, $H=H(x,y,t)$ is a solution of Eqs. (23), then so is,

$$u=\delta_3\sqrt{\delta_2\sqrt{F'}\eta_0'}U(\delta_2\sqrt{F'}x,\eta_0,\eta_0+F+C),$$
$$v=-\frac{F''}{8F'}x^2+A_1+A_2+V(\delta_2\sqrt{F'}x,\eta_0,\eta_0+F+C),$$
$$h=(-\frac{F'''}{8F'}+\frac{3F''^2}{16F'^2})x^2+$$
$$A_2'+F'H(\delta_2\sqrt{F'}x,\eta_0,\eta_0+F+C). \tag{30}$$

Remark 2 From theorem 2, it is easy to see that the symmetry group G_{MS} of the Maccari's system is divided into two sectors: the Lie point symmetry group which corresponds to $\delta_1=\delta_2=1$ and a coset of the Lie group which is related to $\delta_1=1,\delta_2=-1$. The coset is equivant to the reflected transformation of x, ie. $x\to-x$ company with the usual Lie point symmetry transformation. In other words, if we denote by S the Lie point symmetry group of the Maccari's system, by δ_x the reflection of x, by I the identity transformation and by $R\equiv\{I,\delta_x\}$ the discrete reflection group, then the full Lie symmetry group G_{MS} of the Maccari's system given by theorem 2 can be expressed as

$$G_{MS}=R\otimes S. \tag{31}$$

By means of Theorem 2, considering q is complex while h is real, we can obtain many kinds of group-invariant solutions starting from simple solutions in for the Maccari's system, we can easily obtain one Jacobi elliptic function solutions for Maccari's system as follows,

$$q(x,y,t)=\pm\sqrt{\frac{2(\lambda+k^2-\mu)(n-2k)}{m^2+1}}\cdot$$
$$sn(\varphi,m)e^{i(kx+\alpha y+\lambda t+l)}, \tag{32}$$

$$h(x,y,t)=\frac{2(\mu-\lambda-k^2)}{m^2+1}sn^2(\varphi,m)+\mu, \tag{33}$$

where

$$\varphi=\pm\sqrt{\frac{\mu-\lambda-k^2}{m^2+1}}(x+ny-2kt+\xi_0), \tag{34}$$

$0\leqslant m\leqslant1,n<2k$, $\mu,\alpha,\lambda,\xi_0,l$ are arbitrary constants, then the application of symmetry group theorem 2 on the solution (32) and (33).

$$q=\sqrt{\frac{2(\lambda+k^2-\mu)(n-2k)}{m^2+1}}\cdot \delta_3\sqrt{\delta_2\sqrt{F'}\eta_0'}sn(\Psi,m)e^{i\Theta}, \quad (35)$$

$$h=(-\frac{F'''}{8F'}+\frac{3F''^2}{16F'^2})x^2+A_2'+\frac{2(\mu-\lambda-k^2)}{m^2+1}F'sn^2(\varphi,m)+\mu F', \quad (36)$$

where

$$\Psi=\pm\sqrt{\frac{\mu-\lambda-k^2}{m^2+1}}[\delta_2\sqrt{F'}x+n\eta_0-2k(\eta_0+F+C)+\xi_0], \quad (37)$$

$$\Theta=-\frac{F''}{8F'}x^2+A_1+A_2+k\delta_2\sqrt{F'}x+\alpha\eta_0+\lambda F+\lambda\eta_0+\lambda C+l. \quad (38)$$

3 Conclusion

In this paper, the Lie symmetry of a (3+1)-dimensional nonlinear evolution equation and Maccari's system are obtained By the classical Lie approach and symbolic computation. Then making use of the new symmetry group method proposed by Lou et al. and symbolic computation, finite symmetry transformation groups for a (3+1)-dimensional nonlinear evolution equation and Maccari's system are obtained by solving the corresponding determining equations-a huge number of nonlinear partial differential equations. From the finite symmetry groups, not only the Lie symmetry group can be given, but a group of discrete transformations can be also obtained simultaneously. Further, one exact solution of a (3+1)-dimensional nonlinear evolution equation by a subequation expansion method and some simple exact solutions of Maccari's system in terms of Jacobi elliptic functions are presented. Finally, some group invariant solutions with rich structure for these two equations are derived by their finite symmetry transformation groups.

However, when nonlinear terms of dependent variables occur in the Lie symmetry algebra of a given NLPDE, we have to assume the finite symmetry group transformations to have a more complex form, even a general form. In turn, the price to pay is to solve a very large number of nonlinear NLPDEs. How to seek for an appropriate finite symmetry group transformation for a given NLPDE? How to integrate a large number of NLPDEs resulting from the above finite transformations? These problems should be studied further.

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一个 3+1 维非线性发展方程和 Maccari 系统的对称群及精确解

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摘要：利用广义对称群方法和符号计算，首先得到了一个 3+1 维非线性发展方程和 Maccari 系统的李群以及非李对称变换群，然后利用它们求出的对称群以及一些简单的种子解构造出新解.

关键词：对称群; 3+1 维非线性发展方程; Maccari 系统

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